Bull. Korean Math. Soc. **42** (2005), No. 2, pp. 213–230

SOME COMPANIONS OF OSTROWSKI'S INEQUALITY FOR ABSOLUTELY CONTINUOUS FUNCTIONS AND APPLICATIONS

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Abstract. Companions of Ostrowski's integral inequality for absolutely continuous functions and applications for composite quadrature rules and for p.d.f.'s are provided.

1. Introduction

In [1], Guessab and Schmeisser have proved among others, the following companion of Ostrowski's inequality.

Theorem 1. *Let* $f : [a,b] \to \mathbb{R}$ *be such that*

$$|f(t) - f(s)| \le M |t-s|^k, \text{ for any } t, s \in [a,b] \tag{1.1}$$

with $k \in (0,1]$, *i.e.,* $f \in Lip_M(k)$. *Then, for each* $x \in \left[a, \frac{a+b}{2}\right]$, *we have the inequality*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{2^{k+1}(x-a)^{k+1} + (a+b-2x)^{k+1}}{2^k (k+1)(b-a)} \right] M. \tag{1.2}$$

This inequality is sharp for each admissible x. *Equality is obtained if and only if* $f = \pm M f_* + c$ *with* $c \in \mathbb{R}$ *and*

$$f_*(t) = \begin{cases} (x-t)^k & \text{for } a \le t \le x; \\ (t-x)^k & \text{for } x \le t \le \frac{1}{2}(a+b); \\ f_*(a+b-t) & \text{for } \frac{1}{2}(a+b) \le t \le b. \end{cases} \tag{1.3}$$

Received March 24, 2003.
2000 Mathematics Subject Classification: Primary 26D15; Secondary 41A55.
Key words and phrases: Ostrowski's inequality, trapezoid rule, midpoint rule.

We remark that for $k=1$, i.e., $f \in \mathrm{Lip}_M$, since

$$\frac{4(x-a)^2+(a+b-2x)^2}{4(b-a)}=\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a),$$

then we have the inequality

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)M, \tag{1.4}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

The constant $\frac{1}{8}$ is best possible in (1.4) in the sense that it cannot be replaced by a smaller constant.

We must also observe that the best inequality in (1.4) is obtained for $x=\frac{a+3b}{4}$, giving the trapezoid type inequality

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}(b-a)M. \tag{1.5}$$

The constant $\frac{1}{8}$ is sharp in (1.5) in the sense mentioned above.

For a recent monograph devoted to Ostrowski type inequalities, see [2].

In this paper we improve the above results and also provide other bounds for absolutely continuous functions whose derivatives belong to the Lebesgue spaces $L_p[a,b]$, $1 \leq p \leq \infty$. Some natural applications are also provided.

2. Some integral inequalities

The following identity holds.

Lemma 1. *Assume that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$. Then we have the equality*

$$\text{(2.1)}\quad \frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^x (t-a)\,f'(t)\,dt+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\,f'(t)\,dt,$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. Using the integration by parts formula for Lebesgue integrals, we have

$$\int_a^x (t-a)\,f'(t)\,dt=f(x)(x-a)-\int_a^x f(t)\,dt,$$

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)f'(t)\,dt$$
$$=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x} f(t)\,dt,$$

and

$$\int_{a+b-x}^b (t-b)\,f'(t)\,dt=(x-a)\,f(a+b-x)-\int_{a+b-x}^b f(t)\,dt.$$

Summing the above equalities, we deduce the desired identity (2.1). □

REMARK 1. The identity (2.1) was obtained in [1, Lemma 3.2] for the case of piecewise continuously differentiable functions on $[a,b]$.

The following result holds.

THEOREM 2. *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *Then we have the inequality*

$$\text{(2.2)}\quad \left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\le\frac{1}{b-a}\left[\int_a^x (t-a)\left|f'(t)\right|dt\right.$$
$$\left.+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt+\int_{a+b-x}^b (b-t)\left|f'(t)\right|dt\right]$$
$$:=M(x),$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

