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COMPARISON FOR SOLUTIONS OF A SPDE DRIVEN BY MARTINGALE MEASURE

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ABSTRACT. We derive a comparison theorem for solutions of the following stochastic partial differential equations in a Hilbert space H.

$$Lu^i = \alpha(u^i)\dot{M}(t,x) + \beta^i(u^i), \text{ for } i = 1, 2,$$

where $Lu^i = \frac{\partial u^i}{\partial t} - Au^i$, A is a linear closed operator on H and $\dot{M}(t,x)$ is a spatially homogeneous Gaussian noise with covariance of a certain form. We are going to show that if $\beta^1 \leq \beta^2$ then $u^1 \leq u^2$ under some conditions.

1. Introduction

We want to derive a comparison theorem for solutions of stochastic partial differential equations(SPDEs) driven by martingale measures. Let $\dot{M}$ be a Gaussian noise, typically white in time but possibly with some spatial correlation. Following the same approach as in Dalang[2], $M_t(B) \equiv M([0,t] \times B)$ is a worthy martingale measure with covariance measure defined by

$$Q([0,t]\times A\times B) = \langle M(A), M(B)\rangle_t = t\int_{R^d} dx \int_{R^d} dy 1_A(x)f(x-y)1_B(y),$$

for some function f.

Let U be a bounded open set in R^d. Assume that for $i = 1, 2,\ \alpha, \beta^i$ are globally Lipschitz with constant K, which implies $|\alpha(u)| \leq K(1 + |u|)$ and $|\beta^i(u)| \leq K(1 + |u|)$. We consider this extension of the martingale

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measure stochastic integral to a comparison problem for the solutions to the following SPDE:

$$Lu^i = \alpha(u^i)\dot{M}(t,x) + \beta^i(u^i), \text{ for } i = 1,2, \ t > 0, \ x \in U, \tag{1.1}$$

where $Lu^i = \frac{\partial u^i}{\partial t} - Au^i$, A is a linear closed operator which generates a strongly continuous semigroup on $H = L_2(U)$ and $\dot{M}(t,x)$ is a spatially homogeneous Gaussian noise with covariance of the form

$$E(\dot{M}(t,x)\dot{M}(s,y)) = \delta(t-s)f(x-y). \tag{1.2}$$

In this equation, $\delta(\cdot)$ denotes the Dirac-Delta function and $f : R^d \to R_+$ is continuous on $R^d - \{0\}$. The case $f(x) = \delta(x)$ would correspond to the case of space-time white noise.

$u^i(t,x)$ solves (1.1) in the following sense;

$$\begin{aligned} u^i(t,x) &= \int_U \Gamma(0, x-y)u_0^i(y)dy \\ &+ \int_0^t \int_U \Gamma(t-s, x-y)\beta^i(u^i(s,y))dyds \\ &+ \int_0^t \int_U \Gamma(t-s, x-y)\alpha(u^i(s,y))M(ds,dy), \end{aligned} \tag{1.3}$$

where $\Gamma(t,x)$ is the fundamental solution of $Lu = 0$ with the hypothesis mentioned later and the above stochastic integral is defined with respect to the martingale measure in the sense of Walsh[11]. Since we consider a solution of an SPDE at any point $(t,x) \in R_+ \times R^d$ as a continuous random field $u(t,x)$ this problem is related with a class of function valued SPDEs.

Dalang[2] showed that even though the integrand Γ may be a distribution under some conditions, the value of the stochastic integral is always an ordinary real valued random variable, and the stochastic integral process is a square-integrable martingale.

We are going to show that if $\beta^1 \leq \beta^2$, then $u^1 \leq u^2$ a.s. under some conditions. We follow Dalang's argument in [3] to assert the existence of solutions for (1.1), which is specially applied to the heat equation both linear and nonlinear, and parabolic equations. He found the condition on the covariance function f in (1.2) under which the stochastic integrals of Green functions are defined. It turns out that the condition is the same for both the heat and wave equations.

