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OSCILLATION AND NONOSCILLATION THEOREMS OF SOLUTIONS FOR SOME NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study oscillation and nonoscillation criteria of solutions for the following nonlinear differential equation

$$\left[\frac{1}{p(t)}(x'(t))^{\mu}\right]' + q(t)x(t)^{\mu} = 0.$$

where μ with $\mu \geq 1$ is a quotient of odd integers.

1. Introduction

The purpose of this paper is to study oscillatory or nonoscillatory properties of solutions of some differential equation

$$\left[\frac{1}{p(t)}\left(x'(t)\right)^{\mu}\right]' + q(t)x(t)^{\mu} = 0 \qquad (E)$$

where

- (C_1) the function $p \in C[t_0, \infty)$ is positive.
- (C_2) $q(t)$ is positive for all $t \in [t_0, \infty)$.
- (C_3) μ with $\mu \geq 1$ is a quotient of odd integers.

In this paper we always define a function $\rho(t)$ as

$$\rho(t) = \int_{t_0}^{t} p(s)^{1/\mu}\, ds, \qquad t_0 \leq t,$$

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and assume that

$$\int_{t_0}^{\infty} p(s)^{1/\mu}\, ds = \infty \tag{H_1}$$

and that

$$\int_{t_0}^{\infty} q(s)\, ds = \infty \tag{H_2}$$

By a solution of (E) is meant a function $x(t) \in C^2[T,\infty)$, $T \geq t_0$, satisfying $x'(t)^\nu \in C^1[T,\infty)$ and satisfying (E) for all $t \geq T$. There are many papers devoted to either oscillation or nonoscillation of solutions(See [1],[2],[5]-[8]). It will be always assumed that nonconstant solutions of (E) exist on some ray $[T,\infty)$, $T \geq t_0$. A solution $x(t)$ is oscillatory if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of zeros of $x(t)$ such that $t_n \to \infty$ as $n \to \infty$. Otherwise it is said to be nonoscillatory. Equation (E) is called oscillatory if all solutions are oscillatory.

2. Main Results

THEOREM 1. *Let a function $a(t)$ be positive, increasing and differentiable for $t \geq t_0$. Then under the assumption (H_1) the equation (E) is oscillatory if the inequality*

$$\int^{\infty} \left[a(s)q(s) - \frac{a'(s)^{\mu+1}}{p(s)a(s)^{\mu}} \left(\frac{1}{\mu+1} \right)^{\mu+1} \right] ds = \infty \tag{1}$$

is valid.

Proof. We assume that (E) is nonoscillatory. Then there exists a solution $x(t)$ eventually of one sign. We may assume that $x(t) > 0$, $t \geq T$ for some $T \geq t_0$. The similar argument is valid for the case when $x(t)$ is eventually negative. We define a function $w(t)$ by

$$w(t) = \frac{a(t)}{p(t)} \frac{\left[x'(t)\right]^{\mu}}{x(t)^{\mu}}. \tag{2}$$

Then

$$\frac{x'(t)}{x(t)} = \left[\frac{p(t)w(t)}{a(t)} \right]^{1\mu}. \tag{3}$$

It follows that $\dfrac{1}{p(t)}\left[x'(t)\right]^{\mu}$ is decreasing.
We can easily show that

$$w(t) > 0 \tag{4}$$

for $t \geq T$. We have then from (2) and (3)

$$\begin{aligned} w'(t) &= -a(t)q(t) + \frac{a'(t)}{a(t)}w(t) - \mu w(t)\left[\frac{p(t)w(t)}{a(t)}\right]^{1/\mu} \\ &= -a(t)q(t) + \frac{a'(t)}{a(t)}w(t) - \mu\left[\frac{p(t)}{a(t)}\right]^{1/\mu} w(t)^{1+1/\mu} \end{aligned} \tag{5}$$

We seek the maximum of

$$F(z,t) = \frac{a'(t)}{a(t)}z - \mu\left[\frac{p(t)}{a(t)}\right]^{1/\mu} z^{1+1/\mu}.$$

It is obvious that F has the maximum at

$$z_0 = \frac{a'(t)^{\mu}}{p(t)a(t)^{\mu-1}}\left(\frac{1}{\mu+1}\right)^{\mu}.$$

for all t. Thus we have

$$F(z,t) \leq \frac{a'(t)^{\mu+1}}{p(t)a(t)^{\mu}}\left(\frac{1}{\mu+1}\right)^{\mu+1} \tag{6}$$

for all t. Therefore we obtain

$$w'(t) \leq -a(t)q(t) + \frac{a'(t)^{\mu+1}}{p(t)a(t)^{\mu}}\left(\frac{1}{\mu+1}\right)^{\mu+1}. \tag{7}$$

