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CHARACTERIZATIONS OF WEAKLY PRECOMPACTNESS OF OPERATORS ACTING BETWEEN BANACH SPACES

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Abstract. We give an extensive presentation of results about the properties of Banach space operators with weakly precompact adjoints. Further we give a description of operators having weakly precompact adjoints on abstract continuous function spaces

1. Introduction

We point out that in general the adjoint of a weakly precompact operator need not be weakly precompact. For an easy example, take any bounded linear surjection $T : \ell_1 \to c_0$. Since c_0 does not contain a copy of ℓ_1, T is weakly precompact. But $T^* : \ell_1 \to \ell_\infty$ is an isomorphism and thus T^* fails to be weakly precompact.

Of course the question arises whether every operator having a weakly precompact adjoint must itself be weakly precompact. It is natural to ask about the nature of weakly precompact operators acting between Banach spaces.

In this paper we survey geometric structures of Banach space operators with weakly precompact adjoints. Here, we present C. Abbott, E. Bator, R. Bilyeu and P. Lewis' approach to this subject (cf. [1], [2]).

We first handle the above question. And then we compare operators whose duals are weakly precompact with unconditionally converging operators.

Next we characterize operators with weakly precompact adjoints in terms of the weak Radon-Nikodym property.

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Next we provide the operator version of Josefson-Nissenzweig theorem [8]. And then we consider a result due to W. Johnson and H. Rosenthal [10] in the framework of operators. As a corollary we obtain that the adjoint of a weakly precompact operator with a Grothendieck space domain is always weakly precompact.

We turn to the study of operators having weakly precompact adjoints defined on the space $C(K,X)$ of all continuous functions from a compact Hausdorff space K into a Banach space X in connection with the above problems.

We discuss the relationship between operators having weakly precompact adjoints defined on the space $C(K,X)$ and their representing measures. And then we characterize operators having weakly precompact adjoints defined on the space $C(K,X)$ under the hypothesis that the dual of a Banach space X contains no isomorphic copy of ℓ_1.

Finally we find usable conditions on an operator defined on the space $C(K,X)$ which imply that X has the weak* Radon-Nikodym property.

2. Definitions and Notation

We present some of the definitions and notation to be used in this paper. Throughout this paper X and Y denote Banach spaces with duals X^* and Y^* respectively.

A subset E of a Banach space X is called weakly precompact if every bounded sequence in E has a weakly Cauchy subsequence.

We say that a bounded sequence (x_n) of a Banach space X is a copy of (or is equivalent to) the usual ℓ_1-basis if there exists a constant $\delta > 0$ such that $\|\sum a_k x_k\| \geq \delta \cdot \sum |a_k|$ for all finitely non-zero sequences (a_k) of real numbers.

H. Rosenthal's theorem [14] states that a bounded subset E of a Banach space X is weakly precompact if and only if it contains no copy of the usual ℓ_1-basis.

Notation. (1) The adjoint operator of an operator T is denoted by T^*.
(2) The closed unit ball of X is denoted by B_X.
(3) The closed unit sphere of X is denoted by S_X.
(4) The closed linear span of the sequence (x_n) is denoted by $[x_n]$.
(5) $\mathcal{B}(X,Y)$ denotes the set of all bounded linear operators

from X into Y.

(6) $\mathcal{C}(K,X)$ denotes the space of all continuous X-valued functions defined on a compact Hausdorff space K.

An operator $T \in \mathcal{B}(X,Y)$ is weakly precompact if $T(B_X)$ is weakly precompact in Y.

An operator $T \in \mathcal{B}(X,Y)$ is unconditionally converging if T maps weakly unconditionally Cauchy series into unconditionally convergent series.

Let $(\Omega, \sum, \mu)$ be a finite measure space. A function $f : \Omega \to X$ is scalarly measurable if the scalar function $x^* f(\cdot)$ is measurable for each $x^* \in X^*$. A function $f : \Omega \to X$ is called Pettis integrable if it is scalarly integrable and if for each measurable set A in $\sum$, there exists an element x_A of X such that $x^*(x_A) = \int_A x^* f \, d\mu$ for each $x^* \in X^*$. In this case we write $x_A = \text{Pettis}-\int_A f \, d\mu$. The Pettis norm of a Pettis integrable function f is defined to be $\sup\{\int_\Omega |x^* f| \, d\mu : x^* \in B_{X^*}\}$.

A Banach space X has the weak Radon-Nikodym property if for every finite measure space $(\Omega, \sum, \mu)$, every bounded linear operator $T : L_1(\mu) \to X$ is Pettis representable, i.e., there exists a Pettis integrable function $\phi : \Omega \to X$ such that $T(f) = \text{Pettis} - \int_\Omega f \phi \, d\mu$ for every $f \in L_1(\mu)$.

A Banach space X has the weak* Radon-Nikodym property if for every finite measure space $(\Omega, \sum, \mu)$ and for every bounded linear operator $T : L_1(\mu) \to X$, there exists a Pettis integrable function $\phi : \Omega \to X^{**}$ such that $T(f) = \text{Pettis} - \int_\Omega f \phi \, d\mu$ for every $f \in L_1(\mu)$.

Let (φ_n) be a sequence of Pettis integrable functions from a probability space $(\Omega, \mathcal{F}, P)$ into X and let $(\mathcal{F}_n)$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$. The sequence (φ_n) is said to be a martingale relative to $(\mathcal{F}_n)$ if each φ_n is $\mathcal{F}_n$-scalarly measurable and Pettis-$\int_A \varphi_n \, dP$ =Pettis-$\int_A \varphi_{n+1} \, dP$ for each $A \in \mathcal{F}_n$ and each $n \in bN$.

Let $T \in \mathcal{B}(X,Y)$. We say that a sequence (y_n^*) in Y^* is T-weak* null if for each $x \in X$, $\lim_{n\to\infty} \langle Tx, y_n^* \rangle = 0$.

A Banach space X with the property that weak* and weak sequential convergence in X^* coincide is called a Grothendieck space.

An operator ideal $\mathcal{U}$ is said to be separably determined provided that an operator $T : X \to Y$ belongs to $\mathcal{U}(X,Y)$ if and only if its restriction to any separable subspace E of X belongs to $\mathcal{U}(E,Y)$.

The Riesz representation theorem tells us that if $T \in \mathcal{B}(C(K,X),Y)$

then there is a unique finitely additive set function m of bounded semivariation defined on the σ-algebra $\sum$ of Borel sets of K with values in $\mathcal{B}(X, Y^{**})$ such that $T(f) = \int f\,dm$ for $f \in \mathcal{C}(K, X)$. We refer to the set function m as the representing measure for T.

Let $U_X(\sum)$ denote the collection of all X-valued functions f over the σ-algebra $\sum$ of Borel sets of K for which f is the uniform limit of a sequence (f_n) of X-valued simple functions. The expression $\int f\,dm$ makes sense for $f \in U_X(\sum)$ and hence this integral defines a natural extension of T to $U_X(\sum)$. We denote this extension by $\hat{T}$.

For a set $A \in \sum$, we put $\tilde{m}(A) = \sup_I \|\sum_{i\in I} m(A_i)\,x_i\|$, where the supremum is taken over all finite collections $\{A_i\}_{i\in I}$ of pairwise disjoint members of $\sum$ such that $\cup_{i\in I} A_i = A$ and over all finite collections $\{x_i\}_{i\in I}$ of elements of B_X. The number $\tilde{m}(A)$ is called the semivariation of m on the set A.

