Kangweon-Kyungki Math. Jour. **12** (2004), No. 1, pp. 67–75

INEQUALITIES FOR JACOBI POLYNOMIALS

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Abstract. Paul Turan observed that the Legendre polynomials satisfy the inequality $P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) > 0, -1 \leq x \leq 1$. And G. Gasper(ref. [6], ref. [7]) proved such an inequality for Jacobi polynomials and J. Bustoz and N. Savage (ref. [2]) proved $P_n^\alpha(x)P_{n+1}^\beta(x) - P_{n+1}^\alpha(x)P_n^\beta(x) > 0, \frac{1}{2} \leq \alpha < \beta \leq \alpha + 2, 0 < x < 1$, for the ultraspherical polynomials (respectively, Laguerre ploynomials). The Bustoz-Savage inequalities hold for Laguerre and ultraspherical polynomials which are symmetric. In this paper, we prove some similar inequalities for non-symmetric Jacobi polynomials.

1. Introduction

A distribution function $\alpha(x)$ is a non-decreasing function defined on $(-\infty, \infty)$ such that the moments $\int_{-\infty}^{\infty} x^n d\alpha(x)$ are finite for $n = 0, 1, 2, \cdots$. A sequence of polynomials$\{P_n(x)\}$ with degree $P_n(x) = n$ is said to be orthogonal if

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\alpha(x) = k_n\delta_{mn}, m, n = 0, 1, 2, \cdots . \tag{1.1}$$

where $k_n > 0$.

Probably the best known orthogonal polynomials are the classical orthogonal polynomials. These include the Jacobi, Laguerre and Hermite polynomials. The ultraspherical polynomial is a special cases of the Jacobi polynomial and in turn the Legendre and Chebyshev polynomials are special ultraspherical polynomials. The Hungarian mathematican Paul Turan observed that the Legendre polynomials satisfy the inequality

$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) > 0, -1 \leq x \leq 1. \tag{1.2}$$

Received February 11, 2004.
2000 Mathematics Subject Classification: 33C45.
Key words and phrases: Jacobi polynomial, orthogonal polynomial.

Gabor Szego(ref. [10]) gave two very beautiful proofs of (1.2). In the years since Szego's paper appeared, it has been proved by various people that the classical orthogonal polynomials satisfy (1.2) (ref. [1], ref. [3], ref. [4], ref. [8], ref. [9], ref. [11]). In particular, G. Gasper(ref. [6], ref. [7]) proved such an inequality for Jacobi polynomials and J. Bustoz and N. Savage(ref. [2]) proved

(1.3)
$$P_n^{\alpha}(x)P_{n+1}^{\beta}(x) - P_{n+1}^{\alpha}(x)P_n^{\beta}(x) > 0, \frac{1}{2} \le \alpha < \beta \le \alpha + 2, 0 < x < 1,$$

for the ultraspherical polynomials (respectively, Laguerre ploynomials). The Bustoz-Savage inequalities hold for Laguerre and ultraspherical polynomials which are symmetric.

In this paper, we prove some similar inequalities for non-symmetric Jacobi polynomials.

2. Main results

We will need the following inequalities(ref. [5], ref. [11]). We will frequently suppress the independent variable and write $P_n^{a,b}$ for $P_n^{a,b}(x)$.

$$\begin{aligned} &2(n+1)(n+a+b+1)(2n+a+b)P_{n+1}^{a,b} \\ &= (2n+a+b+1)[(2n+a+b)(2n+a+b+2)x + a^2 - b^2]P_n^{a,b} \\ &\quad - 2n(n+a)(n+b)(2n+a+b+2)P_{n-1}^{a,b}, \ n = 1, 2, 3, \cdots . \end{aligned} \tag{2.1}$$

$$P_n^{a,b} = 2^{-n}\sum\nolimits_{m=0}^{n} \binom{n+a}{m}\binom{n+b}{n-m}(x-1)^{n-m}(x+1)^m. \tag{2.2}$$

$$(1-x)P_n^{a+1,b} + (1+x)P_n^{a,b+1} = 2P_n^{a,b}. \tag{2.3}$$

$$(2n+a+b)P_n^{a+1,b} = (n+a+b)P_n^{a,b} - (n+b)P_{n-1}^{a,b}. \tag{2.4}$$

$$(2n+a+b)P_n^{a,b-1} = (n+a+b)P_n^{a,b} + (n+a)P_{n-1}^{a,b}. \tag{2.5}$$

$$P_n^{a,b-1} - P_n^{a-1,b} = P_{n-1}^{a,b}. \tag{2.6}$$

$$(n + \frac{a}{2} + \frac{b}{2} + 1)(1-x)P_n^{a+1,b} = (n+a+1)P_n^{a,b} - (n+1)P_{n+1}^{a,b}. \tag{2.7}$$

$$(2.8)\quad (n+\frac{a}{2}+\frac{b}{2}+1)(1+x)P_n^{a,b+1}=(n+b+1)P_n^{a,b}+(n+1)P_{n+1}^{a,b}.$$