If $f' \in L_{\infty}[a,b]$, *then we have the inequalities*

$$
\begin{aligned}
M(x) &\le \frac{1}{b-a}\left[\frac{(x-a)^2}{2}\|f'\|_{[a,x],\infty} + \left(\frac{a+b}{2}-x\right)^2\|f'\|_{[x,a+b-x],\infty} \right. \\
&\qquad\qquad \left. + \frac{(x-a)^2}{2}\|f'\|_{[a+b-x,b],\infty}\right]
\end{aligned} \tag{2.3}
$$

$$
\le \begin{cases}
\left[\frac{1}{8} + 2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty}; \\[2ex]
\left[\frac{1}{2^{\alpha-1}}\left(\frac{x-a}{b-a}\right)^{2\alpha} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right]^{\frac{1}{\alpha}} \\
\times\left[\|f'\|^{\beta}_{[a,x],\infty} + \|f'\|^{\beta}_{[x,a+b-x],\infty} + \|f'\|^{\beta}_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}(b-a) \\
\qquad\qquad \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\max\left\{\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2, \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\} \\
\times\left[\|f'\|_{[a,x],\infty} + \|f'\|_{[x,a+b-x],\infty} + \|f'\|_{[a+b-x,b],\infty}\right](b-a);
\end{cases}
$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

The inequality (2.2), *the first inequality in* (2.3) *and the constant* $\frac{1}{8}$ *are sharp.*

Proof. The inequality (2.2) follows by Lemma 1 on taking the modulus and using its properties.

If $f' \in L_{\infty}[a,b]$, then

$$
\int_a^x (t-a)\left|f'(t)\right|dt \le \frac{(x-a)^2}{2}\|f'\|_{[a,x],\infty},
$$

$$
\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt \le \left(\frac{a+b}{2}-x\right)^2\|f'\|_{[x,a+b-x],\infty},
$$

$$
\int_{a+b-x}^{b}(b-t)\left|f'(t)\right|dt \le \frac{(x-a)^2}{2}\|f'\|_{[a+b-x,b],\infty}.
$$

and the first inequality in (2.3) is proved.

Denote

$$\tilde{M}(x) := \frac{(x-a)^2}{2}\left\|f'\right\|_{[a,x],\infty} + \left(\frac{a+b}{2}-x\right)^2\left\|f'\right\|_{[x,a+b-x],\infty} + \frac{(x-a)^2}{2}\left\|f'\right\|_{[a+b-x,b],\infty},$$

for $x \in \left[a, \frac{a+b}{2}\right]$.

Firstly, observe that

$$\begin{aligned}\tilde{M}(x) &\leq \max\left\{\left\|f'\right\|_{[a,x],\infty}, \left\|f'\right\|_{[x,a+b-x],\infty}, \left\|f'\right\|_{[a+b-x,b],\infty}\right\} \\ &\quad\times\left[\frac{(x-a)^2}{2} + \left(\frac{a+b}{2}-x\right)^2 + \frac{(x-a)^2}{2}\right] \\ &= \left\|f'\right\|_{[a,b],\infty}\left[\frac{1}{8}(b-a)^2 + 2\left(x - \frac{3a+b}{4}\right)^2\right]\end{aligned}$$

and the first inequality in (2.3) is proved.

Using Hölder's inequality for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we also have

$$\begin{aligned}\tilde{M}(x) \leq &\left\{\left[\frac{(x-a)^2}{2}\right]^\alpha + \left(x - \frac{a+b}{2}\right)^{2\alpha} + \left[\frac{(x-a)^2}{2}\right]^\alpha\right\}^{\frac{1}{\alpha}} \\ &\times\left[\left\|f'\right\|^\beta_{[a,x],\infty} + \left\|f'\right\|^\beta_{[x,a+b-x],\infty} + \left\|f'\right\|^\beta_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}\end{aligned}$$

giving the second inequality in (2.3).

Finally, we also observe that

$$\begin{aligned}\tilde{M}(x) \leq \max&\left\{\frac{(x-a)^2}{2}, \left(x - \frac{a+b}{2}\right)^2\right\} \\ &\times\left[\left\|f'\right\|_{[a,x],\infty} + \left\|f'\right\|_{[x,a+b-x],\infty} + \left\|f'\right\|_{[a+b-x,b],\infty}\right].\end{aligned}$$

The sharpness of the inequalities mentioned follows from Theorem 1 for $k = 1$. We omit the details. □

REMARK 2. If in Theorem 2 we choose $x = a$, then we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4}(b-a)\left\|f'\right\|_{[a,b],\infty} \tag{2.4}$$

with $\frac{1}{4}$ as a sharp constant (see for example [2, p.25]).

If in the same theorem we now choose $x=\frac{a+b}{2}$, then we get

$$\begin{aligned}(2.5)\qquad &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{8}(b-a)\left[\left\|f'\right\|_{\left[a,\frac{a+b}{2}\right],\infty}+\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\ &\leq \frac{1}{4}(b-a)\left\|f'\right\|_{[a,b],\infty}\end{aligned}$$

with the constants $\frac{1}{8}$ and $\frac{1}{4}$ being sharp. This result was obtained in [3].