We say that m is strongly bounded if $\lim_{n\to\infty} \tilde{m}(A_n) = 0$ whenever (A_n) is a sequence of disjoint sets from $\sum$.

An operator $T : \mathcal{C}(K, X) \to Y$ is said to be strongly bounded provided that its representing measure m is strongly bounded.

For every $z \in Y^*$, m defines a finitely additive set function $m_z : \sum \to X^*$ by $\langle m_z(A), x\rangle = \langle m(A)x, z\rangle,\ A \in \sum,\ x \in X$. The variation of m_z is the extended nonnegative function $|m_z|$ whose value on a set $A \in \sum$ is given by $|m_z|(A) = \sup_I \sum_{i\in I} \|m_z(A_i)\|$, where the supremum is taken over all finite collections $\{A_i\}_{i\in I}$ of pairwise disjoint members of $\sum$ such that $\cup_{i\in I} A_i = A$. It is easy to check that $\tilde{m}(A) = \sup\{|m_z|(A) : z \in B_{Y^*}\},\ A \in \sum$.

3. Results

We begin by proving that every operator having a weakly precompact adjoint must itself be weakly precompact.

Theorem 1. *If the operator $T : X \to Y$ has a weakly precompact adjoint then T is weakly precompact.*

Proof. We argue by contradiction. Suppose that $T : X \to Y$ is not weakly precompact. Then X contains an isomorphic copy of ℓ_1 on which T acts as an isomorphism. Let $J : \ell_1 \to X$ be an isomorphic embedding. Then $TJ : \ell_1 \to Y$ is an isomorphism. Therefore $(TJ)^* :$

$Y^* \to \ell_\infty$ is a surjection and so $J^*T^*(B_{Y^*}) \supseteq \delta B_{\ell_\infty}$ for some $\delta > 0$. Since T^* is weakly precompact, the same is true of J^*T^*. This allows us to have that ℓ_∞ contains no isomorphic copy of ℓ_1. This contradiction proves the assertion. □

We investigate the link between an operator with weakly precompact adjoint and an unconditionally converging operator.

THEOREM 2. *If the operator $T : X \to Y$ has a weakly precompact adjoint then T is unconditionally converging.*

Proof. Arguing contrapositively, we assume that $T : X \to Y$ is not unconditionally converging. A result due to C. Bessaga and A. Pelczynski [3] assures us that there is a subspace E of X isomorphic to c_0 such that the restriction of T to E is an isomorphism. Let $J : c_0 \to E \subset X$ be an isomorphic embedding. Then $TJ : c_0 \to Y$ is an isomorphism. Consequently $(TJ)^* : Y^* \to \ell_1$ is a surjection and thus $J^*T^*(B_{Y^*}) \supseteq \delta B_{\ell_1}$ for some $\delta > 0$. This leads us to have that $T^*(B_{Y^*})$ cannot be weakly precompact in X^*, i.e., T^* fails to be weakly precompact. □

Using the factorization construction of W. Davis, T. Figiel, W. Johnson and A. Pelczynski [5], we establish the following useful characterization of operators whose duals are weakly precompact.

THEOREM 3. *The following statements about an operator $T : X \to Y$ are equivalent.*

(i) *$T^* : Y^* \to X^*$ is weakly precompact.*
(ii) *$T^{**} : X^{**} \to Y^{**}$ factors through a Banach space with the weak Radon-Nikodym property.*

Proof. (i) $\Rightarrow$ (ii). The hypothesis (i) informs us that $T^*(B_{Y^*})$ is weakly precompact in X^*. Let P_n be the gauge of the set $U_n = 2^n T^*(B_{Y^*}) + 2^{-n} B_{X^*}$ for $n = 1, 2, \cdots$. Define, for $x^* \in X^*$, $|||x^*||| = (\sum_{n=1}^\infty P_n(x^*)^2)^{1/2}$, let $V = \{x^* \in X^*,\ |||x^*||| < \infty\}$. We deduce from lemma 1 of [5] that the inclusion map $J : V \to X^*$ is continuous and B_V is weakly precompact. Note that if $y^* \in B_{Y^*}$ then $2^n T^* y^* \in U_n$ and hence $P_n(T^*y^*) < 2^{-n}$, $n = 1, 2, \cdots$. This gives that if $\|y^*\| \le 1$ then $|||T^*y^*|||^2 = \sum_{n=1}^\infty P_n(T^*y^*)^2 < \infty$. So the map $S : Y^* \to V$ given by

$Sy^* = T^*y^*$, $y^* \in Y^*$, is a bounded linear operator. Certainly $T^* = JS$ and thus T^{**} admits a factorization $T^{**} : X^{**} \xrightarrow{J^*} V^* \xrightarrow{S^*} Y^{**}$. Since V contains no isomorphic copy of ℓ_1, Janicka's theorem [9] steps in to ensure that V^* has the weak Radon-Nikodym property.

(ii) $\Rightarrow$ (i). Assume that T^* is not weakly precompact. We choose a sequence $(T^*y_n^*)$ in $T^*(B_{Y^*})$ that is a copy of the usual ℓ_1-basis (e_n). We denote by M the closed linear span of the sequence $(T^*y_n^*)$. Then the map $S : M \to \ell_1$ given by $S(T^*y_n^*) = e_n$ is an isomorphism. Let (e_n^*) be the usual ℓ_∞-basis. By setting $x_n = S^*(e_n^*)$, we get $\langle x_n, T^*y_k^*\rangle = \langle e_n^*, ST^*y_k^*\rangle = \langle e_n^*, e_k\rangle$. We consider a map $f_n : [0,1] \to M^*$ which is defined by $f_n(\cdot) = \|S\|^{-1}\sum_{k=1}^n r_k(\cdot)x_k$, where r_k is the k-th Rademacher function. It is plain that (f_n) is a martingale relative to the dyadic partitions of the interval $[0,1]$. We use this sequence to define an operator $R : L_1[0,1] \to M^*$ via $R(g) = \lim_n \int gf_n \, d\lambda$ for all $g \in L_1[0,1]$, where λ is Lebesgue measure on $[0,1]$. Since $L_1[0,1]$ has the lifting property, there exists an operator $\hat{R} : L_1[0,1] \to X^{**}$ such that $R = i^*\hat{R}$ and $\|\hat{R}\| = \|R\|$, where $i : M \to X^*$ is the natural injection. If π_n denotes the dyadic partition of $[0,1]$ into intervals of length $1/2^n$, then the functions $g_n : [0,1] \to X^{**}$ defined by $g_n(\cdot) = \sum_{A\in\pi_n} \frac{\hat{R}(\chi_A)}{\lambda(A)}\chi_A(\cdot)$ form a martingale satisfying $\langle f_n(\cdot), x^*\rangle = \langle g_n(\cdot), x^*\rangle$ for all $x^* \in M$, and $\sup_n \|f_n\| = \sup_n \|g_n\|$. Hence $g_n([0,1]) \subset B_{X^{**}}$ and so $(T^{**}g_n)$ is an Y^{**}-valued martingale with $T^{**}g_n([0,1]) \subset T^{**}(B_{X^{**}})$. On account of hypothesis (ii), we invoke theorem 5 of [11] to infer that $(T^{**}g_n)$ converges with respect to the Pettis norm. On the other hand, estimating the Pettis norm of $T^{**}g_{n+1} - T^{**}g_n$, we obtain

$$\sup\{\int |\langle z, T^{**}g_{n+1} - T^{**}g_n\rangle| \, d\lambda : z \in Y^{***}, \|z\| \le 1\}$$
$$\ge \int |\langle T^{**}g_{n+1} - T^{**}g_n, y_{n+1}^*\rangle| \, d\lambda$$
$$= \int |\langle f_{n+1} - f_n, T^*y_{n+1}^*\rangle| \, d\lambda = \|S\|^{-1} \int |r_{n+1}| \, d\lambda = \|S\|^{-1}.$$

As a result $(T^{**}g_n)$ is not a Cauchy sequence with respect to the Pettis norm. This contradiction shows that T^* is weakly precompact. □

In the following we treat Josefson-Nissenzweig theorem [8] in the framework of operators.

