Kangweon-Kyungki Math. Jour. **12** (2004), No. 1, pp. 33–40

CONDUCTANCE AND CAPACITY INEQUALITIES FOR CONFORMAL MAPPINGS

Bo-Hyun Chung

Abstract. Let $E, F \subset (R^*)^n$ be non-empty sets and let Γ be the family of all closed curves which join E to F in $(R^*)^n$. In this paper, we shall study the problems of finding properties for the conductance $C(\Gamma)$. And we obtain the inequalities in connection with capacity of condensers.

1. Introduction

The conductance of a curve family is a basic tool in the theory of quasiconformal and quasiregular mappings ([8]). The numerical value of the conductance is known only for a few curve families. Therefore good estimates are of importance. Several estimates are given in the paper ([1], [5], [6], [9]). And in Gehring [3], he has shown that the capacity is related to the conductance of a family of surfaces that separate the boundary components of a space ring A. In this paper, we consider the capacity of A that is related to the conductance of a family of curves which join the boundary components of A.

Throughout this paper, n is a fixed integer and $n \geq 2$. We denote the n-dimensional Euclidean space by $\mathbb{R}^n$ and its one-point compactification by $(R^*)^n = \mathbb{R}^n \cup \{\infty\}$. All topological operations are performed with respect to $(R^*)^n$. Balls and spheres centered at $x \in \mathbb{R}^n$ and with radius $r > 0$ are denoted, respectively, by

$$B^n(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$$

Received December 17, 2003. Revised February 17, 2004.
2000 Mathematics Subject Classification: 30C20, 30C85, 30D40.
Key words and phrases: conformal mapping, conductance, boundary behavior.

$$S^{n-1}(x,r) = \partial B^n(x,r) = \{y \in \mathbb{R}^n : |y-x| = r\}$$

We employ the abbreviations

$$B^n(r) = B^n(0,r), \quad B^n = B^n(1),$$

$$S^{n-1}(r) = S^{n-1}(0,r), \quad S^{n-1} = S^{n-1}(1).$$

As a measure in $\mathbb{R}^n$ we use the n-dimensional Lebesque measure m_n, the element of volume, where the subscript n may be omitted. And we abbreviate $\omega_n = m_n(B^n)$, where $\omega_q = \frac{\pi^{\frac{q}{2}}}{\Gamma(1+\frac{q}{2})}$. The standard unit coordinate vectors are $e_1, \cdots, e_n$.

2. Conductance of a curve family

Definition 2.1.([9]) Given a family, Γ, of nonconstant curves γ in $(R^*)^n$, we let $af(\Gamma)$ denote the family of Borel measurable functions $\rho : R^n \to [0,\infty)$ such that

$$\int_\gamma \rho \, ds \geq 1 \tag{1}$$

for all locally rectifiable $\gamma \in \Gamma$, where ds is the element of arc length. We call

$$C(\Gamma) = inf_{\rho \in af(\Gamma)} \int_{R^n} \rho^n \, dm \tag{2}$$

the conductance of Γ.

Example 2.2. If Γ is the family of curves γ joining two parallel faces of area and distance d apart, then

$$C(\Gamma) = a \cdot d^{1-n}. \tag{3}$$

In fact, choose $\rho \in af(\Gamma)$ and let γ_y be the vertical segment from y in the base B of parallel faces. Then $\gamma_y \in \Gamma$ and

$$1 \leq \left(\int_{\gamma} \rho \; ds\right)^n \leq d^{n-1} \int_{\gamma_y} \rho^n \; ds.$$

This holds for all such y and hence

$$\int_{R^n} \rho^n \; dm \geq \int_B \left(\int_{\gamma_y} \rho^n \; ds\right) dm_{n-1} \geq a \cdot d^{1-n}.$$

Since ρ is arbitrary,

$$C(\Gamma) \geq a \cdot d^{1-n}.$$

Next, let $\rho = \frac{1}{d}$ inside the parallelepiped and $\rho = 0$ otherwise.