There is a lot of literature on comparison theorems of SPDEs. Specially for infinite dimension, you may refer to Assing, and Manthey[1], Geiß and Manthey[6], Kotelenez[8], Pardoux[10], and Kallianpur and Xiong[9]. The problem we have interest in this paper is motivated by the work of C. Donati-Martin and E. Pardoux[4] and Kotelenez[7]. Donati-Martin and Pardoux[4] proved a comparison theorem for white noise driven SPDEs using an Ito's formula and an approximation method. Kotelenez[7] proved a comparison theorem for solutions of SPDEs in a Hilbert space.

2. Preliminaries and assumptions

Let μ be a nonnegative tempered measure on R^d whose Fourier transform is f. The relationship between μ and f is, by the definition of Fourier transform of tempered distributions that for any test function ϕ

$$\int f(x)\phi(x)dx = \int \mathcal{F}\phi(\xi)\mu(d\xi),$$

$$\mathcal{F}\mu = \int \exp(-2i\pi\xi \cdot x)\mu(d\xi) = f(x). \tag{2.1}$$

Following the definition of Walsh[11], we consider a worthy martingale measure $M_t(B) \equiv M([0,t] \times B)$, $B \in \mathcal{B}(U)$ with covariance measure defined by

$$\begin{aligned} Q([0,t] \times A \times B) &= \langle M(A), M(B)\rangle_t \\ &= t\int_U dx \int_U dy 1_A(x) f(x-y) 1_B(y) \end{aligned}$$

and dominating measure $K \equiv Q$. By construction of Dalang[2], $t \longmapsto M_t(B)$ is a continuous martingale. We also denote that for any test function ϕ

$$M_t(\phi) \equiv \int_0^t \int_U \phi(x) M(dt, dx). \tag{2.2}$$

Then

$$\begin{aligned} E[\langle M_t(\phi)\rangle] &= E\left[\left(\int_0^t\int_U \phi(x)M(dt,dx)\right)^2\right] \\ &= \int_0^t\int_U \phi(x)f(x-y)\phi(y)dtdxdy \\ &= \int_0^t ds\int_U \mu(d\xi)|\mathcal{F}\phi(\xi)|^2, \end{aligned}$$

where $\mathcal{F}\phi$ is the Fourier transformation of ϕ.

HYPOTHESIS A. Let $S(t) = \Gamma(t,\cdot)$ is the fundamental solution of $Lu = 0$.

(1) $\Gamma(t,x)$ is a deterministic function with values in the space of non-negative functions with rapid decrease such that for any $T > 0$

$$\int_0^T\int_U \Gamma(t,x)^p dxdt \le C_T < \infty, \text{ for } p,\ 0 < p < 3.$$

(2) (i) $$\lim_{h\to 0}\int_0^T dt\int_U \mu(d\xi)\sup_{t<r<t+h}|\mathcal{F}S(r)(\xi) - \mathcal{F}S(t)(\xi)|^2 = 0.$$

(ii) $t \longmapsto \mathcal{F}S(t)(\xi)$ is continuous, for all $\xi \in U$.

(iii) there is $\epsilon > 0$ and a function $t \longmapsto k(t)$ with values in the space of non-negative functions with rapid decrease such that for all $t \ge 0$ and $h \in [0,\epsilon]$

$$|\mathcal{F}S(t+h)(\xi) - \mathcal{F}S(t)(\xi)| \le |\mathcal{F}k(t)(\xi)|, \text{ and}$$
$$\int_0^T dt\int_U \mu(d\xi)|\mathcal{F}k(t)(\xi)|^2 < \infty$$

(3) There exists a predictable process $h(x,s)$ satisfying

$$\langle M(A,t)\rangle \le \int_{A\times[0,t]} h(x,s)dx\,ds, \text{ for all } A \in \mathcal{B}(U),$$

and $\sup_{x\in\bar{U}}\sup_{0<s<T} h(x,s) < \infty$ a.s., where $\bar{U}$ is the closure of U.