Proposition 1. *Let $T : X \to Y$ be an operator. Suppose Y^* contains a copy of ℓ_1 but that no T-weak* null sequence in Y^* is equivalent to the usual ℓ_1-basis. Then T is not weakly precompact.*

Proof. Suppose (y_n^*) is a sequence in B_{Y^*} equivalent to the usual ℓ_1-basis. Define $\delta(y_n^*) = \sup\{\lim_n \sup |\langle Tx, y_n^*\rangle| : \|x\| = 1\}$. Our hypothesis guarantees that (y_n^*) is not T-weak* null and hence $\delta(y_n^*) > 0$. Set $\delta = \delta(y_n^*)$. Let $\epsilon > 0$ be given. There is an $x_1 \in S_X$ and an infinite set N_1 in $\mathbb{N}$ such that for any $n \in N_1$, $\langle Tx_1, y_n^*\rangle < -\delta + \epsilon$. Suppose $0 < \epsilon' < \epsilon/3$. Partition N_1 into two disjoint infinite subsets enumerated by the increasing sequences (n_k) and (m_k) of positive integers. The sequence $(\frac{1}{2}(y_{n_k}^* - y_{m_k}^*))$ is a normalized ℓ_1-block of (y_n^*). Thus there is an $x_2 \in S_X$ and an infinite set of k for which

$$\langle Tx_2, \frac{1}{2}(y_{n_k}^* - y_{m_k}^*)\rangle > \delta - \epsilon'. \tag{1}$$

Of course, $(y_{n_k}^*)$ and $(y_{m_k}^*)$ are normalized ℓ_1-blocks of (y_n^*) which for all but finitely many k must satisfy

$$|\langle Tx_2, y_{n_k}^*\rangle|, \ |\langle Tx_2, y_{m_k}^*\rangle| < \delta + \epsilon'. \tag{2}$$

Suppose k satisfies (1)and (2) but $\langle Tx_2, y_{n_k}^*\rangle \leq \delta - 3\epsilon'$. Then we would have

$$\delta - \epsilon' < \frac{1}{2}(\langle Tx_2, y_{n_k}^*\rangle - \langle Tx_2, y_{m_k}^*\rangle) < \frac{1}{2}(\delta - 3\epsilon' + \delta + \epsilon') = \delta - \epsilon'.$$

This contradiction proves that $\langle Tx_2, y_{n_k}^*\rangle > \delta - 3\epsilon' > \delta - \epsilon$ for those k enjoying the above estimates (1) and (2). Now suppose k satisfies (1) and (2) but $|\langle Tx_2, y_{m_k}^*\rangle| \geq -\delta + 3\epsilon'$. Then we would have

$$\delta - \epsilon' < \frac{1}{2}(\langle Tx_2, y_{n_k}^*\rangle - \langle Tx_2, y_{m_k}^*\rangle) < \frac{1}{2}(\delta + \epsilon' + \delta - 3\epsilon') = \delta - \epsilon'.$$

This contradiction yields that $\langle Tx_2, y_{m_k}^*\rangle < -\delta + 3\epsilon' < -\delta + \epsilon$ for those k enjoying the above estimates (1) and (2). We see that the sets

$N_2 = \{n_k : \langle Tx_2, y^*_{n_k}\rangle > \delta - \epsilon\}$ and $N_3 = \{m_k : \langle Tx_2, y^*_{m_k}\rangle < -\delta + \epsilon\}$ are infinite disjoint subset of N_1. Let $0 < \epsilon' < \epsilon/7$. We can decompose N_2 into two disjoint infinite subsets which we enumerate as increasing sequences $(n_k(1))$ and $(n_k(2))$ of positive integers and similarly decompose N_3 into sequences $(m_k(1))$ and $(m_k(2))$.The sequence $(\frac{1}{4}(y^*_{n_k(1)} - y^*_{n_k(2)} + y^*_{m_k(1)} - y^*_{m_k(2)}))$ is a normalized ℓ_1-block of (y^*_n). Therefore there is an $x_3 \in S_X$ such that for infinitely many k,

$$(3) \qquad \langle Tx_3, \frac{1}{4}(y^*_{n_k(1)} - y^*_{n_k(2)} + y^*_{m_k(1)} - y^*_{m_k(2)})\rangle > \delta - \epsilon'.$$

Of course, each of the sequences $(y^*_{n_k(1)})$, $(y^*_{n_k(2)})$, $(y^*_{m_k(1)})$, and $(y^*_{m_k(2)})$ are normalized ℓ_1-blocks of (y^*_n). So for all but a finite number of k we must have

$$(4)\ |\langle Tx_3, y^*_{n_k(1)}\rangle|, |\langle Tx_3, y^*_{n_k(2)}\rangle|, |\langle Tx_3, y^*_{m_k(1)}\rangle|, |\langle Tx_3, y^*_{m_k(2)}\rangle| < \delta+\epsilon'.$$

Suppose k satisfies (3) and (4) yet $\langle Tx_3, y^*_{n_k(1)}\rangle \le \delta - 7\epsilon'$. Then we would have

$$\begin{aligned} \delta-\epsilon' &< \frac{1}{4}(\langle Tx_3, y^*_{n_k(1)}\rangle - \langle Tx_3, y^*_{n_k(2)}\rangle + \langle Tx_3, y^*_{m_k(1)}\rangle - \langle Tx_3, y^*_{m_k(2)}\rangle) \\ &< \frac{1}{4}(\delta - 7\epsilon' + \delta + \epsilon' + \delta + \epsilon' + \delta + \epsilon') = \delta - \epsilon'. \end{aligned}$$

This contradiction shows that $\langle Tx_3, y^*_{n_k(1)}\rangle > \delta - 7\epsilon' > \delta - \epsilon$ for those k satisfying the above estimates (3) and (4). In a similar way, we derive that $\langle Tx_3, y^*_{n_k(2)}\rangle < -\delta + \epsilon$, $\langle Tx_3, y^*_{m_k(1)}\rangle > \delta - \epsilon$, and $\langle Tx_3, y^*_{m_k(2)}\rangle < -\delta + \epsilon$ for those k satisfying (3) and (4). We see that the sets $N_4 = \{n_k(1) : \langle Tx_3, y^*_{n_k(1)}\rangle > \delta - \epsilon\}$ and $N_5 = \{n_k(2) : \langle Tx_3, y^*_{n_k(2)}\rangle < -\delta + \epsilon\}$ are disjoint infinite subsets of N_2, and the sets $N_6 = \{m_k(1) : \langle Tx_3, y^*_{m_k(1)}\rangle > \delta - \epsilon\}$ and $N_7 = \{m_k(2) : \langle Tx_3, y^*_{m_k(2)}\rangle < -\delta + \epsilon\}$ are disjoint infinite subsets of N_3.