Then $\rho \in af(\Gamma)$ and

$$C(\Gamma) \leq \int_{R^n} \rho^n \; dm = a \cdot d^{1-n}.$$

EXAMPLE 2.3.([2]) If Γ is the family of curves joining the sphere with center x_0 and radius r_1 to the concentric sphere of radius r_2, then

$$C(\Gamma) = n\omega_n \left(log\frac{r_2}{r_1}\right)^{1-n}. \tag{4}$$

In fact, choose $\rho \in af(\Gamma)$ and let

$$\gamma_e = \{x | x = re, r_1 < r < r_2\}$$

be the radial segment in Γ and is parallel to the unit vector e. Using Hölder's inequality (See [4], theorem 189, P.140) we obtain

$$1 \leq \left(\int_{\gamma_e} \rho \; ds\right)^n \leq \left(log\frac{r_2}{r_1}\right)^{n-1} \int_{r_1}^{r_2} \rho^n \; r^{n-1} \; dr.$$

Integrating over all e we obtain by Fubini's Theorem in polar coordinates

$$n\omega_n \le \left(log\frac{r_2}{r_1}\right)^{n-1}\int_A \rho^n\ dm,$$

where A is the spherical ring $r_1 < |x| < r_2$. The equality holds for

$$\rho = \frac{1}{|x|log\frac{r_2}{r_1}}.$$

Thus

$$C(\Gamma) = n\omega_n\left(log\frac{r_2}{r_1}\right)^{1-n}.$$

Proposition 2.4. *(i) If each curve γ_1 in a family Γ_1 contains a subcurve γ_2 in a family Γ_2, then*

$$C(\Gamma_1) \le C(\Gamma_2),$$

(ii) $C\left(\cup_j\Gamma_j\right) \le \sum_j C(\Gamma_j)$.

Proof. (i) Choose $\rho \in af(\Gamma_2)$ and suppose $\gamma_1 \in \Gamma_1$ is locally rectifiable. Then

$$\int_{\gamma_1}\rho\ ds \ge \int_{\gamma_2}\rho\ ds$$

where γ_2 is the subcurve in Γ_2, and $\rho \in af(\Gamma_1)$. Thus

$$C(\Gamma_1) \le \int_{R^n}\rho^n\ dm$$

and taking the infimum over all such ρ yields

$$C(\Gamma_1) \le C(\Gamma_2). \tag{5}$$

Briefly, the set of fewer and longer curves has the smaller conductance.

(ii) We may assume $C(\Gamma_j) < \infty$ for all j. Then given $\varepsilon > 0$ we can choose for each j a $\rho_j \in af(\Gamma_j)$ such that

$$\int_{R^n} (\rho_j)^n \; dm \leq C(\Gamma_j) + 2^{-j}\varepsilon$$

Now let

$$\rho = \sup_j \rho_j, \qquad \Gamma = \cup_j \Gamma_j$$

Then $\rho : R^n \to [0, \infty)$ is Borel measurable. Moreover, if $\gamma \in \Gamma$ is locally rectifiable, then $\gamma \in \Gamma_j$ for some j,

$$\int_\gamma \rho \; ds \geq \int_\gamma \rho_j \; ds \geq 1$$

and hence $\rho \in af(\Gamma)$. Thus

$$C(\Gamma) \leq \int_{R^n} \rho^n \; dm \leq \int_{R^n} \sum_j (\rho_j)^n \; dm \leq \sum_j C(\Gamma_j) + \varepsilon. \tag{6}$$

□

Proposition 2.5. *If $f : (R^*)^n \to (R^*)^n$ is a $1:1$ conformal mapping, then*

$$C(f(\Gamma)) = C(\Gamma). \tag{7}$$

for all curve families Γ in $(R^*)^n$.

Proof. Choose $\rho' \in af(f(\Gamma))$, let

$$\rho(x) = \rho' \circ f(x)|f'(x)|$$

for $x \in R^n - \{f^{-1}(\infty)\}$, and let Γ_0 be the family of $\gamma \in \Gamma$ which pass through $f^{-1}(\infty)$. Then

$$C(\Gamma) = C(\Gamma - \Gamma_0), \quad \rho \in af(\Gamma - \Gamma_0)$$

and hence

$$C(\Gamma) \le \int_{R^n} \rho^n \, dm = \int_{R^n} (\rho' \circ f)^n |f'| \, dm$$
$$= \int_{R^n} (\rho' \circ f)^n J(f) \, dm$$
$$= \int_{R^n} (\rho')^n \, dm.$$

Taking the infimum over every such ρ' gives

$$C(\Gamma) \le C(f(\Gamma)).$$

The result follows by repeating the preceding argument with f replaced by f^{-1}. □

3. Capacity of condensers

A condenser is a ring $R \subset (R^*)^n$ whose complement is the union of two distinguished disjoint compact sets D_0 and D_1. We write

$$R = R(D_0, D_1).$$

A ring is a condenser $R = R(D_0, D_1)$ where D_0 and D_1 are continua. We call D_0 and D_1 the complementary components of R.

