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COMPLETION OF A UNIFORM SPACE IN K_0-PROXIMITY SPACE

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Abstract. We introduce the k_0-proximity space as a generalization of the Efremovič -proximity space. We try to show every ultrafilter in K_0-proximity space generates a cluster and every Cauchy cluster is a point cluster.

1. Introduction

The proximity relation δ was introduced in 1950 by Efremovič and he showed that the proximity relation δ induces a topology $\tau(\delta)$ in X and that the induced topology is completely regular in [1].

He also showed that every completely regular space (X, τ) admits a compatible proximity δ on X such that $\tau(\delta) = \tau$. He axiomatically characterized the proximity relation, A is near B, which is denoted by $A\delta B$, for subsets A and B of any set X. Efremovič axioms of proximity relation δ are as follows;

E1. $A\delta B$ implies $B\delta A$.

E2. $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.

E3. $A\delta B$ implies $A \neq \phi$, $B \neq \phi$.

E4. $A\not\delta B$ implies there exists a subset E such that $A\not\delta E$ and $(X - E)\not\delta B$.

E5. $A \cap B \neq \phi$ implies $A\delta B$.

A binary relation δ satisfying axioms E1-E5 on the power set of X is called a (Efremocič) proximity on X. If δ also satisfies the following;

E6. $x\delta y$ implies $x = y$ then δ is called the separated proximity relation.

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Definition 1.1. Let δ be a binary relation between a set X and its power set $P(X)$ such that

$K_0 1$. $x\delta\{y\}$ implies $y\delta\{x\}$.

$K_0 2$. $x\delta(A \cup B)$ if and only if $x\delta A$ or $x\delta B$.

$K_0 3$. $x \not\delta \phi$ for all $x \in X$.

$K_0 4$. $x \in A$ implies $x\delta A$.

$K_0 5$. For each subset $E \subset X$, if there is a point $x \in X$ such that either $x\delta A$, $x\delta E$ or $x\delta B$, $x\delta(X - E)$, then we have $y\delta A$ and $y\delta B$ for some $y \in X$. The binary relation δ is called the K_0-proximity on X iff δ satisfies the axioms $K_0 1 - K_0 5$. The pair (X, δ) is called a K_0-proximity space.

$K_0 6$. If $x\delta\{y\}$ implies $x = y$, then δ is called the separated K_0-proximity relation.

Lemma 1.2. *In a K_0-proximity space (X, δ) let δ_1 be a binary relation on $P(X)$ defined as follows;*

If we define $A\delta_1 B$ if and only if there is a point $x \in X$ such that $x\delta A$,$x\delta B$, then δ_1 is an Efremovič proximity.

It is well known that a family $\mathcal{L}$ of subsets of a non-empty set X is an ultrafilter if and only if the following condition are satisfied:

(i) If A and B belong to $\mathcal{L}$, then $A \cap B \neq \phi$.

(ii) If $A \cap C \neq \phi$ for every $C \in \mathcal{L}$, then $A \in \mathcal{L}$.

(iii) If $(A \cup B) \in \mathcal{L}$, then $A \in \mathcal{L}$ or $B \in \mathcal{L}$.

Now we consider the family of sets in an K_0-proximity space satisfying condition similar to (i), (ii), (iii), with nearness replacing non-empty intersection and we are led to the following definition:

Definition 1.3. A family σ of subsets of an K_0-proximity space (X, δ) is called a cluster iff the following condition are satisfied;

(1) If A and B belong to σ, then there is a point $x \in X$ such that $x\delta A$ and $x\delta B$.

(2) If for every $C \in \sigma$, there is a point $x \in X$ such that $x\delta A$, $x\delta C$, then $A \in \sigma$.

(3) If $(A \cup B) \in \sigma$, then $A \in \sigma$ or $B \in \sigma$.

Definition 1.4. A subset B of a K_0-proximity space (X, δ) is a δ-neighborhood of A (in symbols $A \ll B$) iff for each $x \in X, x \not\delta A$ or $x \not\delta (X - B)$.

Lemma 1.5. *Let (X, δ) be a K_0-proximity space let $\bar{A}$ and Int A denote, respectively, the closure and interior of A in $\tau(\delta)$. Then*

1. *$A \ll B$ implies $\bar{A} \ll B$, and*
2. *$A \ll B$ implies $A \ll$ Int B.*

Therefore $A \subset$ Int B, showing that a δ-neighborhood is a topological neighborhood.