It is natural to consider the following corollary.

COROLLARY 1. *With the assumptions in Theorem 2, one has the inequality:*

$$(2.6)\qquad \left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}(b-a)\left\|f'\right\|_{[a,b],\infty}.$$

The constant $\frac{1}{8}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The case when $f'\in L_p[a,b]$, $p>1$ is embodied in the following theorem.

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ so that $f'\in L_p[a,b]$, $p>1$. If $M(x)$ is as defined in (2.2), then we have the bounds:*

$$\begin{aligned}(2.7)\qquad M(x) \leq \frac{1}{(q+1)^{\frac{1}{q}}}&\left[\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}}\left\|f'\right\|_{[a,x],p}\right. \\ &+2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\left\|f'\right\|_{[x,a+b-x],p} \\ &\left.+\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}}\left\|f'\right\|_{[a+b-x,b],p}\right](b-a)^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{1}{(q+1)^{\frac{1}{q}}} \times \begin{cases} \left[2\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}} + 2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\right] \\ \times \max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,a+b-x],p}, \|f'\|_{[a+b-x,b],p}\right\}(b-a)^{\frac{1}{q}}; \\ \\ \left[2\left(\frac{x-a}{b-a}\right)^{\alpha+\frac{\alpha}{q}} + 2^{\frac{\alpha}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha+\frac{\alpha}{q}}\right]^{\frac{1}{\alpha}} \\ \times \left[\|f'\|^{\beta}_{[a,x],p} + \|f'\|^{\beta}_{[x,a+b-x],p} + \|f'\|^{\beta}_{[a+b-x,b],p}\right]^{\frac{1}{\beta}}(b-a)^{\frac{1}{q}} \\ \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \max\left\{\left(\frac{x-a}{b-a}\right)^{1+\frac{1}{q}}, 2^{\frac{1}{q}}\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{1+\frac{1}{q}}\right\} \\ \quad \times \left[\|f'\|_{[a,x],p} + \|f'\|_{[x,a+b-x],p} + \|f'\|_{[a+b-x,b],p}\right](b-a)^{\frac{1}{q}}; \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Using Hölder's integral inequality for $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned} \int_a^x (t-a)\left|f'(t)\right| dt &\leq \left(\int_a^x (t-a)^q\, dt\right)^{\frac{1}{q}} \left\|f'\right\|_{[a,x],p} \\ &= \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left\|f'\right\|_{[a,x],p}, \end{aligned}$$

$$\begin{aligned} &\int_x^{a+b-x} \left|t-\frac{a+b}{2}\right| \left|f'(t)\right| dt \\ &\leq \left(\int_x^{a+b-x} \left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}} \left\|f'\right\|_{[x,a+b-x],p} \\ &= \frac{2^{\frac{1}{q}}\left(\frac{a+b}{2}-x\right)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left\|f'\right\|_{[x,a+b-x],p}, \end{aligned}$$

and

$$\int_{a+b-x}^{b} (b-t)\left|f'(t)\right| dt \leq \left(\int_{a+b-x}^{b} (b-t)^{q} dt\right)^{\frac{1}{q}} \left\|f'\right\|_{[a+b-x,b],p}$$
$$= \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left\|f'\right\|_{[a+b-x,b],p}.$$

Summing the above inequalities, we deduce the first bound in (2.7).

The last part may be proved in a similar fashion to the one in Theorem 2, and we omit the details. □

REMARK 3. If in (2.7) we choose $\alpha = q$, $\beta = p$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then we get the inequality

(2.8)
$$M(x) \leq \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \left\|f'\right\|_{[a,b],p},$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

REMARK 4. If in Theorem 3 we choose $x = a$, then we get the trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2} \cdot \frac{(b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p}}{(q+1)^{\frac{1}{q}}}. \tag{2.9}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant (see for example [2, p.42]).