Writing $R_n = R_n(x) = P_n^{a,b}(x), \quad S_n = S_n(x) = P_n^{c,d}(x)$ and letting a prime denote differentiation with respect to x, we find from pp.71-72 of ref. [11] that

$$(2.9)\quad \begin{aligned} (1-x^2)R_n^{'} &= E_nR_n + F_nR_{n-1} \\ &= G_nR_n + H_nR_{n+1}, \\ (1-x^2)S_n^{'} &= E_n^*S_n + F_n^*S_{n-1} \\ &= G_n^*S_n + H_n^*S_{n+1}, \end{aligned}$$

where

$$(2.10)\quad \begin{aligned} E_n &= E_n(x) = -nx - \frac{n(b-a)}{2n+a+b}, \\ F_n &= \frac{2(n+a)(n+b)}{2n+a+b}, \\ G_n &= G_n(x) = (n+a+b+1)x + \frac{(n+a+b+1)(a-b)}{2n+a+b+2}, \\ H_n &= \frac{-2(n+1)(n+a+b+1)}{2n+a+b+2}, \\ E_n^* &= E_n^*(x) = -nx - \frac{n(d-c)}{2n+c+d}, \\ F_n^* &= \frac{2(n+c)(n+d)}{2n+c+d}, \\ G_n^* &= G_n^*(x) = (n+c+d+1)x + \frac{(n+c+d+1)(c-d)}{2n+c+d+2}, \\ H_n^* &= \frac{-2(n+1)(n+c+d+1)}{2n+c+d+2}. \end{aligned}$$

Note that E_n, G_n, E_n^* and G_n^* are linear in x, while F_n, H_n, F_n^* and H_n^* are independent of x. Define $\delta_n = \delta_n(x; a, b, c, d) = R_nS_{n+1} - R_{n+1}S_n$. Since $(1-x^2)\delta_n^{'} = (1-x^2)(R_n^{'}S_{n+1} + S_{n+1}^{'}R_n - R_{n+1}^{'}S_n - R_{n+1}S_n^{'})$ from

(2.9), we obtain

$$
\begin{aligned}
&(1-x^2)\delta_n' \\
&= (E_nR_n + F_nR_{n-1})S_{n-1} + (E_{n+1}^*S_{n+1} + F_{n+1}^*S_n)R_n \\
&- (E_{n+1}R_{n+1} + F_{n+1}R_n)S_n - (E_n^*S_n + F_n^*S_{n-1})R_{n+1} \\
&= (G_n + E_{n+1}^*)R_nS_{n+1} - (G_n^* + E_{n+1})R_{n+1}S_n \\
&+ (H_n - H_n^*)R_{n+1}S_{n+1} + (F_{n+1}^* - F_{n+1})R_nS_n.
\end{aligned}
\tag{2.11}
$$

If we set $c = a + k, d = b - k$ in (2.11), we get

$$
\begin{aligned}
(1-x^2)\triangle_n' = (C_nA_{n+1}^*)R_nQ_{n+1} - (C_n^* + A_{n+1})R_{n+1}Q_n \\
+ (B_{n+1}^* - B_{n+1})R_nQ_n,
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
&\triangle_n = R_nQ_{n+1} - R_{n+1}Q_n, R_n = P_n^{a,b}, Q_n = P_n^{a+k,b-k}, \\
&A_n = E_n, A_n^* = -nx - \frac{n(b-a-2k)}{2n+a+b}, \\
&C_n = G_n, C_n^* = (n+a+b+1)x + \frac{(n+a+b+1)(a-b+2k)}{2n+a+b+2}, \\
&B_n = F_n, B_n^* = \frac{2(n+a+k)(n+b-k)}{2n+a+b} \text{ and } k = \pm 1.
\end{aligned}
$$

After adding and substracting $(C_n^* + A_{n+1})R_nQ_{n+1}$ from (2.12), we get

(2.13)

$$
\begin{aligned}
&(1-x^2)\triangle_n' \\
&= (C_n^* + A_{n+1})(R_nQ_{n+1} - R_{n+1}Q_n) + (C_n + A_{n+1}^* - C_n^* - A_{n+1}) \\
&\qquad R_nQ_{n+1} + (B_{n+1}^* - B_{n+1})R_nQ_n \\
&= (C_n^* + A_{n+1})\triangle_n + R_n[(C_n - C_n^* + A_{n+1}^* - A_{n+1})Q_{n+1} \\
&\qquad + (B_{n+1}^* - B_{n+1})Q_n].
\end{aligned}
$$