We continue in this fashion. Letting $\Omega_n = \{y^*_k : k \in N_n\}$, we get a tree of subsets of B_{Y^*}. Furthermore, (Tx_n) has been so selected from $T(S_X)$ that if $2^{n-1} \le k < 2^n$, then $(-1)^k \langle Tx_n, y^*\rangle > \delta - \epsilon$ for all $y^* \in \Omega_k$. We summon up Pelczynski's result [12] to conclude that (Tx_n) is a copy of the usual ℓ_1-basis. This means that T is not weakly precompact. □

The proof of our proposition given below is nearly identical to the proof of proposition 1.

Proposition 2. *Let $T : X \to Y$ be an operator. Suppose $T^* : Y^* \to X^*$ is not weakly precompact but that no weak* null sequence in $T^*(B_{Y^*})$ is equivalent to the usual ℓ_1-basis. Then T is not weakly precompact.*

Proof. Since T^* is not weakly precompact, we find a sequence $(T^*y_n^*)$ in $T^*(B_{Y^*})$ that is a copy of the usual ℓ_1-basis. Define $\delta(y_n^*) = \sup\{\lim_n \sup|\langle Tx, y_n^*\rangle| : \|x\| = 1\}$. Our hypothesis ensures that $(T^*y_n^*)$ is not weak* null and so $\delta(y_n^*) > 0$. The remaining assertions are established by arguing exactly as in the proof of proposition 1. □

Theorem 4. *If $T : X \to Y$ is not a compact operator then there exists a T-weak* null sequence (y_n^*) in B_{Y^*} such that* $\lim_n \inf \|T^*y_n^*\| > 0$.

Proof. Suppose that in $T^*(B_{Y^*})$, weak* null sequences are norm null. Then either $T^*(B_{Y^*})$ contains a copy of the usual ℓ_1-basis or it does not. If not, then each sequence in $T^*(B_{Y^*})$ has a weakly Cauchy subsequence. Thus each sequence in $T^*(B_{Y^*})$ has a weak* convergent subsequence which is norm convergent in view of our supposition. This means that the operator T^*, and so T, is compact. This contradiction proves the assertion.

We pass now to the case where $T^*(B_{Y^*})$ contains a copy of the usual ℓ_1-basis. Then our supposition ensures that no weak* null sequence in $T^*(B_{Y^*})$ can be equivalent to the usual ℓ_1-basis. An appeal to proposition 2 reveals that T is not weakly precompact. We pick a sequence (Tx_n) in $T(B_X)$ that is a copy of the usual ℓ_1-basis (e_n^*). We consider the bounded linear operator $R : [Tx_n] \to L_\infty[0,1]$ defined by $R(Tx_n) = r_n$, where r_n is the n-th Rademacher function. The injectivity of $L_\infty[0,1]$ permits us to have a bounded extension $\tilde{R}$ of R to all of Y. Notice that the operator $L : L_\infty[0,1] \to c_0$ given by $Lf = (\int_0^1 f(t)r_n(t)\,dt)$ is a bounded linear operator. Since $(L\tilde{R}T)^* : \ell_1 \to X^*$ is weak*-to-weak* continuous and (e_n^*) is a weak* null sequence in ℓ_1, it follows that $((L\tilde{R}T)^*e_n^*)$ is a weak* null sequence in $T^*(B_{Y^*})$. Then our supposition indicates that $((L\tilde{R}T)^*e_n^*)$ is norm null. However,

$\langle (L\tilde{R}T)^* e_n^*, x_n\rangle = \langle e_n^*, L\tilde{R}Tx_n\rangle = \langle e_n^*, e_n\rangle = 1$ for each n. This is a contradiction.

□

In the theorem stated below we deal with a result due to W. Johnson and H. Rosenthal [10] in the framework of operators. For this purpose the next elementary fact is required.

LEMMA 1. *Let (x_i^*) be a sequence in X^* equivalent to the usual ℓ_1-basis. Then given $\epsilon > 0$, $n \in \mathbb{N}$, and a subsequence $(x_{i_j}^*)$ of (x_i^*) there is a finite set of vectors $x_1, \cdots, x_m$ from CB_X such that $\{(\langle x_{i_j}^*, x_p\rangle)_{j=1}^n : 1 \le p \le m\}$ forms an ϵ-net for $B_{\ell_\infty^n}$.*

Proof. Given $\epsilon > 0$, we choose $\alpha_1 = (\alpha_{1i})_{i=1}^\infty, \cdots, \alpha_m = (\alpha_{mi})_{i=1}^\infty$ in B_{ℓ_∞} such that $\{(\alpha_{pj})_{j=1}^n : 1 \le p \le m\}$ is an $\epsilon/2$-net for $B_{\ell_\infty^n}$. For simplicity, we relabel $(x_{i_j}^*)$ by (x_j^*). Pick $z_1, \cdots, z_m$ in $CB_{X^{**}}$ so that $\langle z_p, x_j^*\rangle = \alpha_{pj}$ for $1 \le p \le m$, $j \in \mathbb{N}$. Since CB_X is weak* dense in $CB_{X^{**}}$, for each $1 \le p \le m$, there is an $x_p \in CB_X$ such that $|\langle x_j^*, x_p\rangle - \langle z_p, x_j^*\rangle| < \epsilon/2$ for $1 \le j \le n$. Therefore $\{(\langle x_j^*, x_p\rangle)_{j=1}^n : 1 \le p \le m\}$ is an ϵ-net for $B_{\ell_\infty^n}$. □

THEOREM 5. *The following statements about an operator $T : X \to Y$ are equivalent.*

(i) *$T^* : Y^* \to X^*$ is not weakly precompact.*
(ii) *There is a subspace E of X and an operator $S : Y \to \ell_\infty$ such that $ST(E) = c_0$.*

Proof. (i) ⇒ (ii). The hypothesis (i) guarantees the existence of a sequence (y_n^*) in B_{Y^*} such that $(T^* y_n^*)$ is a copy of the usual ℓ_1-basis. Let $\mathcal{A}$ denote the collection of all such sequences. We deal first with the case where $\mathcal{A}$ has an element that is T-weak* null. Let (y_n^*) be a T-weak* null sequence in B_{Y^*} such that $(T^* y_n^*)$ is a copy of the usual ℓ_1-basis. We can assume $\|T^* y_n^*\| = 1$ for each n. Let $n_1 = 1$. Choose $x_1 \in CB_X$ so that $\langle T^* y_{n_1}^*, x_1\rangle = 1$. Using the fact that $(\langle T^* y_n^*, x\rangle)$ is null for each $x \in X$, pick $n_2 > n_1$ so that $|\langle T^* y_n^*, x_1\rangle| < 1/2$ for all $n \ge n_2$. Lemma 1 provides us with a finite set $\{x_2^i : 1 \le i \le m_2\}$ of vectors from CB_X such that $\{(\langle T^* y_{n_j}^*, x_2^i\rangle)_{j=1}^2 : 1 \le i \le m_2\}$ forms a $\frac{1}{4}$-net for $B_{\ell_\infty^2}$. By another use of the fact that $(\langle T^* y_n^*, x\rangle)$

is null for each $x \in X$, we choose $n_3 > n_2$ so that $n \geq n_3$ implies $|\langle T^* y_n^*, x_2^i \rangle| < \frac{1}{4}$ for $1 \leq i \leq m_2$. It takes another appeal to lemma 1 to obtain a finite set $\{x_3^i : 1 \leq i \leq m_3\}$ of vectors from CB_X for which $\{(\langle T^* y_{n_j}^*, x_3^i \rangle)_{j=1}^3 : 1 \leq i \leq m_3\}$ forms a $(\frac{1}{2})^3$-net for $B_{\ell_\infty^3}$. Our procedure is clear. We extract an increasing sequence (n_j) of positive integers and a set $\{x_p^i : 1 \leq i \leq m_p,\ p = 2, 3, \cdots\}$ of vectors from CB_X such that