By means of (7) we have

$$w(t) \leq w(T) - \int_T^t \left[a(s)q(s) - \frac{a'(s)^{\mu+1}}{p(s)a(s)^{\mu}}\left(\frac{1}{\mu+1}\right)^{\mu+1}\right] ds, \tag{8}$$

which contradicts (4). Therefore our theorem is proved. □

COROLLARY 1. *Under the same assumptions as in theorem 1 the equation (E) is oscillatory if the inequality*

$$\liminf_{t\to\infty}\left[p(s)q(s)\frac{a(s)^{\mu+1}}{a'(s)^{\mu+1}} - \left(\frac{1}{\mu+1}\right)^{\mu+1}\right] > 0 \tag{9}$$

is valid.

THEOREM 2. *The equation (E) with $p(t) \equiv 1$ is oscillatory if the inequality*

$$\int^{\infty}\left[s^{\mu}q(s) - \frac{1}{s}\left(\frac{\mu}{\mu+1}\right)^{\mu+1}\right]ds = \infty \tag{10}$$

is valid.

Proof. In the proof of theorem 1 we choose functions $a(t) = t^{\mu}$ and $p(t) = 1$. Then it is obvious that

$$w'(t) \le -t^{\mu}q(t) + \frac{1}{t}\left(\frac{\mu}{\mu+1}\right)^{\mu+1}. \tag{11}$$

The rest of proof is the same as in the proof of theorem 1. □

As a consequence we obtain the following result.

COROLLARY 2. *The equation (E_1) is oscillatory if the inequality*

$$\liminf_{t\to\infty}\left[t^{\mu+1}q(t) - \left(\frac{\mu}{\mu+1}\right)^{\mu+1}\right] > 0$$

is valid.

COROLLARY 3. *Assume that (H_1), (H_2) are valid. The equation (E) is oscillatory..*

Proof. In the proof of theorem 1 we choose a function $w(t)$ as follows

$$w(t) = \frac{x'(t)^\mu}{p(t)\, x(t)^\mu}. \tag{12}$$

Since then $w(t) > 0$ for large t, it is obvious that

$$\begin{aligned} w'(t) &= -q(t) - \mu p(t)^{1/\mu} w(t)^{1+1/\mu} \\ &\leq -q(t). \end{aligned} \tag{13}$$

Therefore our theorem follows. □

THEOREM 3. *Assume that (H_1) is valid and that $\int_{t_0}^\infty q(s)\, ds < \infty$. Then the following are equivalent*

(a) *the equation (E) is nonoscillatory.*
(b) $\lim_{t\to\infty} w(t) = 0$ *where $w(t)$ is the same as given in (12).*
(c) *There exist a $T \geq t_0$ and a continuous and positive function $w(t)$ such that for $T \leq t$*

$$w(t) = \int_t^\infty p(s)^{1/\mu} w(s)^{1+1/\mu}\, ds + \int_t^\infty q(s)\, ds. \tag{14}$$

Proof. (a)$\Longrightarrow$ (b): Assume that (a) is valid. There exist a $T \geq t_0$ and a solution $x(t)$ of (E) such that $x(t) > 0$ for $t \geq T$. The similar argument is valid for the case when $x(t)$ is eventually negative.
It follows that $x'(t) > 0$ and that $x'(t)^\mu / p(t)$ is decreasing. Therefore we have

$$\lim_{t\to\infty} \frac{x'(t)^\mu}{p(t)} \geq 0.$$

Assume that

$$\lim_{t\to\infty} \frac{x'(t)^\mu}{p(t)} = \alpha > 0. \tag{15}$$

Since then there exists a $T_1 > T$ such that

$$x(t) \geq x(T_1) + \left(\frac{\alpha}{2}\right)^{1/\mu} \int_{T_1}^t p(s)^{1/\mu}\, ds \tag{16}$$

we have

$$\lim_{t\to\infty} x(t) = \infty. \tag{17}$$

Therefore It follows that

$$\lim_{t\to\infty} w(t) \le \lim_{t\to\infty} \frac{x'(T)^\mu}{p(T)x(t)^\mu} = 0. \tag{18}$$

Assume that

$$\lim_{t\to\infty} \frac{x'(t)^\mu}{p(t)} = 0. \tag{19}$$

On the other hand, since $x'(t) > 0$, there exist a $T_2 > T$ and a constant $c > 0$ such that $x(t) > c$ for $T_2 \le t$. Therefore It follows that

$$\lim_{t\to\infty} w(t) \le c^\mu \lim_{t\to\infty} \frac{x'(t)^\mu}{p(t)} = 0. \tag{20}$$