Further, since

$$
\begin{aligned}
[(1-x)^\alpha(1+x)^\beta\triangle_n]' =& (1-x)^{\alpha-1}(1+x)^{\beta-1}[\{-\alpha(1+x) \\
&+ \beta(1-x)\triangle_n\} + (1-x^2)\triangle_n].
\end{aligned}
$$

We get from (2.13) that

(2.14)
$$[(1-x)^{\alpha}(1+x)^{\beta}\triangle_n]' = (1-x)^{\alpha-1}(1+x)^{\beta-1}[\{-\alpha(1+x)+\beta(1-x)\triangle_n + (C_n^* + A_{n+1})\triangle_n\} + R_n\{(C_n - C_n^* + A_{n+1}^* - A_{n+1})Q_{n+1} + (B_{n+1}^* - B_{n+1})Q_n\}].$$

Therefore, setting

$$\alpha(n,k) = \frac{2a + a^2 + ab + 2an + k(1+a+b+n)}{2n+a+b+2}$$

and

$$\beta(n,k) = \frac{2b + b^2 + ab + 2bn - k(1+a+b+n)}{2n+a+b+2}.$$

We get for $-1 < x < 1$ and $n \geq 0$ the identity

$$\begin{aligned}&[(1-x)^{\alpha}(1+x)^{\beta}\triangle_n]' \\ &= (1-x)^{\alpha-1}(1+x)^{\beta-1}R_n[(C_n - C_n^* + A_{n+1}^* - A_{n+1})Q_{n+1} \\ &+ (B_{n+1}^* - B_{n+1})Q_n]\end{aligned} \tag{2.15}$$

upon which our proof of the following Theorem 2.1 will be based.

THEOREM 2.1. *If* $b \geq a \geq 0$, *then*

$$\triangle_n = P_n^{a,b}P_{n+1}^{a+1,b-1} - P_{n+1}^{a,b}P_n^{a+1,b-1} > 0 \text{ for } -1 < x < 1.$$

Proof. We get from (2.15) that

$$\begin{aligned}&[(1-x)^{\alpha}(1+x)^{\beta}\triangle_n]' \\ &= (1-x)^{\alpha-1}(1+x)^{\beta-1}P_n^{a,b}[(C_n - C_n^* + A_{n+1}^* - A_{n+1})P_{n+1}^{a+1,b-1} \\ &\qquad + (B_{n+1}^* - B_{n+1})P_n^{a+1,b-1}],\end{aligned}$$

where

$$\alpha = \frac{2a + a^2 + ab + 2an + (1 + a + b + n)}{2n + a + b + 2},$$
$$\beta = \frac{2b + b^2 + ab + 2bn - (1 + a + b + n)}{2n + a + b + 2},$$
$$C_n(x) = (n + a + b + 1)x + \frac{(n + a + b + 1)(a - b)}{2n + a + b + 2},$$
$$C_n^*(x) = (n + a + b + 1)x + \frac{(n + a + b + 1)(a - b + 2)}{2n + a + b + 2},$$
$$A_{n+1}(x) = -(n + 1)x - \frac{(n + 1)(b - a)}{2n + a + b + 2},$$
$$A_{n+1}^*(x) = -(n + 1)x - \frac{(n + 1)(b - a - 2)}{2n + a + b + 2},$$
$$B_{n+1}(x) = \frac{2(n + a + 1)(n + b + 1)}{2n + a + b + 2},$$
$$B_{n+1}^*(x) = \frac{2(n + a + 2)(n + b)}{2n + a + b + 2}.$$

(Case 1) If $P_n^{a,b}(x) = 0$ for some x, then

$$\triangle_n(x) = -P_{n+1}^{a,b} P_n^{a+1,b-1}. \tag{2.16}$$

From (2.7) and (2.8), we have

$$(n + \frac{a}{2} + \frac{b}{2} + 1)\{(1 + x)P_n^{a,b+1} - (1 - x)P_n^{a+1,b}\} = 2(n + 1)P_{n+1}^{a,b}. \tag{2.17}$$

From (2.3) and $P_n^{a,b}(x) = 0$, we have

$$P_n^{a,b+1} = -(\frac{1 - x}{1 + x})P_n^{a+1,b}.$$

From (2.17) and the above equation, we have

$$\triangle_n(x) = \frac{(n + \frac{a}{2} + \frac{b}{2} + 1)(1 - x)}{n + 1} P_n^{a+1,b} P_n^{a+1,b-1}.$$

From (2.4) and (2.16), we have

$$(n + a + b + 1)P_n^{a+1,b} = (n + b)P_{n-1}^{a+1,b}$$

and

$$P_n^{a+1,b-1} = P_{n-1}^{a+1,b}. \tag{2.18}$$

From (2.18), we have

$$\triangle_n(x) = \frac{(n+\frac{a}{2}+\frac{b}{2}+1)(1-x)(n+a+b+1)}{(n+1)(n+b)}(P_n^{a+1,b})^2 > 0,$$

for $-1 < x < 1$.