(a) $\{(\langle T^* y_{n_j}^*, x_p^i \rangle)_{j=1}^p : 1 \leq i \leq m_p\}$ forms a $(\frac{1}{2})^p$-net for $B_{\ell_\infty^p}$.
(b) $n \geq n_{p+1}$ implies $|\langle T^* y_n^*, x_p^i \rangle| < (\frac{1}{2})^p$ for $1 \leq i \leq m_p$.

Now we define a linear map $U : X \to c_0$ by $Ux = (\langle T^* y_{n_j}^*, x \rangle)_j$ for all $x \in X$. Plainly $\|U\| \leq 1$. Let us take $\alpha = (\alpha_n) \in B_{C_0}$. Given $\epsilon > 0$, select $N \in \mathbb{N}$ so that $(\frac{1}{2})^N < \epsilon/2$ and $|\alpha_n| < \epsilon/2$ for all $n \geq N$. Since $(\alpha_j)_{j=1}^N \in B_{\ell_\infty^N}$, it follows from property (a) that $\|(\alpha_j - \langle T^* y_{n_j}^*, x_N^k \rangle)_{j=1}^N\| < (\frac{1}{2})^N < \epsilon/2$ for some $x_N^k \in CB_X$ with $1 \leq k \leq m_N$. If $j > N$ then $n_j \geq n_{N+1}$ and thus $|\langle T^* y_{n_j}^*, x_N^k \rangle| < (\frac{1}{2})^N < \epsilon/2$ in view of property (b). Consequently $|\langle T^* y_{n_j}^*, x_N^k \rangle - \alpha_j| < \epsilon$ for all j, that is $\|U x_N^k - \alpha\| < \epsilon$. This forces that $B_{c_0} \subset \overline{U(CB_X)}$ and so U is a surjection. If $S : Y \to \ell_\infty$ is defined by $Sy = (\langle y_{n_j}^*, y \rangle)$ for all $y \in Y$ then $ST = U$ and $ST(X) = c_0$.

Now we treat the case where $\mathcal{A}$ has no element that is T-weak* null. We invoke proposition 2 to infer that T is not weakly precompact. We find a sequence (Tx_n) in $T(B_X)$ that is a copy of the usual ℓ_1-basis. Let $L : [Tx_n] \to c_0$ be a bounded linear surjection. The injectivity of ℓ_∞ makes an extension $\tilde{L} : Y \to \ell_\infty$ of $L : [Tx_n] \to \ell_\infty$. Putting $E = [x_n]$ yields that $\tilde{L}T(E) = c_0$.
(ii) $\Rightarrow$ (i). Suppose that there exists a subspace E of X and an operator $S : Y \to \ell_\infty$ such that $ST(E) = c_0$. Consider the natural inclusion map $I : E \to X$. Then $(STI)^* : \ell_1 \to E^*$ is an isomorphism and hence $I^* T^* S^* = (STI)^*$ is not weakly precompact. Thus T^* is not weakly precompact because the class of all weakly precompact operators is an operator ideal. □

The preceding theorem permits us to find a special Banach space with the property that for every operator with dimain such a space the converse of theorem 1 is true.

Corollary. *Suppose X is a Grothendieck space. If an operator*

$T : X \to Y$ is weakly precompact then its adjoint $T^* : Y^* \to X^*$ is weakly precompact.

Proof. Suppose that T^* is not weakly precompact. Since T is weakly precompact, we know from the proof of theorem 5 that there exists an operator $S : Y \to \ell_\infty$ such that $ST(X) = c_0$. Then $(ST)^* : \ell_1 \to X^*$ is an isomorphism. Observe that (e_n^*) is a weak* null sequence in ℓ_1 and so $((ST)^* e_n^*)$ is a weak* null sequence in X^*. Since X is a Grothendieck space, it follows that $((ST)^* e_n^*)$ is a weakly null sequence in X^* and hence (e_n^*) is a weakly null sequence in ℓ_1. This contradiction completes the proof. □

We pass to the study of operators $T : C(K, X) \to Y$ with weakly precompact adjoints. In the next theorem we describe property of the representing measure for an operator $T : C(K, X) \to Y$ having a weakly precompact adjoint. To this end, the following auxiliary result is needed.

LEMMA 2. *The set* $\mathcal{U} = \{T \in \mathcal{B} \,|\, T^* \text{ is weakly precompact}\}$ *is a closed separably determined operator ideal.*

Proof. First we claim that $\mathcal{U}$ is closed. Assume to the contrary that there is a sequence (T_n) in $\mathcal{U}(X, Y)$ converging to $T \in \mathcal{B}(X, Y)$ with respect to the operator norm but $T \notin \mathcal{U}(X, Y)$. Then there is a subspace V of Y^* isomorphic to ℓ_1 such that the restriction of T^* to V is an isomorphism. Given $\epsilon > 0$, we choose a natural number N with $\|T^* - T_N^*\| < \epsilon$. Hence the restriction of T_N^* to V is an isomorphism, in other words T_N^* is not weakly precompact, which is a contradiction.

Next we assert that $\mathcal{U}$ is separably determined. If $T \in \mathcal{U}(X, Y)$ and if E is any separable subspace of X, then certainly $T|_E \in \mathcal{U}(E, Y)$. Suppose now that $T : X \to Y$ is such that $T|_E \in \mathcal{U}(E, Y)$ for every separable subspace E of X. Suppose $T \notin \mathcal{U}(X, Y)$. Theorem 5 guarantees the existence of a subspace M of X and an operator $S : Y \to \ell_\infty$ for which $ST(M) = c_0$. Since we can find a separable subspace E of M such that $ST(E) = ST|_E(E) = c_0$, another appeal to theorem 5 establishes that ${T|_E}^*$ is not weakly precompact. This contradiction proves that $T \in \mathcal{U}(X, Y)$. □

THEOREM 6. *Let $T : C(K,X) \to Y$ be an operator with representing measure m. If T has a weakly precompact adjoint then $m(A)^* : Y^* \to X^*$ is weakly precompact for each $A \in \sum$.*