REMARK. (1) It is known that (see [2]), there is an integer $p \geq 1$ such that

$$\int_{R^d} \frac{1}{(1+|x|^2)^p}\mu(dx) < \infty.$$

(2) You may refer several examples for $S(t)$ to Dalang's paper[2].

(3) Specifically in the case of heat equation or parabolic equation $\Gamma(t,x) = (2\pi t)^{-d/2}\exp(-|x|^2/2t)$ or $\Gamma(t,x) \leq C(t)^{-d/2}\exp(-C|x|^2/t)$, respectively.

THEOREM 2.1. (Theorem 13 in [2]) *If Hypothesis* A(2) *is satisfied and* $\alpha(\cdot)$ *and* $\beta^i(\cdot)\,(i = 1,2)$ *are Lipshitz functions, then* (1.1) *has a unique solution* $u^i(t,x)$.

Moreover, this solution is L^2*−continuous and for any* $T > 0$ *and* $p \geq 1$

$$\sup_{0\leq t\leq T}\sup_{x\in\bar{U}} E(|u^i(t,x)|^p) < \infty.$$

Let $H = L_2(U)$ and let N be a H-valued martingale process. We first consider the following stochastic evolution equations on H: for $i = 1,2$

$$du^i = (Au^i + \beta^i(u^i))dt + \alpha(u^i)dN, \quad u^i(0) = u_0^i \in H, \tag{2.3}$$

where $\alpha(u^i) = \alpha(u^i(\cdot))(u^i = u^i(\cdot) \in H)$ acts as a multiplication operator on H. Similarly, $\beta^i(u^i) = \beta^i(u^i(\cdot))$ i.e. $\beta^i(r)$ and $\alpha(r)$ are real valued functions of $r \in R$. $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the usual norm and the scalar product on H, respectively.

Kotelenez[7] derived a comparison theorem for the mild solutions of (2.3), i.e., for the integral solutions of the integral equations; for $i = 1,2$,

$$\begin{aligned} u^i(t,x) = &\int_U \Gamma(0,x-y)u_0^i(y)dy \\ &+ \int_0^t\int_U \Gamma(t-s,x-y)\beta^i(u^i(s,y))dyds \\ &+ \int_0^t \Gamma(t-s,x-\cdot)\alpha(u^i(s,\cdot))dN \end{aligned} \tag{2.4}$$

Existence, uniqueness, and smoothness for equations of type (2.3) have been studied by Walsh[11], Funaki[5], Pardoux[10], Geiß and Manthey[6], and Kotelenez[8].

Constants will be denoted by C or K with possible subindices and the same letter may denote different constants in the course of one proof.

Let $U(t,s)f \equiv U_{t-s}f = \int \Gamma(t-s,x-y)f(y)dy \in H$ and make some assumptions on (2.3) and (2.4).

HYPOTHESIS B.

B1) $U(t,s)$ is a positivity preserving and strongly continuous semigroup.

B2) u_0^i is $\mathcal{F}_0$-measurable and $E\|u_0^i\|^2 < \infty$, $i = 1, 2$.

B3) α, $\beta^i : H \to R$. For any $T > 0$ there is a finite constant K satisfying

$$|\beta^i(0)| = |\alpha(0)| = 0, \text{ for } i = 1, 2; \tag{2.5}$$

$$|\alpha(x) - \alpha(y)| + |\beta^i(x) - \beta^i(y)| \le K|x - y|, \tag{2.6}$$

for all $x, y \in R$, $0 \le t \le T$, and $i = 1, 2$.