Indeed, if we assume that (2.9) holds with a constant $C > 0$, instead of $\frac{1}{2}$, i.e.,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq C \cdot \frac{(b-a)^{\frac{1}{q}} \|f'\|_{[a,b],p}}{(q+1)^{\frac{1}{q}}}, \tag{2.10}$$

then for the function $f:[a,b]\to\mathbb{R}$, $f(x)=k\left|x-\frac{a+b}{2}\right|$, $k>0$, we have

$$\frac{f(a)+f(b)}{2}=k\cdot\frac{b-a}{2},$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt=k\cdot\frac{b-a}{4},$$
$$\|f'\|_{[a,b],p}=k(b-a)^{\frac{1}{p}};$$

and by (2.10) we deduce

$$\left|\frac{k(b-a)}{2}-\frac{k(b-a)}{4}\right|\le\frac{C\cdot k(b-a)}{(q+1)^{\frac{1}{q}}},$$

giving $C\ge\frac{(q+1)^{\frac{1}{q}}}{4}$. Letting $q\to 1+$, we deduce $C\ge\frac{1}{2}$, and the sharpness of the constant is proved.

Remark 5. If in Theorem 3 we choose $x=\frac{a+b}{2}$, then we get the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.11}$$
$$\le\frac{1}{2}\cdot\frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\|f'\|_{[a,\frac{a+b}{2}],p}+\|f'\|_{[\frac{a+b}{2},b],p}\right]$$
$$\le\frac{1}{2}\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p},\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1.$$

In both inequalities the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

To show this fact, assume that (2.11) holds with $C,D>0$, i.e.,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.12}$$
$$\le C\cdot\frac{(b-a)^{\frac{1}{q}}}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\|f'\|_{[a,\frac{a+b}{2}],p}+\|f'\|_{[\frac{a+b}{2},b],p}\right]$$
$$\le D\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p}.$$

For the function $f:[a,b]\to\mathbb{R}$, $f(x)=k\left|x-\frac{a+b}{2}\right|$, $k>0$, we have

$$f\left(\frac{a+b}{2}\right)=0,\ \ \frac{1}{b-a}\int_a^b f(t)\,dt=\frac{k(b-a)}{4},$$

$$\left\|f'\right\|_{\left[a,\frac{a+b}{2}\right],p}+\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],p}=2\left(\frac{b-a}{2}\right)^{\frac{1}{p}}k=2^{\frac{1}{q}}(b-a)^{\frac{1}{p}}k,$$

$$\left\|f'\right\|_{[a,b],p}=(b-a)^{\frac{1}{p}}k;$$

and then by (2.12) we deduce

$$\frac{k(b-a)}{4}\le C\cdot\frac{k(b-a)}{(q+1)^{\frac{1}{q}}}\le D\cdot\frac{k(b-a)}{(q+1)^{\frac{1}{q}}},$$

giving $C,D\ge\frac{(q+1)^{\frac{1}{q}}}{4}$ for any $q>1$. Letting $q\to1+$, we deduce $C,D\ge\frac{1}{2}$ and the sharpness of the constants in (2.11) are proved.

The following result is useful in providing the best quadrature rule in the class for approximating the integral of an absolutely continuous function whose derivative is in $L_p[a,b]$.

Corollary 2. *Assume that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function so that $f'\in L_p[a,b]$, $p>1$. Then one has the inequality*

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{[a,b],p},\tag{2.13}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

Proof. The inequality follows by Theorem 3 and Remark 3 on choosing $x=\frac{3a+b}{4}$.

To prove the sharpness of the constant, assume that (2.13) holds with a constant $E>0$, i.e.,

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le E\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{[a,b],p}.\tag{2.14}$$

Consider the function $f : [a, b] \to \mathbb{R}$,

$$f(x) = \begin{cases} \left| x - \dfrac{3a+b}{4} \right| & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \left| x - \dfrac{a+3b}{4} \right| & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then f is absolutely continuous and $f' \in L_p[a, b]$, $p > 1$. We also have

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] = 0, \quad \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{b-a}{8},$$

$$\|f'\|_{[a,b],p} = (b-a)^{\frac{1}{p}},$$

and then, by (2.14), we obtain:

$$\frac{b-a}{8} \leq E\frac{(b-a)}{(q+1)^{\frac{1}{q}}}$$

giving $E \geq \frac{(q+1)^{\frac{1}{q}}}{8}$ for any $q > 1$, i.e., $E \geq \frac{1}{4}$, and the corollary is proved. □

If one is interested in obtaining bounds in terms of the 1−norm for the derivative, then the following result may be useful.