Proof. Let $\hat{T}$ denote a natural extension of T to $U_X(\sum)$. Take any separable subspace H of $U_X(\sum)$. Let $\{f_i : i \in \mathbb{N}\}$ be a dense subset of H. Lusin's theorem [6] assures us that for every $n \in \mathbb{N}$, there exists a compact subset K_n of K with $\tilde{m}(K - K_n) < 1/n$ such that the restriction of f_i to K_n is continuous for all i. Set $H_n = [f_i|_{K_n}]$, $n \in \mathbb{N}$. First, look at the map $R_n : H \to H_n$ defined by $R_n f = f|_{K_n}$, $f \in H$. Consider the map $L_n : K_n \to \mathcal{B}(H_n, X)$ given by $L_n(t) = \delta_t$, $t \in K_n$, where $\delta_t(f) = f(t)$ for $f \in H_n$. Then L_n is continuous when $\mathcal{B}(H_n, X)$ has the pointwise convergence topology and $L_n(K_n) \subseteq \{T \in \mathcal{B}(H_n, X) : \|T\| \leq 1\} = M$. Since H_n is separable and complete, it follows that M is a complete convex metrizable subset of $\mathcal{B}(H_n, X)$ and hence L_n has a continuous extension $\tilde{L}_n : K \to M$. Now define a map $S_n : H_n \to C(K,X)$ via $S_n(f)(t) = \tilde{L}_n(t)(f)$, $f \in H_n$, $t \in K$. For $f \in H_n$ and $t \in K_n$, we have $S_n(f)(t) = L_n(t)(f) = f(t)$. Evidently S_n is isometric. If $\| \sum_{i=1}^{p} \alpha_i f_i \| \leq 1$ then we have

$$
\begin{aligned}
&\|TS_nR_n(\sum_{i=1}^{p} \alpha_i f_i) - \hat{T}(\sum_{i=1}^{p} \alpha_i f_i)\| \\
&\qquad = \| \int_K S_n(\sum_{i=1}^{p} \alpha_i f_i|_{K_n})\, dm - \int_K (\sum_{i=1}^{p} \alpha_i f_i)\, dm\| \\
&= \| \int_{K_n} S_n(\sum_{i=1}^{p} \alpha_i f_i|_{K_n})\, dm + \int_{K-K_n} S_n(\sum_{i=1}^{p} \alpha_i f_i|_{K_n})\, dm \\
&\qquad - \int_{K_n} (\sum_{i=1}^{p} \alpha_i f_i)\, dm - \int_{K-K_n} (\sum_{i=1}^{p} \alpha_i f_i)\, dm\| \\
&= \| \int_{K-K_n} S_n(\sum_{i=1}^{p} \alpha_i f_i|_{K_n})\, dm - \int_{K-K_n} (\sum_{i=1}^{p} \alpha_i f_i)\, dm\| \leq 2/n.
\end{aligned}
$$

The denseness of $\{f_i : i \in \mathbb{N}\}$ in H implies that $\lim_{n\to\infty} \|\hat{T}|_H - TS_nR_n\| = 0$. If we set $\mathcal{U} = \{T \in \mathcal{B}\,|\, T^* \text{ is weakly precompact }\}$ then $T \in \mathcal{U}(C(K,X), Y)$ and so $TS_nR_n \in \mathcal{U}(H, Y)$. We know from lemma

2 that $\mathcal{U}(H,Y)$ is closed and thus $\hat{T}|_H \in \mathcal{U}(H,Y)$. Again lemma 2 tells us that $\mathcal{U}$ is separably determined and so $\hat{T} \in \mathcal{U}(U_X(\sum), Y)$. Since T^* is weakly precompact, we use theorem 2 to derive that T is unconditionally converging. Therefore m is strongly bounded with the help of Swartz's result [15] and hence $m : \sum \to \mathcal{B}(X,Y)$. Select any $\emptyset \neq A \in \sum$. Let $\phi : X \to U_X(\sum)$ be an isometric embedding which is given by $\phi(x) = \chi_A\, x,\ x \in X$. It is obvious that $m(A)x = \hat{T}\phi(x)$ for $x \in X$. Accordingly we end up with $m(A) \in \mathcal{U}(X,Y)$. This means that $m(A)^* : Y^* \to X^*$ is weakly precompact. □

We may conjecture that the converse of the preceding theorem is true, but we do not know the answer. The following proposition is an intrinsic characterization of a strongly bounded operator $T : C(K,X) \to Y$.

Proposition 3. *The following statements about an operator $T : C(K,X) \to Y$ are equivalent.*

(i) *T is strongly bounded.*
(ii) *The sequence (Tf_n) converges to 0 whenever $(f_n)$is a bounded sequence in $C(K,X)$ such that $(f_n(t))$ converges to zero for every $t \in K$.*

Proof. (i) $\Rightarrow$ (ii). The hypothesis (i) indicates that the representing measure m for T is strongly bounded. Thus for each decreasing sequence (A_n) of Borel sets satisfying $\lim_n A_n = \emptyset$, we have $\lim_n \tilde{m}(A_n) = 0$. Then $\{|m_z| : z \in B_{Y^*}\}$ is uniformly countably additive. As a result $\{|m_z| : z \in B_{Y^*}\}$ is weakly conditionally compact and there exists a control measure λ for $\tilde{m}$. Let (f_n) be a bounded sequence in $C(K,X)$ such that $(f_n(t))$ converges to zero for every $t \in K$. Suppose $\sup_n \|f_n\| \le \sigma$. We take account of Egoroff's theorem [6] to infer that (f_n) converges to zero uniformly λ-a.e., hence $\tilde{m}$–a.e.. This means that for each $\epsilon > 0$ there is a set $A \in \sum$ so that $\tilde{m}(A) < \epsilon/\sigma$ and (f_n) converges to zero uniformly on $K - A$. We choose a natural number N such that $\|f_n(t)\| < \epsilon/(\tilde{m}(K) + 1)$ for all $n \ge N$ and $t \in K - A$. Thus we have

$$\|T(f_n)\| = \| \int_K f_n \, dm\| \leq \| \int_A f_n \, dm\| + \| \int_{K-A} f_n \, dm\|$$
$$\leq \sigma \, \tilde{m}(A) + \frac{\epsilon \cdot \tilde{m}(K-A)}{\tilde{m}(K)+1} < 2\epsilon \quad \text{for all } n \geq N,$$

that is $(T(f_n))$ converges to zero.
(ii) $\Rightarrow$ (i). For this step, we will take the contrapositive route. Assume that (i) is false. Then the representing measure m for T is not strongly bounded. Then there exists a $\delta > 0$ and a sequence (A_n) of disjoint Borel sets such that $\tilde{m}(A_n) > \delta$ for each n. Since $\tilde{m}(A_n) = \sup\{|m_z|(A_n) : z \in B_{Y^*}\}$, we can pick a sequence in (z_n) in B_{Y^*} with $|m_{z_n}|(A_n) > \delta$ for each n. Hence $\{|m_{z_n}|\}$ is not uniformly countably additive and so $\{|m_{z_n}|\}$ is not weakly conditionally compact. Thanks to Grothendieck's theorem [7], we may assume that there exists an $\epsilon > 0$ and a sequence (O_n) of disjoint open subsets of K so that $|m_{z_n}|(O_n) > \epsilon$ for each n. The regularity property of m enables us to select a sequence (f_n) in $C(K,X)$ such that $\|f_n\| \leq 1$, support$(f_n) \subset O_n$ and $\|Tf_n\| > \epsilon$ for each n. Then for each $t \in K$, $(f_n(t))$ converges to zero but a sequence (Tf_n) will find it impossible to converge to zero. □

Next we give useful descriptions of operators $T : C(K,X) \to Y$ having weakly precompact adjoints under some restriction to the underlying Banach space X.