It was shown in Kotelenez[8] that the above assumptions imply the existence of unique solutions u^i of (2.4), which are Markov processes. Since $U(t,s)$ is strongly continuous semigroup, taking $A_n \equiv n(U_{1/n} - I)$ and $U_n \equiv \exp(A_n)$, U_n is obviously positive preserving and the following (2.7) comes from Trotter's theorem: for any $T > 0$ and $h \in H$

$$\sup_{0 \le s \le t \le T} \|(U_n(t,s) - U(t,s))h\| \to 0 \text{ as } n \to \infty. \tag{2.7}$$

3. A comparison theorem

We rewrite (1.1) as the following SPDE:

$$\frac{\partial u(t,x)}{\partial t} = Au(t,x) + \alpha(u)\dot{M}(t,x) + \beta(u). \tag{3.1}$$

Let $\{e_k\}$ be an orthonormal basis of H and

$$M_t^k = \int_0^t \int_U e_k(x) M(ds, dx). \tag{3.2}$$

$\{M^k\}_{k=1}^{\infty}$ is a family of mutually independent martingale process. For $n \ge 1$, let B^m be the H-valued martingale process defined by

$$B_t^m = \sum_{k=1}^{m} M_t^k e_k. \tag{3.3}$$

We are going to take an approximation using a Kotelenez' result. We replace α and β^i, $i = 1, 2$ by smooth functions α_l and β_l^i, respectively(e.g. by cutting off and using mollifiers) with the following properties:

(3.4) (i) the j-th derivatives of α_l and β_l^i are bounded continuous for $j = 1, 2$;
(ii) $\beta_l^1(t,x) \le \beta_l^2(t,x)$ for all $(t,x) \in [0,\infty) \times U$;
(iii) $\beta_l^i \to \beta^i(t,x)$, $\alpha_l(t,x) \to \alpha(t,x)$ as $l \to \infty$ for all $(t,x) \in [0,T] \times U$, $i = 1, 2$.

We consider the following approximate SPDE's for $i = 1, 2$ with A_n described under Hypothesis B. We consider the following equations;

$$du_{n,l}^{m,i} = [A_n u_{n,l}^{m,i} + \beta_l^i(u_{n,l}^{m,i})]dt + \alpha_l(u_{n,l}^{m,i})\, dB_t^m, \tag{3.5}$$

$$du_n^{m,i} = [A_n u_n^{m,i} + \beta^i(u_n^{m,i})]dt + \alpha(u_n^{m,i})\, dB_t^m, \tag{3.6}$$

$$du^{m,i} = [A u^{m,i} + \beta^i(u^{m,i})]dt + \alpha(u^{m,i})\, dB_t^m. \tag{3.7}$$

Obviously, (3.5)–(3.7) have a unique solution which is a Markov process for all l and n including the limiting case. We adapt the following lemma in [7]. $u_{n,l}^{m,i}(\cdot,\cdot)$ denotes the solution of (3.5) with initial value u_0^i.

LEMMA 3.1(LEMMA 2.2 [7]). *Assume Hypotheses A, B and* (3.4) *in addition to* $u_0^1 \le u_0^2$ a.s.. *Then for any* t, $0 \le t \le T$

$$u_{n,l}^{m,1}(t,\cdot) \le u_{n,l}^{m,2}(t,\cdot) \text{ a.s..} \tag{3.8}$$

For $f \in H$, let $\|f\| = \int_U f(x)^2 dx$.

LEMMA 3.2. *Under the assumptions of Lemma 3.1, there exists a constant C_1 such that for a predictable process $f(s,\cdot)$ with*

$$E\left[\int_0^T \|f(s,\cdot)\|^2\, ds\right] < \infty,$$

one has

$$E\left[\sup_{t\le T}\left|\int_0^t f(s,\cdot)\, dB_s^m\right|^2\right] \le C_1 E\left[\int_0^T \|f(s,\cdot)\|^2\, ds\right], \tag{3.9}$$

where we consider $f(s,\cdot)$ as a multiplication operator.

Proof. It is well known that we can choose $\{e_k\}_{k=1}^{\infty}$ which is an orthonormal basis for H consisting of eigenfunctions for $\mathcal{F}$; namely $\mathcal{F}e_k = (-i)^k e_k$ and $\sup_k \sup_{x\in R^d} |e_k(x)| < \infty$.