THEOREM 4. *Assume that the function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$. If $M(x)$ is as in equation (2.2), then we have the bounds*

$$\text{(2.15)} \quad M(x) \leq \left(\frac{x-a}{b-a}\right)\|f'\|_{[a,x],1} + \left(\frac{\frac{a+b}{2}-x}{b-a}\right)\|f'\|_{[x,a+b-x],1} + \left(\frac{x-a}{b-a}\right)\|f'\|_{[a+b-x,b],1}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left|\frac{x - \frac{3a+b}{4}}{b-a}\right|\right] \left\|f'\right\|_{[a,b],1}, \\ \\ \left[2\left(\frac{x-a}{b-a}\right)^{\alpha} + \left(\frac{\frac{a+b}{2} - x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times \left[\left\|f'\right\|_{[a,x],1}^{\beta} + \left\|f'\right\|_{[x,a+b-x],1}^{\beta} + \left\|f'\right\|_{[a+b-x,b],1}^{\beta}\right]^{\frac{1}{\beta}} \\ \qquad\qquad if\ \ \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \left[\frac{x + \frac{b-3a}{2}}{b-a}\right] \max\left[\left\|f'\right\|_{[a,x],1}, \left\|f'\right\|_{[x,a+b-x],1}, \left\|f'\right\|_{[a+b-x,b],1}\right]. \end{cases}$$

The proof is as in Theorem 2 and we omit it.

REMARK 6. By the use of Theorem 3, for $x = a$, we get the trapezoid inequality (see for example [2, p.55])

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left\|f'\right\|_{[a,b],1}. \tag{2.16}$$

If in (2.15) we also choose $x = \frac{a+b}{2}$, then we get the mid point inequality (see for example [2, p.56])

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left\|f'\right\|_{[a,b],1}. \tag{2.17}$$

The following corollary also holds.

COROLLARY 3. *With the assumption in Theorem 3, one has the inequality:*

$$\left|\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{4}\left\|f'\right\|_{[a,b],1}. \tag{2.18}$$

3. A composite quadrature formula

We use the following inequalities obtained in the previous section:

$$(3.1)\quad \left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \begin{cases} \frac{1}{8}(b-a)\left\|f'\right\|_{[a,b],\infty} & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{[a,b],p} & \text{if } f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}\left\|f'\right\|_{[a,b],1} & \text{if } f'\in L_1[a,b]. \end{cases}$$

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $h_i := x_{i+1} - x_i$ $(i = 0,\ldots,n-1)$ and $\nu(I_n) := \max\{h_i | i = 0,\ldots,n-1\}$.

Consider the composite quadrature rule

$$(3.2)\qquad Q_n(I_n,f) := \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i+x_{i+1}}{4}\right)+f\left(\frac{x_i+3x_{i+1}}{4}\right)\right]h_i.$$

The following result holds.

THEOREM 5. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have*

$$(3.3)\qquad \int_a^b f(t)\,dt = Q_n(I_n,f) + R_n(I_n,f),$$

where $Q_n(I_n,f)$ is defined by formula (3.2), and the remainder satisfies the estimates

(3.4)

$$|R_n(I_n,f)| \leq \begin{cases} \frac{1}{8}\left\|f'\right\|_{[a,b],\infty}\sum_{i=0}^{n-1}h_i^2 & \textit{if } f'\in L_\infty[a,b]; \\ \frac{1}{4(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{[a,b],p}\left(\sum_{i=0}^{n-1}h_i^{q+1}\right)^{\frac{1}{q}} & \textit{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}\left\|f'\right\|_{[a,b],1}\nu(I_n). \end{cases}$$

Proof. Applying inequality (3.1) on the intervals $[x_i, x_{i+1}]$, we may state that

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right] h_i\right| \leq \begin{cases} \frac{1}{8} h_i^2 \|f'\|_{[x_i,x_{i+1}],\infty}; \\ \frac{1}{4(q+1)^{\frac{1}{q}}} h_i^{1+\frac{1}{q}} \|f'\|_{[x_i,x_{i+1}],p}, & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} h_i \|f'\|_{[x_i,x_{i+1}],1}; \end{cases} \tag{3.5}$$

for each $i \in \{0, \ldots, n-1\}$.

Summing the inequality (3.5) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$|R_n(I_n, f)| \leq \begin{cases} \frac{1}{8} \sum_{i=0}^{n-1} h_i^2 \|f'\|_{[x_i,x_{i+1}],\infty}; \\ \frac{1}{4(q+1)^{\frac{1}{q}}} \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \|f'\|_{[x_i,x_{i+1}],p}, & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} \sum_{i=0}^{n-1} h_i \|f'\|_{[x_i,x_{i+1}],1}. \end{cases} \tag{3.6}$$

Now, we observe that

$$\sum_{i=0}^{n-1} h_i^2 \|f'\|_{[x_i,x_{i+1}],\infty} \leq \|f'\|_{[a,b],\infty} \sum_{i=0}^{n-1} h_i^2.$$