Theorem 7. *Suppose X^* does not contain a copy of ℓ_1. Then the following statements about an operator $T : C(K,X) \to Y$ are equivalent.*

(i) *T^* is weakly precompact.*
(ii) *T is unconditionally converging.*
(iii) *T is strongly bounded.*

Proof. (i) $\Rightarrow$ (ii) follows from theorem 2.
(ii) $\Rightarrow$ (iii) follows from Swartz's result [15].
(iii) $\Rightarrow$ (i). Let m be the representing measure for T. Assume that T^* is not weakly precompact. We find a sequence (y_n^*) in B_{Y^*} such

that $(T^*y_n^*) = (m_{y_n^*})$ is a copy of the usual ℓ_1-basis. We can assume $\|m_{y_n^*}\| \leq 1$ for each n. Relabel $m_{y_n^*}$ by $m_n,\ n = 1, 2, \cdots$. The hypothesis (iii) signifies that m is strongly bounded and so $\{|m_n|\}$ is uniformly countably additive. Let $\nu' = \sum_{n=1}^{\infty} |m_n|/2^n$ and let $(K, \sum_0, \nu)$ denote the completion of $(K, \sum, \nu')$. Then there exists a constant C such that $|m_n| \leq C\nu$ for each n. Let ρ be a lifting of $L_\infty(\nu)$. According to theorem 5 of [6], for each n there exists a function $g_n : K \to X^*$ such that

(a) $\|g_n(\cdot)\|$ is ν-integrable,
(b) $|m_n|(A) = \int_A \|g_n(\cdot)\|\, d\nu$ for $A \in \sum_0$,
(c) $\int f\, dm_n = \int \langle g_n(t), f(t)\rangle\, d\nu(t)$ for $f \in L_1(|m_n|, X)$,
(d) For each $x \in X,\ \langle g_n, x\rangle \in L_\infty(\nu)$ and $\rho\langle g_n, x\rangle = \langle g_n, x\rangle$.

Take any finite sequence $(a_n)_{n=1}^p$ of real numbers. Suppose $\alpha > 0$ is chosen so that $\|\sum_{n=1}^p a_n m_n\| > \alpha \sum_{n=1}^p |a_n|$. Let $0 < \epsilon < \alpha$. Then we can find $\delta > 0$ so that if $\nu(A) < \delta$ then $|m_n|(A) < \epsilon$ for all n. Put $A_n = \{t : \|g_n(t)\| > 1/\delta\}$. As $\|m_n\| = |m_n|(K) = \int_K \|g_n(\cdot)\|\, d\nu \leq 1$, we have $\nu(A_n) < \delta$ and thus $|m_n|(A_n) < \epsilon$ for all n. For each $B \in \sum_0$ and n, by setting $\hat{m}_n(B) = m_n(B - A_n)$ we get

$$|\hat{m}_n|(B) = \int_{B-A_n} \|g_n(\cdot)\|\, d\nu \leq 1/\delta \cdot \nu(B - A_n) \leq 1/\delta \cdot \nu(B).$$

It takes another appeal to theorem 5 of [6] to yield that for each n, there exists a function $\hat{g}_n : K \to X^*$ having the following properties :

(a) $\|\hat{g}_n(\cdot)\|$ is ν-integrable,
(b) $|\hat{m}_n|(A) = \int_A \|\hat{g}_n(\cdot)\|\, d\nu$ for $A \in \sum_0$,
(c) $\int f\, d\hat{m}_n = \int \langle \hat{g}_n(t), f(t)\rangle\, d\nu(t)$ for $f \in L_1(|\hat{m}_n|, X)$,
(d) For each $x \in X,\ \langle \hat{g}_n, x\rangle \in L_\infty(\nu)$ and $\rho\langle \hat{g}_n, x\rangle = \langle \hat{g}_n, x\rangle$.

Define $\mu = \sum_{n=1}^p a_n \hat{m}_n$. We can call on theorem 5 of [6] again to derive that there exists a function $g : K \to X^*$ such that

(a) $\|g(\cdot)\|$ is ν-integrable,
(b) $|\mu|(A) = \int_A \|g(\cdot)\|\, d\nu$ for $A \in \sum_0$,
(c) $\int f\, d\mu = \int \langle g(t), f(t)\rangle\, d\nu(t)$ for $f \in L_1(|\mu|, X)$,
(d) For each $x \in X,\ \langle g, x\rangle \in L_\infty(\nu)$ and $\rho\langle g, x\rangle = \langle g, x\rangle$.

If $h = \sum_{n=1}^p a_n \hat{g}_n$ then for each $A \in \sum_0$ and $x \in X$, we have

$$\int_A \langle h(t), x\rangle\, d\nu = \sum_{n=1}^p a_n \langle \hat{m}_n(A), x\rangle = \langle \mu(A), x\rangle = \int_A \langle g(t), x\rangle\, d\nu$$

and so $\langle g(\cdot), x\rangle = \langle h(\cdot), x\rangle$ ν-a.e.. Notice that for each $x \in X$, $\rho\langle h, x\rangle = \sum_{n=1}^{p} a_n\, \rho\langle \hat{g}_n, x\rangle = \sum_{n=1}^{p} a_n \langle \hat{g}_n, x\rangle = \langle h, x\rangle$. From the definition of a lifting ρ, it follows that for each $x \in X$, $\langle g(\cdot), x\rangle = \langle h(\cdot), x\rangle$ everywhere. Consequently

$$\|\sum_{n=1}^{p} a_n \hat{m}_n\| = \|\mu\| = |\mu|(K) = \int_K \|g(\cdot)\|\, d\nu = \int_K \|\sum_{n=1}^{p} a_n \hat{g}_n(\cdot)\|\, d\nu.$$

Now we consider the space $\mathcal{F}$ of all finitely non-zero sequences of real numbers and denote by e_n the n-th unit vector. For $t \in K$, define $|\cdot|_t$ on $\mathcal{F}$ by $|(a_n)|_t = \|\sum a_n \hat{g}_n(t)\|$, $(a_n) \in \mathcal{F}$. If $(a_n) \in \mathcal{F}$ then we get

$$\begin{aligned}\int_K |(a_n)|_t\, d\nu = \|\sum a_n \hat{m}_n\| &\geq \|\sum a_n m_n\| - \sum |a_n| \cdot \|m_n - \hat{m}_n\| \\ &= \|\sum a_n m_n\| - \sum |a_n| \cdot |m_n - \hat{m}_n|(K) \\ &= \|\sum a_n m_n\| - \sum |a_n| \cdot |m_n|(A_n) \\ &> \alpha \sum |a_n| - \epsilon \sum |a_n| = (\alpha - \epsilon) \sum |a_n|.\end{aligned}$$

Define $|||\cdot|||$ on $\mathcal{F}$ by $|||(a_n)||| = \int_K |(a_n)|_t\, d\nu$, $(a_n) \in \mathcal{F}$. Then we have

$$\begin{aligned}(\alpha - \epsilon) \sum |a_n| \leq |||\sum a_n e_n||| &= \|\sum a_n \hat{m}_n\| \\ &\leq \sum |a_n| \cdot |\hat{m}_n|(K) \leq 1/\delta \cdot \nu(K) \cdot \sum |a_n|.\end{aligned}$$

This gives that (e_n) is a copy of the usual ℓ_1-basis for $|||\cdot|||$. Then Bourgain's theorem [4] steps in to ensure that there exist $t \in K$ and a subsequence (e_{n_k}) of (e_n) which is a copy of the usual ℓ_1-basis for $|\cdot|_t$. That is, there exists $\eta > 0$ such that $\eta \sum |a_k| \leq |\sum a_k e_{n_k}|_t = \|\sum a_k \hat{g}_{n_k}(t)\|$ for all finitely non-zero sequences (a_k) of real numbers. This forces that $(\hat{g}_{n_k}(t))$ is a sequence in X^* that is a copy of the usual ℓ_1-basis. We reach a contradiction. $\square$

We provide a certain kind of surjective operators defined on the space of continuous X-valued functions in order to find conditions which imply that X has the weak* Radon-Nikodym property.

