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# ON THE QUOTIENT BOOLEAN ALGEBRA $\wp(S)/I$

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Abstract. In this paper we introduce the notion of quotient Boolean algebra and study the relation between the ideals of Boolean algebra $\wp(S)$ and the ideals of quotient Boolean algebra $\wp(S)/I$.

## 1. Introduction

It is regarded that the concept of quotient Boolean algebra had been studied for long period of time. The research on this subject has been done by many mathematicians including Roman Sikorski[ 9 ] and Dwinger [ 6 ]. In particular, Sabine Koppelberg [ 8 ] gave order relation on Boolean algebra and defined ideals on it. Using the concept of ideals, quotient Boolean algebra was introduced. Let $S$ be a nonempty set. Then the triple $(\wp(S), \cup, \cap)$ is a Boolean algebra. For any nonempty subset $I$ of $\wp(S)$ the triple $(I, \cup, \cap)$ is said to be a Boolean ideal of $\wp(S)$ if and only if i)$A, B \in I$ imply $A \cup B \in I$, ii) $A \in I$ and $X \in \wp(S)$ imply $A \cap X \in I$. Every ideal of $\wp(S)$ contains $\phi$( zero element ). And we have that if $X \cup Y \in I$ then $X, Y \in I$. We studied the properties of ideals of Boolean algebra $\wp(S)$[ 3 ]. In this paper we introduce the notion of an equivalence relation on $\wp(S)$ and the notion of quotient Boolean algebra, and we study the properties of ideals of the quotient Boolean algebra.

## 2. Quotient Boolean Algebra $\wp(S)/I$

Let $I$ be an ideal of Boolean algebra $\wp(S)$. For any $X, Y \in \wp(S)$, we define a relation $\sim$ on $\wp(S)$ by $X \sim Y$ if and only if there exist

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$I_1, I_2 \in I$ such that $X \cup I_1 = Y \cup I_2$. Now we prove that $\sim$ is an equivalence relation on $\wp(S)$. Since $\phi \in I$, $X \sim X$ for any $X \in \wp(S)$. This means that $\sim$ is reflexive. By the definition of the relation, $\sim$ is symmetric. If $X \sim Y$ and $Y \sim Z$, then there are $I_1, I_2, I_3$ and $I_4 \in I$ such that $X \cup I_1 = Y \cup I_2$ and $Y \cup I_3 = Z \cup I_4$. Hence $(X \cup I_1) \cup I_3 = (Y \cup I_2) \cup I_3 = Y \cup (I_2 \cup I_3) = Y \cup (I_3 \cup I_2) = (Y \cup I_3) \cup I_2 = (Z \cup I_4) \cup I_2$. That is $X \cup (I_1 \cup I_3) = Z \cup (I_4 \cup I_2)$. Thus $X \sim Z$. Hence $\sim$ is transitive. Therefore $\sim$ is an equivalence relation on $\wp(S)$.

Furthermore we have the following Lemma ;

LEMMA 2.1. *If $A \sim B$, then $A \cup X \sim B \cup X$ and $A \cap X \sim B \cap X$ for all $X \in \wp(S)$.*

*Proof.* Since $A \sim B$, there exist $I_1, I_2 \in I$ such that $A \cup I_1 = B \cup I_2$. Thus for any $X \in \wp(S)$, $(A \cup I_1) \cup X = (B \cup I_2) \cup X$. Hence $(A \cup X) \cup I_1 = (B \cup X) \cup I_2$. Therefore $A \cup X \sim B \cup X$. Similarly, $(A \cup I_1) \cap X = (B \cup I_2) \cap X$ and so $(A \cap X) \cup (I_1 \cap X) = (B \cap X) \cup (I_2 \cap X)$. Since $I$ is an ideal of $\wp(S)$, $I_1 \cap X, I_2 \cap X \in I$. Thus $A \cap X \sim B \cap X$. □

We denote by $[X]_I = \{Y \in \wp(S) | X \sim Y\}$ the equivalence class of $X$ determined by an ideal $I$.