Definition 3.1.([7], [9]) We let $d(x, y)$ denote the chordal distance between points $x, y \in (R^*)^n$. That is

$$d(x, y) = |x - y| \cdot [(1 + |x|^2)(1 + |y|^2)]^{-\frac{1}{2}}, \quad x, y \neq \infty$$

Let $af(R) \neq \varnothing$ denote the family of functions $u : (R^*)^n \to R^1$ with the following conditions :

(i) u is continuous in $(R^*)^n$ and u has distribution derivatives in R^1,

(ii) $u = 0$ on D_0, $u = 1$ on D_1,

(iii) $u(x) = \min\{\frac{d(x, D_0)}{d(D_1, D_0)}, 1\} \in af(R)$.

We call

$$Cap(R) = \inf_{u \in af(R)} \int_R |\bigtriangledown u|^n \, dm \tag{8}$$

the capacity of R.

Theorem 3.2. *If $R = R(D_0, D_1)$ is a condenser and if Γ is the family of curves γ joining D_0 and D_1 in R, then*

$$Cap(R) \leq C(\Gamma). \tag{9}$$

Proof. Choose a bounded continuous $\rho \in af(\Gamma)$ and let

$$u(x) = \min\{1, \inf_{\gamma} \int_{\gamma} \rho \, ds\}$$

for $x \in R$, where the infimum is taken over all locally rectifiable γ joining D_0 to x in R. Then u has distribution derivatives and

$$\lim_{x \to D_0} u(x) = 0, \qquad \lim_{x \to D_1} u(x) = 1.$$

Hence we can extend u to $(R^*)^n$ so that $u \in af(R)$. Then since $| \bigtriangledown u| = \rho$ in R,

$$Cap(R) \leq \int_R \rho^n \, dm \leq \int_{R^n} \rho^n \, dm$$

Another smoothing argument shows the infimum over such ρ gives $C(\Gamma)$. Thus

$$Cap(R) \leq C(\Gamma).$$

□

As an immediate consequences of Theorem 3.2 and Example 2.3 we have

Corollary 3.3. *If $A = \{x | r_1 < |x| < r_2\}$ is the condenser in $\mathbb{R}^n$ bounded by concentric sphere of radii r_1 and r_2, then*

$$Cap(A) \leq n\omega_n \left(\log \frac{r_2}{r_1} \right)^{1-n}.$$

If $n = 2$,

$$Cap(A) \leq \frac{2\pi}{\log \frac{r_2}{r_1}}.$$

References

1. P. Caraman, *n-Dimensional Quasiconformal Mappings*, Editura Academic Bucuresti, Romania (1974).
2. Enrique Villamor, *Geometric proofs of some classical results on boundary values for analytic functions*, Canadian Mathematical Bulletin **37** (1994), 263–269.
3. F. W. Gehring, *Quasiconformal Mappings*, Complex analysis and its applications **11** (1976), 213–268 Internat. Atomic Energy Agency.
4. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Univ. Press, Cambridge (1988).
5. O. Martio, S. Rickman and J. Vaisala, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. **448** (1969), 1–40.
6. R. Nakki, *Extension of Loewner's capacity theorem*, Trans. Amer. Math. Soc. **180** (1973), 229–236.
7. M. D. O'neill, R. E. Thurman, *Extremal problems for Robin capacity*, Complex Variables Theory and Applications, **41** (2000).
8. Shen Yu-Liang, *Extremal problems for quasiconformal mappings*, Journal of Mathematical Analysis and Applications **247** (2000), 27–44.
9. J. Vaisala, *Lectures on n-Dimensional Quasiconformal Mappings*, Springer-Verlag, New York (1971).

Mathematics Section, College of Science and Technology
Hong-Ik University
Chochiwon, 339-701 Korea
E-mail: `bohyun@hongik.ac.kr`