Proposition 4. *If $L : X \to Y$ is an operator and $L(X)$ contains a subspace F of Y isomorphic to c_0, then there is a bounded linear surjection $T : C(\{-1,1\}^{\mathbb{N}}, X) \to F$ with representing measure m so that m is strongly bounded and $m(A) : X \to F$ is compact for each $A \in \sum$.*

Proof. Let (y_n) be an unconditional basis for F which is equivalent to the usual c_0-basis (e_n), and suppose (y_n^*) is the sequence of coefficient functionals. Let z_n^* be a Hahn-Banach extension of y_n^* to all of Y. For $t = (t_n) \in \{-1,1\}^{\mathbb{N}} = \triangle$, we define the map $St : F \to F$ by $St(y) = \sum t_n\, z_n^*(y)\, y_n,\ y \in F$. Evidently St is linear and bounded. Define on Y an equivalent norm $|||y||| = \sup\{\sum |z_n^*(y) y^*(y_n)| : y^* \in B_{F^*}\}$. The uniform boundedness principle implies that $\sup\{\|St\| : t \in \triangle\} < \infty$. We use the letter μ for the canonical probability measure (Haar measure) $(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes\mathbb{N}}$ defined on $\triangle$. Let (r_n) be the sequence of Rademacher functions on $\triangle$. Consider the map $m : \sum \to \mathcal{B}(X, F)$ given by $m(A)x = \sum_{n=1}^{\infty} (\int_A r_n\, d\mu)\, z_n^*(Lx)\, y_n,\ A \in \sum,\ x \in X$. Then we have $\|m(A)x\| \le |||Lx||| \cdot \mu(A)$. Accordingly m is dominated and so m is strongly bounded. For each $A \in \sum$, since $\lim_{n\to\infty} \int_A r_n\, d\mu = 0$, it follows that $m(A)$ is the limit in operator norm of a sequence of finite rank operators. This leads us to have that $m(A)$ is compact. Let $T : C(\triangle, X) \to F$ be the operator defined by $T(f) = \int_\triangle f\, dm,\ f \in C(\triangle, X)$. Now pick $\tau > 0$ for which $\{St(B_F) : t \in \triangle\} \subseteq L(\tau B_X)$. Let us take any $y \in B_F$. Given $\epsilon > 0$ we choose a natural number N with $|z_n^*(y)| < \epsilon$ for all $n > N$. By $D_{Ni},\ 1 \le i \le 2^N$, we mean the dyadic partition at the N-th stage. For $t^i \in D_{Ni},\ 1 \le i \le 2^N$, there exists $x^i \in \tau B_X$ so that $St^i y = Lx^i$. By letting $f = \sum_{i=1}^{2^N} \chi_{D_{Ni}}\, x^i \in \tau B_{C(\triangle, X)}$, we get

$$T(f) = \sum_{i=1}^{2^N} m(D_{Ni}) x^i = \sum_{i=1}^{2^N} \sum_{n=1}^{\infty} (\int_{D_{Ni}} r_n\, d\mu)\, z_n^*(St^i y)\, y_n.$$

Observe that if $n > N$ then $\int_{D_{Ni}} r_n\, d\mu = 0$ for $1 \le i \le 2^N$, and if $1 \le n \le N$ then $(\int_{D_{Ni}} r_n\, d\mu)\, z_n^*(St^i y)\, y_n = 1/2^N \cdot z_n^*(y)\, y_n$. Therefore we have $T(f) = \sum_{n=1}^{N} z_n^*(y)\, y_n$ and $\|y - Tf\| = \|\sum_{n=N+1}^{\infty} z_n^*(y)\, y_n\| < \alpha \|\sum_{n=N+1}^{\infty} z_n^*(y)\, e_n\| < \alpha\, \epsilon$ for some constant $\alpha > 0$. This yields that $B_F \subseteq \overline{T(\tau B_{C(\triangle, X)})}$ and hence T is a surjection. □

Having this preliminary result we draw the following theorem.

THEOREM 8. *Suppose that the operator $T : C(K, X) \to Y$ has a weakly precompact adjoint whenever the representing measure m for T is strongly bounded and $m(A) : X \to Y$ is compact for each $A \in \sum$. Then X has the weak* Radon-Nikodym property.*

Proof. Assume that the assertion fails. Then X^{**} does not have the weak Radon-Nikodym property. By virtue of Janicka's theorem [9], X^* contains a copy of ℓ_1. Then either X contains a copy of ℓ_1 or it does not. If not, then we use a result due to J. Hagler and W. Johnson [8] to see that there exists a weak* null sequence (x_n^*) in X^* equivalent to the usual ℓ_1-basis. we can assume $\|x_n^*\| = 1$ for each n. Proceeding in the same way as in the proof of theorem 5, we get an increasing sequence (n_j) of positive integers and a set $\{x_p^i : 1 \leq i \leq m_p, p = 2, 3, \cdots\}$ of vectors from CB_X such that
(a) $\{(\langle x_{n_j}^*, x_p^i\rangle)_{j=1}^p : 1 \leq i \leq m_p\}$ forms a $(\frac{1}{2})^p$-net for $B_{\ell_\infty^p}$,
(b) $n \geq n_{p+1}$ implies $|\langle x_n^*, x_p^i\rangle| < (\frac{1}{2})^p$ for $1 \leq i \leq m_p$.
If we define a linear map $S : X \to c_0$ by $Sx = (\langle x_{n_j}^*, x\rangle)_j$ for all $x \in X$, then $\|S\| \leq 1$ and $S(X) = c_0$.

Next consider the case in which X contains a copy of ℓ_1. Let $L : \ell_1 \to c_0$ be any bounded linear surjection. The injectivity of ℓ_∞ makes an extension $\tilde{L} : X \to \ell_\infty$ of $L : \ell_1 \to \ell_\infty$ such that $\tilde{L}(\ell_1) = c_0 \subseteq \tilde{L}(X)$.

Now we apply proposition 4 to derive that there exists a bounded linear surjection $T : C(\{-1, 1\}^{\mathbb{N}}, X) \to c_0$ with representing measure m so that m is strongly bounded and $m(A) : X \to c_0$ is compact for each $A \in \sum$. As a result T^* is an isomorphism on ℓ_1, i.e., T^* is not weakly precompact. This contradiction completes the proof. □

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