**Note.**

1). For each $X \in \wp(S)$, $X \in [X]_I$.
2). $[X]_I = [Y]_I \Leftrightarrow X \sim Y$.
3). $X \in [Y]_I \Leftrightarrow [X]_I = [Y]_I \Leftrightarrow Y \in [X]_I$.
4). For any $X, Y \in \wp(S)$, the exactly one of the followings hold ;
  i). $[X]_I \cap [Y]_I = \phi$
  ii). $[X]_I = [Y]_I$.

Now we have the following lemma.

LEMMA 2.2. *For any ideal $I$ of $\wp(S)$, $I = [\phi]_I$.*

*Proof.* If $X \in I$, then $X \cup \phi = \phi \cup X$. Thus $X \sim \phi$. Hence $X \in [\phi]_I$. i.e. $I \subseteq [\phi]_I$. Conversely, if $X \in [\phi]_I$, then $\phi \sim X$ and so there are $I_1, I_2 \in I$ such that $\phi \cup I_1 = X \cup I_2$. Thus $I_1 = X \cup I_2$. Since $I$ is an ideal of $\wp(S)$ and $X \cup I_2 \in I, X \in I$. Hence $[\phi]_I \subseteq I$. Therefore $I = [\phi]_I$. □

Denote by $\wp(S)/I = \{[X]_I | X \in \wp(S)\}$ the set of all equivalence classes $[X]_I$ determined by an ideal $I$, and we define two operations on $\wp(S)/I$ as followings :

$[A]_I + [B]_I = [A \cup B]_I$ and $[A]_I * [B]_I = [A \cap B]_I$.

Since $\sim$ is an equivalence relation on $\wp(S)$, the operation " + " and " $*$ " are well-defined. In fact, let $[A]_I = [A^*]_I$ and $[B]_I = [B^*]_I$. Then $A \sim A^*$ and $B \sim B^*$. Thus $A \cup B \sim A^* \cup B^*$ and $A \cap B \sim A^* \cap B^*$. Hence $[A]_I + [B]_I = [A \cup B]_I = [A^* \cup B^*]_I = [A^*]_I + [B^*]_I$ and $[A]_I * [B]_I = [A \cap B]_I = [A^* \cap B^*]_I = [A]_I * [B]_I$. Therefore the operation $+$ and $*$ are well-defined.

THEOREM 2.3. *Let $I$ be an ideal of Boolean algebra $\wp(S)$. Then $(\wp(S)/I, +, *)$ is also a Boolean algebra with zero element $[\phi]_I = I$ and unity element $[S]_I$.*

*Proof.* Let $[A]_I, [B]_I, [C]_I \in \wp(S)/I$. Then $[A]_I + [B]_I = [A \cup B]_I = [B \cup A]_I = [B]_I + [A]_I$ and $[A]_I * [B]_I = [A \cap B]_I = [B \cap A]_I = [B]_I * [A]_I$. Also $[A]_I + ([B]_I * [C]_I) = [A]_I + ([B \cap C]_I) = [A \cup (B \cap C)]_I = [(A \cup B) \cap (A \cup C)]_I = [A \cup B]_I * [A \cup C]_I = ([A]_I + [B]_I) * ([A]_I + [C]_I)$. Similarly $[A]_I * ([B]_I + [C]_I) = [A]_I * ([B \cup C]_I) = [A \cap (B \cup C)]_I = [(A \cap B) \cup (A \cap C)]_I = [A \cap B]_I + [A \cap C]_I = ([A]_I * [B]_I) + ([A]_I * [C]_I)$. There exist $[\phi]_I = I$ and $[S]_I$ in $\wp(S)/I$ such that $[A]_I + [\phi]_I = [A \cup \phi]_I = [A]_I$ and $[A]_I * [S]_I = [A \cap S]_I = [A]_I$ for all $[A]_I \in \wp(S)/I$. For each element $[A]_I \in \wp(S)/I$, there exists an element $[A']_I$ in $\wp(S)/I$ such that $[A]_I + [A']_I = [A \cup A']_I = [S]_I$ and $[A]_I * [A']_I = [A \cap A']_I = [\phi]_I$. Therefore $(\wp(S)/I, +, *)$ is a Boolean algebra. □

The Boolean algebra $\wp(S)/I$ described in Theorem 2.3. is called a quotient Boolean algebra of Boolean algebra $\wp(S)$ by an ideal $I$.