DEFINITION 1.6. A uniform structure (or uniformity) $\mathcal{U}$ on a set X is a collection of subsets (called entourages) of $X \times X$ satisfying the following conditions:

(1) Every entourage contains the diagonal Δ.
(2) If $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
(3) Given $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$.
(4) If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.
(5) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a uniform space.

A subfamily β of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ iff each entourage in $\mathcal{U}$ contains a member of β.

A family φ is a subbase for $\mathcal{U}$ iff the family of finite intersections of members of φ is a base for $\mathcal{U}$.

It can be shown that for each $x \in X$, $\{U[x] : U \in \mathcal{U}\}$ is a neighborhood filter. Thus $\mathcal{U}$ generates a topology $\mathcal{T} = \mathcal{T}(\mathcal{U})$ on X.

As is well known, this topology is always completely regular.

If $\mathcal{U}$ satisfies the additional condition

(6) $\bigcap_{U \in \mathcal{U}} U = \Delta$,

Then $\mathcal{U}$ is called a Hausdorff or separated uniformity.

In this case, $\mathcal{T}(\mathcal{U})$ is Tychonoff. Conversely, every (Tychonoff) completely regular space $(X, \mathcal{T})$ has a compactible(separated) uniformity, i.e. a uniformity $\mathcal{U}$ such that $\mathcal{T} = \mathcal{T}(\mathcal{U})$.

Every uniformity has a base consisting of open(closed) symmetric members, and it is frequently more convenient to work with such a base for $\mathcal{U}$ rather than with $\mathcal{U}$ itself.

LEMMA 1.7. *Every uniform space $(X, \mathcal{U})$ has an associated K_0-proximity $\delta = \delta(\mathcal{U})$ defined by that there is a point $x \in X$ such that $x\delta A$, $x\delta B$ iff $(A \times B) \cap U \neq \phi$ for every $U \in \mathcal{U}$.*

Furthermore, $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\delta)$. If $\mathcal{U}$ is separated, then so is $\delta(\mathcal{U})$.

LEMMA 1.8. *Let $(X, \mathcal{U})$ be a uniform space and let $\delta = \delta(\mathcal{U})$. Then $A \ll B$ if and only if there is an entourage U such that $U[A] \subset B$.*

Lemma 1.9. *If* $f : (X,\mathcal{U}_1) \to (Y,\mathcal{U}_2)$ *is uniformly continuous, then* $f : (X,\delta_1) \to (Y,\delta_2)$ *is* K_0*-proximally continuous where* $\delta_i = \delta(\mathcal{U}_i)$ *for* $i = 1, 2$.

2. Main Results

Let $(X,\mathcal{U})$ be a uniform space and $\delta = \delta(\mathcal{U})$. We have seen that every ultrafilter in a K_0-proximity space generates a cluster and that given a set A in a cluster σ, there exists an ultrafilter containing A which generates σ. It therefore seems natural to call a cluster Cauchy if it is generated by a Cauchy ultrafilter. One can readily convince oneself that every Cauchy cluster can be considered to be a point cluster, determined by a point of the completion of X.

Throughout this section, we shall suppose that $\mathcal{U}$ is an open symmetric base for a separated uniformity.

Definition 2.1. A cluster σ in $(X,\mathcal{U},\delta)$ is Cauchy iff there exists a round Cauchy filter $\mathcal{M} \subset \sigma$ such that $M \cap C \neq \phi$ for each $M \in \mathcal{M}$ and $C \in \sigma$.

Given a round Cauchy filter $\mathcal{M}$, $U \in \mathcal{U}$ and $n \in N$, there exists an $M \in \mathcal{M}$ and $V \in \mathcal{U}$ such that $\overset{n}{V}[M] \times \overset{n}{V}[M] \subset U$ and $\overset{n}{V} \subset U$.

This can be seen from the following argument. Since $\mathcal{M}$ is Cauchy and round, there exists sets $M', M \in \mathcal{M}$ such that $M' \times M' \subset U$ and $M \ll M'$. Let $V_1 \in \mathcal{U}$ be such that $V_1[M] \subset M'$.