Using Hölder's discrete inequality, we may write that

$$\begin{aligned}\sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \left\|f'\right\|_{[x_i,x_{i+1}],p} &\le \left(\sum_{i=0}^{n-1} h_i^{\left(1+\frac{1}{q}\right)q}\right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \left\|f'\right\|_{[x_i,x_{i+1}],p}^{p} dt\right)^{\frac{1}{p}} \\ &= \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left|f'(t)\right|^p dt\right)^{\frac{1}{p}} \\ &= \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}} \left\|f'\right\|_{[a,b],p}.\end{aligned}$$

Also, we note that

$$\begin{aligned}\sum_{i=0}^{n-1} h_i \left\|f'\right\|_{[x_i,x_{i+1}],1} &\le \max_{0\le i\le n-1}\{h_i\} \sum_{i=0}^{n-1} \left\|f'\right\|_{[x_i,x_{i+1}],1} \\ &= \nu(I_n) \left\|f'\right\|_{[a,b],1}.\end{aligned}$$

Consequently, by the use of (3.6), we deduce the desired result (3.4). □

For the particular case where the division I_n is equidistant, i.e.,

$$I_n : x_i = a + i \cdot \frac{b-a}{n}, \quad i = 0, \dots, n,$$

we may consider the quadrature rule:

$$\begin{aligned}(3.7) \quad Q_n(f) := \frac{b-a}{2n} \sum_{i=0}^{n-1} \Bigg\{ & f\left[a + \left(\frac{4i+1}{4n}\right)(b-a)\right] \\ & + f\left[a + \left(\frac{4i+3}{4n}\right)(b-a)\right]\Bigg\}.\end{aligned}$$

The following corollary will be more useful in practice.

COROLLARY 4. *With the assumption of Theorem 5, we have*

$$\int_a^b f(t)\,dt = Q_n(f) + R_n(f), \tag{3.8}$$

where $Q_n(f)$ is defined by (3.7) and the remainder $R_n(f)$ satisfies the estimate:

$$
|R_n(I_n, f)| \leq \begin{cases} \frac{1}{8}\|f'\|_{[a,b],\infty} \frac{(b-a)^2}{n}; \\ \\ \frac{1}{4(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],p} \cdot \frac{(b-a)^{1+\frac{1}{q}}}{n}; \\ \\ \frac{1}{4}\|f'\|_{[a,b],1} \cdot \frac{(b-a)}{n}. \end{cases} \tag{3.9}
$$

4. Applications for P.D.F.'s

Summarizing some of the results in Section 2, we may state that for $f : [a,b] \to \mathbb{R}$ an absolutely continuous function, we have the inequality

$$
\left| \frac{1}{2}[g(x) + g(a+b-x)] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{4.1}
$$

$$
\leq \begin{cases} \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|g'\|_{[a,b],\infty} & \text{if } g' \in L_\infty[a,b]; \\ \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{\frac{a+b}{2} - x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|g'\|_{[a,b],p}, \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \text{ and } g' \in L_p[a,b]; \\ \left[\frac{1}{4} + \left|\frac{x - \frac{3a+b}{4}}{b-a}\right|\right]\|g'\|_{[a,b],1}, \end{cases}
$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f : [a,b] \to [0,\infty)$ and with the cumulative distribution function $F(x) = \Pr(X \leq x) = \int_a^x f(t)\,dt$.

The following result holds.

THEOREM 6. *With the above assumptions, we have the inequality*

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{b-E(X)}{b-a}\right| \tag{4.2}$$

$$\leq \begin{cases} \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f\|_{[a,b],\infty} & \text{if } f\in L_\infty[a,b]; \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f\|_{[a,b],p}, & \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \text{ and } f\in L_p[a,b]; & \\ \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right], & \end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, *where* $E(X)$ *is the expectation of* X.

Proof. Follows by (4.1) on choosing $g=F$ and taking into account that

$$E(X)=\int_a^b t\,dF(t)=b-\int_a^b F(t)\,dt. \qquad \square$$

In particular, we have:

COROLLARY 5. *With the above assumptions, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right| \tag{4.3}$$

$$\leq \begin{cases} \frac{1}{8}(b-a)\|f\|_{[a,b],\infty} & \text{if } f\in L_\infty[a,b]; \\ \frac{1}{4}\cdot\frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|f\|_{[a,b],p}, & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \text{ and } f\in L_p[a,b]; \\ \frac{1}{4}. & \end{cases}$$

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