Consequently (b) follows from (18) and (20).
(b)$\Longrightarrow$ (c) : Integrating from t to ∞ after differentiating $w(t)$ we obtain (14).
(c)$\Longrightarrow$ (a) : Differentiating both sides of (14) we obtain (13). Then we have

$$x(t) = x(T)\exp\left[\int_T^t p(s)^{1/\mu} w(s)^{1/\mu}\, ds\right]$$

which is a nonoscillatory solution of (E). □

We consider a differential equation of the type

$$\left[\frac{1}{P(t)}\left(y'(t)\right)^\mu\right]' + Q(t)y(t)^\mu = 0 \tag{E_P}$$

where $P(t)$ is continuous for $t \ge t_0$. Then we obtain the following comparison theorem.

THEOREM 4. *Assume that for $t \geq t_0$*

$$0 \leq p(t) \leq P(t), \qquad q(t) \leq Q(t) \tag{21}$$

and that the following are valid :

$$\int_{t_0}^{\infty} P(s)^{1/\mu}\, ds = \infty, \qquad \int_{t_0}^{\infty} Q(s)\, ds < \infty. \tag{22}$$

Then if (E_P) has a positive solution, (E) has also a positive solution.

Proof. Assume that (E_P) has a positive solution $y(t)$. If we put

$$W(t) = \frac{y'(t)^\mu}{P(t)y(t)^\mu},$$

then it follows that $W(t) > 0$ and

$$W(t) = \int_t^{\infty} Q(s)\, ds + \mu \int_t^{\infty} P(s)^{1/\mu} W(s)^{1+1/\mu}\, ds. \tag{23}$$

Consider a mapping K defined by

$$(Ku)(t) = \int_t^{\infty} q(s)\, ds + \mu \int_t^{\infty} p(s)^{1/\mu} u(s)^{1+1/\mu}\, ds$$

where

$$U = \{u(t) \in C^2[t_0, \infty) \mid 0 \leq u(t) \leq W(t)\}.$$

Then the mapping $K : U \to U$ is a compact mapping and K has a fixed point $u(t)$(see [3]). By means of theorem 3 (E) is nonoscillatory. Then if we choose $T > t_0$ such that $x(T) > 0$, a positive solution of (E) is of the form:

$$x(t) = x(T) \exp\left[\int_T^t p(s)^{1/\mu} u(s)^{1/\mu}\, ds\right].$$

□

We consider the equation[4] :

$$\left[\frac{1}{p(t)}\left(x'(t)\right)^{\mu}\right]' + \rho(t)^{-\mu-1}p(t)^{1/\mu}q(t)x(t)^{\mu} = 0. \qquad (E_1)$$

Put $x = \rho(t)^{\alpha}$. Then since $\rho'(t) = p(t)^{1/\mu}$, we obtain

$$\alpha^{\mu}(\alpha - 1)\mu + q(t) = 0. \qquad (24)$$

It is easy to show that

$$-\alpha^{\mu}(\alpha - 1)\mu \le \left(\frac{\mu}{\mu+1}\right)^{\mu+1}$$

where the equality is valid at $\alpha = \dfrac{\mu}{\mu+1}$. Therefore we obtain :

Example. Let (H_1) be valid. Assume that $q(t)$ is integrable on $[t_0, \infty)$.

(a) (E) is nonoscillatory if for large t

$$\rho(t)^{\mu+1}p(t)^{-1/\mu}q(t) \le \left(\frac{\mu}{\mu+1}\right)^{\mu+1}. \qquad (25)$$

(b) (E) is oscillatory if for large t

$$\rho(t)^{\mu+1}p(t)^{-1/\mu}q(t) > \left(\frac{\mu}{\mu+1}\right)^{\mu+1}. \qquad (26)$$

Proof. We note that equation

$$\left[\frac{1}{p(t)}\left(x'(t)\right)^{\mu}\right]' + \left(\frac{\mu}{\mu+1}\right)^{\mu}\rho(t)^{-\mu-1}p(t)^{1/\mu}x(t)^{\mu} = 0. \qquad (E_2)$$

has a positive solution $x = \rho(t)^{\mu/(\mu+1)}$. It is obvious that $\rho(t)^{-\mu-1}p(t)^{1/\mu}$ is integrable on $[t_0, \infty)$. If we put

$$Q(t) = \left(\frac{\mu}{\mu+1}\right)^{\mu}\rho(t)^{-\mu-1}p(t)^{1/\mu},$$

(a) follows from theorem 4. If (26) is valid, there is no real value α satisfying (24) for all t. Thus (b) holds. □

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