Then V may be chosen to be that entourages satisfying $\overset{n}{V} \subset U \cap V_1$.

Definition 2.2. A uniform space $(X,\mathcal{U})$ is complete iff every Cauchy cluster in $(X,\mathcal{U},\delta)$ is a point cluster σ_x for some $x \in X$.

Lemma 2.3. *A closed subspace* Y *of a complete uniform space* $(X,\mathcal{U})$ *is complete.*

Proof. The trace $\mathcal{U}_Y = \{U \cap (X \times Y) : U \in \mathcal{U}\}$ of $\mathcal{U}$ on Y is a base for the subspace uniformity on Y. If σ_1 is any Cauchy cluster in Y, then σ_1 is a subclass of a unique Cauchy cluster σ_2 in X. Since X is complete, $\sigma_2 = \sigma_x$ for some $x \in X$. But $x\delta B$ for every $B \in \sigma_1$ and since Y is closed, $x \in Y$. Therefore $\{x\} \in \sigma_1$. Let f be a mapping which associates with each point $x \in X$, the point cluster σ_x. Then f is a one-to-one mapping of X onto the space $f(X)$ of all point clusters. Let X^* denote the set of all Cauchy clusters in X. Since every point cluster in X is Cauchy, it follows that $f(X) \subset X^* \subset \mathfrak{X}$.

For each Cauchy cluster σ, let $\mathcal{M}(\sigma)$ be one of the filters given by definition 2.1. For each $U \in \mathcal{U}$, define

$U^* = \{(\sigma_1, \sigma_2) \in X^* \times X^*$: there exist $M \in \mathcal{M}(\sigma_1), N \in \mathcal{M}(\sigma_2)$ such that $M \times N \subset U\}$

To see that U^* is independent of the choice of $\mathcal{M}(\sigma_1)$ and $\mathcal{M}(\sigma_2)$ suppose $(\sigma_1, \sigma_2) \in U^*$. Then there exists an $M \in \mathcal{M}(\sigma_1), N \in \mathcal{M}(\sigma_2)$ and a $V \in \mathcal{U}$ satisfying $V[M] \times V[N] \subset U$.

Given any $\mathcal{M}'(\sigma_1) \neq \mathcal{M}(\sigma_1)$, we can find an $M' \in \mathcal{M}'(\sigma_1)$ such that $M' \times M' \subset V$. Now $M \cap M' \neq \phi$, so that $M' \subset V[M]$.

Hence $V[M] \in \mathcal{M}'(\sigma_1), V[N] \in \mathcal{M}(\sigma_2)$ and $V[M] \times V[N] \subset U$, showing that U^* is well defined. □

LEMMA 2.4. *$\mathcal{U}^* = \{U^* : U \in \mathcal{U}\}$ is a uniformity base on X^*.*

Proof. Every U^* obviously contains the diagonal, and $(U \cap V)^* \subset U^* \cap V^*$.

Given $U^* \in \mathcal{U}^*$, there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$. That $V^* \circ V^* \subset U^*$ follows from the follow argument : if $(\sigma_1, \sigma_2) \in V^* \circ V^*$, then there exists a $\sigma_3 \in X^*$ such that $(\sigma_1, \sigma_3) \in V^*$ and $(\sigma_3, \sigma_2) \in V^*$. Hence there exists an $A \in \mathcal{M}(\sigma_1), B \in \mathcal{M}(\sigma_2)$ and $C', C'' \in \mathcal{M}(\sigma_3)$ such that $A \times C' \subset V$ and $C'' \times B \subset V$. Setting $C = C' \cap C'' \in \mathcal{M}(\sigma_3)$, we have $A \times C \subset V$ and $C \times B \subset V$, Therefore $A \times B \subset V \circ V \subset U$, which implies $(\sigma_1, \sigma_2) \in U^*$. □

Since $f(X) \subset X^* \subset \mathfrak{X}$ and $f(X)$ is dense in $\mathfrak{X}$, we have the following result :

LEMMA 2.5. *$f(X)$ is a dense subset of X^*.*

Let δ^* be the K_0-proximity induced by $\mathcal{U}^*$ on X^*. The restriction $\mathcal{U}_f^*$ of $\mathcal{U}^*$ to $f(X)$ is a uniformity base on $f(X)$ and so induces the K_0-proximity δ_f^* on $f(X)$.