Note that $B_t^m = \sum_{i=1}^m M_t^i e_i$. By the Burkholder's inequality for finite dimensional martingale, we have that

$$
\begin{aligned}
&E\left[\left|\int_0^t f(s,\cdot)dB_s^m\right|^2\right] \\
&\le C\sum_{i=1}^m E\left[\left|\int_0^T\int_U \langle f(s,\cdot),e_i\rangle e_i(x)M(ds,dx)\right|^2\right] \\
&= C\sum_{i=1}^m E\left[\int_0^T\int_U \langle f(s,\cdot),e_i\rangle^2|\mathcal{F}e_i(x)|^2\mu(dx)ds)\right] \\
&\le C_1\sum_{i=1}^m E\int_0^T \langle f(s,\cdot),e_i\rangle^2 ds \quad \text{by Hypothesis A(3)} \\
&\le C_1\int_0^T E[\|f(s,\cdot)\|^2]ds,
\end{aligned}
$$

for some constants C, C_1 where the last two inequalities come from using Fatou's lemma and Parseval's identity, respectively. □

THEOREM 3.3. *Under the assumptions of Lemma 3.1, for any t,*

$$u^1(t,x) \le u^2(t,x) \text{ a.s.}. \tag{3.10}$$

Proof. We divide this proof into three steps.

Step 1 $E|u_n^{m,i}(t,x) - u_{n,l}^{m,i}(t,x)|^2 \to 0$, as $l\to\infty$ for each t.

Let $p > 6$.

$$
\begin{aligned}
&E|u_n^{m,i}(t,x) - u_{n,l}^{m,i}(t,x)|^p \\
&\le C\left(E\left[\left|\int_0^t U^n(t-s)[\beta^i(u_n^{m,i}(s,\cdot)) - \beta_l^i(u_{n,l}^{m,i}(s,\cdot))]ds\right|^p\right]\right. \\
&\quad \left. + E\left[\left|\int_0^t U^n(t-s)[\alpha(u_n^{m,i}(s,\cdot)) - \alpha_l(u_{n,l}^{m,i}(s,\cdot))]dB_s^m\right|^p\right]\right) \\
&= I + II.
\end{aligned}
$$

(3.11)

$$\begin{aligned}
II &\leq C_1 E\left[\int_0^t \|U^n(t-s)[\alpha(u_n^{m,i})-\alpha_l(u_{n,l}^{m,i})]\|^2 ds\right]^{\frac{p}{2}}, \text{ by Lemma 3.2}\\
&\leq C_1 E\left[\int_0^t \|U(t-s)[\alpha(u_n^{m,i})-\alpha_l(u_{n,l}^{m,i})]\|^2 ds\right]^{\frac{p}{2}},\\
&\qquad\qquad \text{by Trotter's theorem}\\
&\leq C_1 E\left[\int_0^t\int_U (\Gamma_{t-s}^2(x-y)[\alpha(u_n^{m,i}(s,y))-\alpha_l(u_{n,l}^{m,i}(s,y))])^2 dyds\right]^{\frac{p}{2}}\\
&\leq C_2\left(\int_0^t\int_U \Gamma_{t-s}^{2q'}(x-y)dyds\right)^{\frac{1}{2q'}} E\int_0^t\int_U |\alpha(u_n^{m,i})-\alpha_l(u_{n,l}^{m,i})|^2 dyds\\
&\leq C_2 E\int_0^t \sup_{x\in\bar U}|u_n^{m,i}(s,\cdot)-u_{n,l}^{m,i}(s,\cdot)|^p ds,
\end{aligned}$$

where $q'=\frac{p/2}{p/2-1}$, $2q'<3$ and for some constants C_1 and C_2. Similarly,