We have the following Theorem related to ideals of $\wp(S)$ and $\wp(S)/I$.

THEOREM 2.4. *If $I$ and $J$ are ideals of Boolean algebra $\wp(S)$ and $I \subseteq J$, then*

*i) $I$ is also an ideal of $J$*

*ii) $J/I = \{[X]_I | X \in J\}$ is an ideal of $\wp(S)/I$.*

*Proof.* i) It is easily proved from the definition of ideal of Boolean algebra $\wp(S)$.

ii) First we have to show that $J/I \subseteq \wp(S)/I$, i.e. each element of $J/I$ is an element of $\wp(S)/I$. To avoid the ambiguity, we denote the element of $J/I$ containing $X$ by $[X]_I^J$. Let $X \in J$ and $Y \in [X]_I$. Then $Y \in \wp(S)$ and $X \sim Y$ with respect to $I$. It follows that there exist $I_1, I_2 \in I$ such that $X \cup I_1 = Y \cup I_2$. Since $I \subseteq J$ and $X \in J, X \cup I_1 = Y \cup I_2 \in J$. Since $J$ is an ideal of $\wp(S), Y \in J$. Thus $Y \in [X]_I^J$, i.e. $[X]_I \subseteq [X]_I^J$. Obviously, $[X]_I^J \subseteq [X]_I$. Therefore $[X]_I = [X]_I^J$. This means that each element of $J/I$ is also an element of $\wp(S)/I$. Now we prove that $J/I$ is an ideal of $\wp(S)/I$. Let $[A]_I^J$ and $[B]_I^J$ be two element of $J/I$. Then $A$ and $B$ are elements of $J$ and so $A \cup B$ is an element of ideal $J$. Thus $[A]_I^J + [B]_I^J = [A]_I + [B]_I = [A \cup B]_I = [A \cup B]_I^J \in J/I$. Finally we let $[C]_I \in \wp(S)/I$ and $[A]_I = [A]_I^J \in J/I$. Then $[C]_I * [A]_I^J = [C]_I * [A]_I = [C \cap A]_I = [C \cap A]_I^J \in J/I$, since $C \cap A \in J$. Therefore $J/I$ is an ideal of $\wp(S)/I$. $\square$

EXAMPLE 2.1. *Let* $S = \{a, b, c\}$.
*Then* $\wp(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, S\}$, *and* $I_1 = \{\phi\}$, $I_2 = \{\phi, \{a\}\}, I_3 = \{\phi, \{b\}\}, I_4 = \{\phi, \{c\}\}, I_5 = \{\phi, \{a\}, \{b\}, \{a, b\}\}$, $I_6 = \{\phi, \{a\}, \{c\}, \{a, c\}\}$, $I_7 = \{\phi, \{b\}, \{c\}, \{b, c\}\}$, $I_8 = \wp(S)$ *are ideals of* $\wp(S)$. *Thus* $[\phi]_{I_2} = \{\phi, \{a\}\} = I_2 = [\{a\}]_{I_2}, [\{b\}]_{I_2} = \{\{b\}, \{a, b\}\} = [\{a, b\}]_{I_2}$, $[\{c\}]_{I_2} = \{\{c\}, \{a, c\}\} = [\{a, c\}]_{I_2}$, $[\{b, c\}]_{I_2} = \{\{b, c\}, S\} = [S]_{I_2}$. *Hence* $\wp(S)/I_2 = \{[\phi]_{I_2}, [\{b\}]_{I_2}, [\{c\}]_{I_2}, [S]_{I_2}\}$. *If* $I = I_2$ *and* $J = I_5$, *then* $I$ *is an ideal of* $J$. *And* $[\phi]_I^J = \{\phi, \{a\}\} = [\{a\}]_I^J = [\{a\}]_I = [\phi]_I = I, [\{b\}]_I^J = \{\{b\}, \{a, b\}\} = [\{a, b\}]_I^J = [\{b\}]_I = [\{a, b\}]_I$. *Thus* $J/I = \{[\phi]_I^J, [\{b\}]_I^J\} = \{[\phi]_I, [\{b\}]_I\}$.