THEOREM 2.6. *$(X,\mathcal{U},\delta)$ and $(F(X),\mathcal{U}_f^*,\delta_f^*)$ are K_0-proximally isomorphic.*

Proof. Clearly f is one-to-one and onto. Suppose for some $x \in X, x\delta A$ and $x\delta B$. Given $U \in \mathcal{U}$, let $U_f^* = U^* \cap (f(X) \times f(X))$.

Then we must show that for some $x \in X, (f(x) \times f(A)) \cap U_f^* \neq \phi$ and $(f(x) \times f(B)) \cap U_f^* \neq \phi$. Let $V \in \mathcal{U}$ be such that $\overset{3}{U} \subset U$.

Since for some $x \in X, x\delta A$ and $x\delta B$, there exist $a \in A, b \in B$ such that $(a, b) \in V$. Therefore $V[a] \times V[b] \subset U$, and the point cluster σ_a, σ_b satisfy the condition $(\sigma_a, \sigma_b) \in U^*_f$. Conversely, if for some $x \in X, f(x)\delta^*_f f(A)$ and $f(x)\delta^*_f f(B)$, then for each U^*_f there exist a $(\sigma_a, \sigma_b) \in U^*_f$, where $\sigma_a \in f(A), \sigma_b \in f(B)$.

Hence $(a, b) \in U$ and we have $(A \times B) \cap U \neq \phi$ for arbitrary $U \in \mathcal{U}$, showing for some $x \in X, x\delta A$ and $x\delta B$. □

It can similarly be proved that $(X,\mathcal{U})$ and $(f(X),\mathcal{U}^*_f)$ are uniformly isomorphic.

Theorem 2.7. *Every Cauchy cluster in* $(X^*,\mathcal{U}^*, \delta^*)$ *is a point cluster.*

Proof. Let σ^* be any Cauchy cluster in X^*. Since $f(X)$ is dense in X^*, σ^* determines a unique Cauchy cluster σ' in $f(X)$ such that $\sigma' \subset \sigma^*$. But σ' is isomorphic to a Cauchy cluster σ in X. In order to show that $\sigma \in \sigma^*$, it is sufficient to verify that for each $U^* \in \mathcal{U}^*$ and each $M \in \sigma', (\sigma \times M) \cap U^* \neq \phi$.

Given $U^* \in \mathcal{U}^*$, there exists a $V \in \mathcal{U}$ and $C \in \mathcal{M}(\sigma)$ such that $\overset{3}{V} \subset U$ and $C \times C \subset V$.

Then $V[C] \times V[C] \subset U$. Setting $M_0 = V[C] \cap f^{-1}(M)$ we have $M_0 \in \sigma$ since $V[C] \in \mathcal{M}(\sigma), f^{-1}(M) \in \sigma$, and we can find an ultrafilter containing both $V[C]$ and $f^{-1}[M]$ which generates σ.

Choose a point $p \in M_0$. Since $V[C]$ is open, there exists a $W \in \mathcal{U}$ such that $W[p] \subset V[C]$. We therefore have $W[p] \times V[C] \subset U$, where $W[p] \in \mathcal{M}(\sigma_p), V[C] \in \mathcal{M}(\sigma)$ and $\sigma_p \in M$. Thus $(\sigma_p, \sigma) \in U^*$ and $(\sigma \times M) \cap U^* \neq \phi$. □

The above result shows that $(X^*,\mathcal{U}^*, \delta^*)$ is complete; for if $\mathcal{F}$ is any Cauchy filter in X^*, then $\mathcal{F}$ is contained in a Cauchy ultrafilter. This ultrafilter generates a Cauchy cluster which, by Theorm 2.7., must be a point cluster σ_{x_0} for some $x_0 \in X^*$. Clearly x_0 is a cluster point of the Cauchy filter $\mathcal{F}$, and thus $\mathcal{F}$ converges to x_0.

Finally, we remark that every Cauchy cluster in X is generated by a Cauchy ultrafilter containing the neighborhood filter of some point in X^*. (If the point is in $X^* - X$, consider the trace of its neighborhood filter on X.) Hence Definition 2.1. is equivalent to: a cluster is Cauchy iff it is generated by a Cauchy ultrafilter.

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