$$\begin{aligned}
I &\leq C_1 E\int_0^t \|U^n(t-s)[\beta^i(u_n^{m,i})-\beta_l^i(u_{n,l}^{m,i})]\|^p ds\\
&\leq C_1 E\Bigg(\int_0^t \|U^n(t-s)[\beta^i(u_n^{m,i})-\beta_l^i(u_n^{m,i})]\|^p\\
&\qquad + \|U^n(t-s)[\beta_l^i(u_n^{m,i})-\beta_l^i(u_{n,l}^{m,i})]\|^p ds\Bigg) \qquad (3.12)\\
&\leq C_2 E\left(\int_0^t |\beta^i(u_n^{m,i})-\beta_l^i(u_n^{m,i})|^p + K|u_n^{m,i}-u_{n,l}^{m,i}|^p ds\right)\\
&= C_3\left(\phi_{n,l}^{m,i}+\int_0^t E|u_n^{m,i}-u_{n,l}^{m,i}|^p ds\right),
\end{aligned}$$

for some constants C_2, C_3 by the same way as the above, where

$$\phi_{n,l}^{m,i} = E\int_0^t |\beta^i(u_n^{m,i})-\beta_l^i(u_{n,l}^{m,i})|^p ds \to 0 \text{ as } l\to\infty$$

by the assumption (3.4) and the dominated convergence theorem. Hence the Gronwall's inequality implies that $E|u_n^{m,i}(t,\cdot)-u_{n,l}^{m,i}(t,\cdot)|^p\to 0$ as $l\to\infty$ and so $E|u_n^{m,i}(t,\cdot)-u_{n.l}^{m,i}(t,\cdot)|^2\to 0$ as $l\to\infty$.

Step 2 $E|u_n^{m,i}(t,\cdot)-u^{m,i}(t,\cdot)|^2 \to 0$, as $n\to\infty$ for each t.

$$
\begin{aligned}
&E|u_n^{m,i}(t,\cdot)-u^{m,i}(t,\cdot)|^p \\
&\leq C\left(E\left|\int_0^t [U(t-s)\beta^i(u^{m,i}(s,\cdot))-U^n(t-s)\beta^i(u_n^{m,i}(s,\cdot))]ds\right|^p\right. \\
&\quad \left.+E\left|\int_0^t [U(t-s)\alpha(u^{m,i}(s,\cdot))-U^n(t-s)\alpha(u_n^{m,i}(s,\cdot))]dB^m(s)\right|^p\right) \\
&=I+II.
\end{aligned}
$$

$$
\begin{aligned}
I &\leq C\left(\int_0^t E|(U-U^n)(t-s)\beta^i(u^{m,i}(s,\cdot))|^2 ds\right)^{\frac{p}{2}} \\
&\quad +\left(\int_0^t E|U^n(t-s)(\beta^i(u^{m,i}(s,\cdot))-\beta^i(u_n^{m,i}(s,\cdot))|^2 ds\right)^{\frac{p}{2}} \\
&\leq C\left(\int_0^t E|(U-U^n)(t-s)\beta^i(u^{m,i}(s,\cdot))|^2 ds\right)^{\frac{p}{2}} \\
&\quad +C_2\left(\int_0^t E|u^{m,i}(s,\cdot)-u_n^{m,i}(s,\cdot)|^p ds\right),
\end{aligned} \tag{3.13}
$$

by the same way as (3.11).

Also,

$$
\begin{aligned}
II &\leq E\left[\left|\int_0^t U_{t-s}(\alpha(u^{m,i}(s,\cdot))-\alpha(u_n^{m,i}(s,\cdot)))dB_s^m\right|^p\right. \\
&\quad \left.+\left|\int_0^t (U_{t-s}-U_{t-s}^n)\alpha(u_n^{m,i}(s,\cdot))dB_s^m\right|^p\right] \\
&\leq E\left(\int_0^t \|U_{t-s}(\alpha(u^{m,i}(s,\cdot))-\alpha(u_n^{m,i}(s,\cdot)))\|^2 ds\right)^{\frac{p}{2}} \\
&\quad +\left(\int_0^t \|(U_{t-s}-U_{t-s}^n)\alpha(u_n^{m,i}(s,\cdot))\|^2 ds\right)^{\frac{p}{2}}, \quad \text{by Lemma 3.2} \\
&\leq C_2 E\int_0^t |u^{m,i}(s,\cdot)-u_n^{m,i}(s,\cdot)|^p ds+(\phi_n^{m,i})^{\frac{p}{2}},
\end{aligned}
$$