(Case 2) We now consider the case when $P_n^{a,b}(x) \neq 0$, and

$$(C_n - C_{n_1}^* + A_{n_1+1}^* - A_{n+1})P_{n+1}^{a+1,b-1} + (B_{n_1+1}^* - B_{n+1})P_n^{a+1,b-1} = 0 \tag{2.19}$$

for some $x \in (-1,1)$, i.e., $P_{n+1}^{a+1,b-1} = \frac{b-a-1}{a+b}P_n^{a+1,b-1}$.

From (2.4) and (2.19), we have

$$\begin{aligned}(2n+a+b)P_n^{a,b-1} &= [(n+a+b) - \frac{(n+b-1)(a+b)}{(b-a-1)}]P_n^{a-1,b-1}\\ &= -\frac{(a^2+ab+n+2an)}{(b-a-1)}P_n^{a+1,b-1}.\end{aligned} \tag{2.20}$$

From (2.5) and (2.19) we have

$$\begin{aligned}&(2n+a+b)P_n^{a,b-1}\\ &= [(n+a+b) + \frac{(n+a)(a+b)}{b-a+1}]P_n^{a,b}\\ &= \frac{(a+b+ab+b^2+n+2bn)}{b-a+1}P_n^{a,b}.\end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21), we have

$$P_n^{a,b} = \frac{(a^2+ab+n+2an)(1-a+b)}{(a+b+ab+b^2+n+2bn)(1+a-b)}P_n^{a+1,b-1}$$

and

$$P_{n+1}^{a,b} = \frac{(1+2a+a^2+ab+n+2an)(1-a+b)}{(1+a+3b+ab+b^2+n+2bn)(1+a-b)}P_{n+1}^{a+1,b-1}. \tag{2.22}$$

From (2.19) and (2.22), we have

$$
\begin{aligned}
&\triangle_n(x)\\
&= P_n^{a,b}P_{n+1}^{a+1,b-1} - P_{n+1}^{a,b}P_n^{a+1,b-1}\\
&= \frac{(1-a^2+2b+b^2)}{(a+b+ab+b^2+n+2bn)(1+a+3b+ab+b^2+n+2bn)}(P_{n+1}^{a+1,b-1})^2\\
&> 0 \text{ if } b \geq a \geq 0.
\end{aligned}
$$

□

The Bustoz-Savage inequalities hold true for Laguerre and ultraspherical polynomials which are symmetric. Here we prove some similar inequalities for non-symmetric Jacobi polynomials. After using Theorem 2.1, we get the following corollary. Thus we prove Paul Turan inequalities for the non-symmetric Jacobi polynomials.

COROLLARY 1. *If $b \geq a \geq 0$, then*

$$\triangle_n(x) = P_n^{a,b}P_n^{a+1,b} - P_{n+1}^{a,b}P_{n-1}^{a+1,b} > 0, \ \textit{for} \ -1 < x < 1.$$

Proof.

$$
\begin{aligned}
&P_n^{a,b}P_n^{a+1,b} - P_{n+1}^{a,b}P_{n-1}^{a+1,b}\\
&= \begin{vmatrix} P_n^{a,b} & P_{n+1}^{a,b} \\ P_{n-1}^{a+1,b} & P_n^{a+1,b} \end{vmatrix}\\
&= \begin{vmatrix} P_n^{a,b} & P_{n+1}^{a,b} \\ P_{n-1}^{a+1,b} + P_n^{a,b} & P_n^{a+1,b} + P_{n+1}^{a,b} \end{vmatrix} \text{(Using the theory of determinant)}\\
&= \begin{vmatrix} P_n^{a,b} & P_{n+1}^{a,b} \\ P_n^{a+1,b-1} & P_{n+1}^{a+1,b-1} \end{vmatrix} \text{(Using (2.6))}\\
&= P_n^{a,b}P_{n+1}^{a+1,b-1} - P_{n+1}^{a,b}P_n^{a+1,b-1} > 0, \text{in view of theorem 2.1.}
\end{aligned}
$$

□

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