*Denote :* $[\phi]_I = \overline{\phi}, [\{b\}]_I = \overline{b}, [\{c\}]_I = \overline{c}, [S]_I = \overline{S}$. *Then we have the following tables and we can check the Theorem 2.3. and 2.4.*

| $+$ | $\overline{\phi}$ | $\overline{b}$ | $\overline{c}$ | $\overline{S}$ |
|---|---|---|---|---|
| $\overline{\phi}$ | $\overline{\phi}$ | $\overline{b}$ | $\overline{c}$ | $\overline{S}$ |
| $\overline{b}$ | $\overline{b}$ | $\overline{b}$ | $\overline{S}$ | $\overline{S}$ |
| $\overline{c}$ | $\overline{c}$ | $\overline{S}$ | $\overline{c}$ | $\overline{S}$ |
| $\overline{S}$ | $\overline{S}$ | $\overline{S}$ | $\overline{S}$ | $\overline{S}$ |

| $*$ | $\overline{\phi}$ | $\overline{b}$ | $\overline{c}$ | $\overline{S}$ |
|---|---|---|---|---|
| $\overline{\phi}$ | $\overline{\phi}$ | $\overline{\phi}$ | $\overline{\phi}$ | $\overline{\phi}$ |
| $\overline{b}$ | $\overline{\phi}$ | $\overline{b}$ | $\overline{\phi}$ | $\overline{b}$ |
| $\overline{c}$ | $\overline{\phi}$ | $\overline{\phi}$ | $\overline{c}$ | $\overline{c}$ |
| $\overline{S}$ | $\overline{\phi}$ | $\overline{b}$ | $\overline{c}$ | $\overline{S}$ |

*Thus* $\overline{\phi} = [\phi]_I = I$ *is an identity with respect to* $+$ *and* $\overline{S} = [\{b, c\}]_I = [S]_I$ *is an identity with respect to* $*$. *Also* $\overline{\phi}' = \overline{S}, \overline{b}' = \overline{c}, \overline{c}' = \overline{b}, \overline{S}' = \overline{\phi}$. *Also* $J/I = \{\overline{\phi}, \overline{b}\}$ *is an ideal of* $\wp(S)/I$.

THEOREM 2.5. *If* $\overline{J}$ *is an ideal of* $\wp(S)/I$, *then* $J = \bigcup\{[X]_I | [X]_I \in \overline{J}\}$ *is an ideal of* $\wp(S)$ *and* $I \subseteq J$.

*Proof.* Let $A, B \in J$. Then there exist $[X]_I$ and $[Y]_I$ in $\overline{J}$ such that $A \in [X]_I$ and $B \in [Y]_I$. Thus $X \sim A$ and $Y \sim B$. By Lemma 2.1. $X \cup Y \sim A \cup B$. Hence $A \cup B \in [X \cup Y]_I = [X]_I + [Y]_I \in \overline{J}$, since $\overline{J}$ is an ideal of $\wp(S)/I$. Thus $A \cup B \in J$. Now let $C \in \wp(S)$ and $A \in J$. Then there exists $[X]_I$ in $\overline{J}$ such that $A \in [X]_I$ and $[C]_I \in \wp(S)/I$. Thus $X \sim A$. By Lemma 2.1. $C \cap X \sim C \cap A$. Hence $C \cap A \in [C \cap X]_I =$

$[C]_I * [X]_I \in \overline{J}$, since $\overline{J}$ is an ideal of $\wp(S)/I$. Thus $C \cap A \in J$. Therefore $J$ is an ideal of $\wp(S)$. Also, $I = [\phi]_I \in \overline{J}$ and so $I \subseteq J$. □

EXAMPLE 2.2. *In Example 2.1., put* $\overline{J} = J/I = \{[\phi]_I, [\{b\}]_I\}$. *Then* $\overline{J}$ *is an ideal of* $\wp(S)/I$. *Also,* $J = [\phi]_I \cup [\{b\}]_I = \{\phi, \{a\}\} \cup \{\{b\}, \{a, b\}\} = \{\phi, \{a\}, \{b\}, \{a, b\}\} = I_5$ *is an ideal of* $\wp(S)$.

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