where $\phi_n^{m,i} = E\int_0^t \|(U(t-s) - U^n(t-s)\alpha(u_n^{m,i}(s,\cdot))\|ds \to 0$ as $n \to \infty$.
Since the first term of (3.13) goes to 0 by (2.7), applying the Gronwall's inequality again, $E|u_n^{m,i}(t,\cdot) - u^{m,i}(t,\cdot)|^2 \to 0$ as $n \to \infty$ for each t.

Step 3 $E|u^{m,i}(t,\cdot) - u^i(t,\cdot)|^2 \to 0$, as $m \to \infty$ for each t.

$$
\begin{aligned}
&E|u^{m,i}(t,x) - u^i(t,x)|^p \\
&= E\left|\int_0^t U(t-s)\beta(u^{m,i}(s,\cdot)) - U(t-s)\beta(u^i(s,\cdot))\,ds\right|^p \\
&\quad + E\left|\int_0^t U(t-s)\alpha(u^{m,i}(s,\cdot))\,dB_s^m \right. \\
&\qquad\qquad \left. - \int_0^t\int_U U(t-s)\alpha(u^i(s,y))M(ds,dy)\right|^p \\
&\le I + II.
\end{aligned}
$$

$$
I \le C_2 \int_0^t E|u^{m,i}(s,\cdot) - u^i(s,\cdot)|^p ds,
$$

by the same way as (3.11).
To estimate II, consider

$$
\begin{aligned}
&\int_0^t U(t-s)\alpha(u^{m,i}(s,\cdot))dB_s^m - \int_0^t\int_U U(t-s)\alpha(u^i(s,\cdot))M(ds,dy) \\
&= \sum_{k=1}^m \int_0^t \left(\int_U [\alpha(u^i(s,y)) - \alpha(u^{m,i}(s,y))]\,\Gamma_{t-s}(x-y)e_k(y)\,dy\right) dM_s^k \\
&\quad + \int_0^t\int_U (\Psi_{t,x}^i(s,y) - \Psi_{t,x}^{m,i}(s,y))M(ds,dy) \\
&= I_m(t,x) + II_m(t,x),
\end{aligned}
$$

where

$$
\begin{aligned}
&\Psi_{t,x}^i(s,y) = \alpha(u^i(s,y))\Gamma_{t-s}(x-y), \quad \text{and} \\
&\Psi_{t,x}^{m,i}(s,y) = \sum_{k=1}^m \left(\int_U \alpha(u^{m,i}(s,z))\,\Gamma_{t-s}(x-z)\,e_k(z)\,dz\right) e_k(y).
\end{aligned}
$$

$$
\begin{aligned}
&E[|I_m(t,x)|^p] \\
&\le CE\left[\int_0^t \sum_{k=1}^m \left(\int_U [\alpha(u^i(s,y)) - \alpha(u^{m,i}(s,y))]\right.\right. \\
&\qquad\qquad \left.\left. \times\ \Gamma_{t-s}(x-y)e_k(y)dy\right)^2 d\langle M^k\rangle_s\right]^{\frac{p}{2}} \\
&\le C_1 E\left[\int_0^t \sum_{k=1}^m \left(\int_U [\alpha(u^i(s,y)) - \alpha(u^{m,i}(s,y))]\right.\right. \\
&\qquad\qquad \left.\left. \times\ \Gamma_{t-s}(x-y)e_k(y)dy\right)^2 ds\right]^{\frac{p}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=1}^m \left\{\int_U \alpha(u^i(s,y)) - \alpha(u^{m,i}(s,y))\, \Gamma_{t-s}(x-y)e_k(y)\,dy\right\}^2 \\
&= \sum_{k=1}^m \langle [\alpha(u^i(s,\cdot)) - \alpha(u^{m,i}(s,\cdot))]\, \Gamma_{t-s}(x-\cdot), e_k(\cdot)\rangle^2 \\
&\le \|[\alpha(u^i(s,\cdot)) - \alpha(u^{m,i}(s,\cdot))]\, \Gamma_{t-s}(x-\cdot)\|^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&E|I_m(t,x)|^p \\
&\le C_1 E\left(\int_0^t \int_U (\alpha(u^i(s,y)) - \alpha(u^{m,i}(s,y)))^2 \Gamma_{t-s}^2(x-y)dyds\right)^{\frac{p}{2}} \\
&\le C_2 E\int_0^t \sup_{y\in\bar{U}} |u^{m,i}(s,y) - u^i(s,y)|^p ds.
\end{aligned}
$$

And

$$
\begin{aligned}
&E\left[\int_U |II_m(t,x)|^p dx\right] \\
&= E\left[\left|\int_0^t \int_U (\Psi_{t,x}^i(s,y) - \Psi_{t,x}^{m,i}(s,y))M(ds,dy)\,dx\right|^2\right]^{\frac{p}{2}} \\
&= E\left[\int_0^t \int_U |\mathcal{F}(\Psi_{t,x}^i(s,y) - \Psi_{t,x}^{m,i}(s,y))|^2 \mu(dy)ds\right]^{\frac{p}{2}} \\
&\le CE\left[\int_0^t \int_U (\Psi_{t,x}^i(s,y) - \Psi_{t,x}^{m,i}(s,y))^2 dyds\right]^{\frac{p}{2}} \to 0,
\end{aligned}
$$

as $m \to \infty$ by the following (3.14)–(3.16) and the Dominated convergence theorem.

Note that

$$\int_U (\Psi^i_{t,x}(s,y) - \Psi^{m,i}_{t,x}(s,y))^2 dy = |\Psi^i_{t,x}(s,\cdot) - \Psi^{m,i}_{t,x}(s,\cdot)|^2 \downarrow 0, \tag{3.14}$$

a.s. as $m \to \infty$,

$$|\Psi^i_{t,x}(s,\cdot) - \Psi^{m,i}_{t,x}(s,\cdot)|^2 \le |\Psi^i_{t,x}(s,\cdot)|^2, \tag{3.15}$$

and almost surely,

(3.16)

$$\begin{aligned}
&E\left(\int_0^t \|\Psi^i_{t,x}(s,\cdot)\|^2 ds\right)^{\frac{p}{2}} \\
&= E\left(\int_0^t \int_U (\alpha(u^i(s,\cdot))\Gamma_{t-s}(x-y))^2 dyds\right)^{\frac{p}{2}} \\
&\le C\left(\int_0^t \int_U \Gamma^{2q'}_{t-s}(x-y) dyds\right)^{\frac{1}{2q'}} E\int_0^t \int_U |\alpha(u^i(s,\cdot)|^p dyds \\
&\le C_1 E\int_0^t \sup_{y\in\bar{U}} |(u^i(s,y)|^p ds \\
&< \infty,
\end{aligned}$$

where $q' = \frac{p/2}{p/2-1}$ and for some constants C and C_1.

By the above three steps, we get

$$E|u^{m,i}_{n,l}(t,\cdot) - u^i(t,\cdot)|^2 \to 0 \text{ as } l \to \infty,\ n \to \infty, \text{ and } m \to \infty.$$

From Lemma 3.1, for all $(t,x) \in [0,T] \times U$, $u^{m,1}_{n,l}(t,x) \le u^{m,2}_{n,l}(t,x)$ a.s.. Hence for all $(t,x) \in [0,T] \times U$

$$u^1(t,x) \le u^2(t,x) \quad \text{a.s..}$$ □

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