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# EXISTENCE OF EQUILIBRIA VIA CONTINUOUS SELECTIONS

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ABSTRACT. The aim of this paper is to give three equilibrium existence theorems for generalized games with lower semicontinuous constraint and preference multimaps by using Michael's continuous selection theorem.

## 1. Introduction

The theory of continuous selections of multimaps is an intensively developing research area in recent decades, and there have been numerous applications in fixed point theory, convex analysis, game theory and other diverse branches of modern mathematics. Among a number of continuous selection theorems, Michael's selection theorem[8] is well-known and very basic in many applications.

In the last five decades, the classical Arrow-Debreu result[1] on the existence of Walrasian equilibria has been generalized in many directions. Mas-Colell[7] has first shown that the existence of equilibrium can be established without assuming preferences to be total or transitive. Next, by using a maximal element existence theorem, Gale and Mas-Colell[5] gave a proof of the existence of a competitive equilibrium without ordered preferences. Using Kakutani's fixed point theorem, Shafer and Sonnenschein[9] proved the powerful result on 'the Arrow-Debreu lemma for abstract economies' for the case where preferences may not be total or transitive but has open graph. As we have seen in the literature including Borglin-Keiding[3], Gale-Mas-Colell[5], Shafer-Sonnenschein[9], Yannelis-Prabhakar[10], in most results on the existence of equilibria for

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abstract economies, constraint and preference multimaps are assumed to have strong open lower sections or open graphs, and also constraint multimaps are assumed to be upper semicontinuous. Here, it should be noted that we will encounter many kinds of constraints and preferences in various economic situations (e.g., see Aubin[2]); so it is important that we shall consider several types of constraints and preferences, and obtain some existence results for such multimaps in general settings.

The main purpose of this paper is to give three new equilibrium existence theorems for generalized games with lower semicontinuous constraint and preference multimaps. First, we shall prove a maximal element existence lemma by using Michael's continuous selection theorem, and obtain a fixed point theorem as a corollary. Next, we shall prove three equilibrium existence theorems for generalized games with lower semicontinuous constraint and preference multimaps without assuming upper semicontinuity nor open lower sections. Hence our results are comparable to the previous equilibrium existence results due to Shafer-Sonnenschein[9], Borglin-Keiding[3], Gale-Mas-Colell[5] and Yannelis-Prabhakar[10].

## 2. Preliminaries

Let $A$ be a subset of a topological space $X$. We shall denote by $2^A$ the family of all subsets of $A$ and by $cl\,A$ the closure of $A$ in $X$. If $A$ is a subset of a vector space, we shall denote by $co\,A$ the convex hull of $A$. If $A$ is a non-empty subset of a topological vector space $X$ and $S, T : A \to 2^X$ are multimaps (or correspondences), then $co\,T$, $cl\,T$, $\overline{co}\,T$, $T \cap S : A \to 2^X$ are multimaps defined by $(co\,T)(x) = co\,T(x)$, $(cl\,T)(x) = cl\,T(x)$, $(\overline{co}\,T)(x) = cl(co\,T(x))$, and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

Let $X, Y$ be non-empty topological spaces and $T : X \to 2^Y$ be a multimap. A multimap $T : X \to 2^Y$ is said to be *upper semicontinuous* (in $X$) if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \subset V$ for each $y \in U$; and a multimap $T : X \to 2^Y$ is said to be *lower semicontinuous* (in $X$) if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. It is also known that $T : X \to 2^Y$ is lower semicontinuous in $X$ if and only if for each closed set $V$ in $Y$, the set $\{x \in X \mid T(x) \subset V\}$ is closed in $X$.

Here we note that by the definition, $T$ is automatically lower semicontinuous at every $x$ where $T(x) = \emptyset$, and also note that if $T$ has open lower sections (i.e., $T^{-1}(y)$ is open for each $y \in Y$), then $T$ is lower semicontinuous, e.g., see Yannelis-Prabhakar ([10, Proposition 4.1]).

Let $X$ and $Y$ be non-empty topological spaces, and $A$ be a non-empty subset of $X$. Let $T : A \to 2^Y$ be lower semicontinuous in $A$, and let $T_1 : X \to 2^Y$ be a multimap defined by

$$T_1(x) = \begin{cases} T(x) & \text{if} \quad x \in A, \\ \emptyset & \text{if} \quad x \notin A. \end{cases}$$

If $A$ is open in $X$, then it is easy to see that $T_1$ is a lower semicontinuous multimap in $X$. However, if $A$ is closed in $X$, then $T_1$ is not necessarily a lower semicontinuous multimap in $X$. For example, let $T : [0,1] \to 2^{[0,2]}$ be defined by $T(x) := [2-x, 2]$ for each $x \in [0,1]$; and let $T_1 : [0,2] \to 2^{[0,2]}$ be defined by

$$T_1(x) = \begin{cases} [2-x, 2] & \text{if} \quad x \in [0,1], \\ \emptyset & \text{if} \quad x \in (1,2]. \end{cases}$$

Then, it is clear that $T$ is lower semicontinuous in $[0,1]$, however $T_1$ is not lower semicontinuous at 1.

Finally we recall the following definition of equilibrium theory in mathematical economics due to Shafer-Sonnenschein[9] or Borglin-Keiding[3]. Let $I$ be a finite or an infinite set of agents. For each $i \in I$, let $X_i$ be a non-empty set of actions. A *generalized game* (or an *abstract economy*) $\Gamma = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples $(X_i, A_i, P_i)$ where $X_i$ is a non-empty topological vector space (a choice set), $A_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a constraint multimap and $P_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a preference multimap. An *equilibrium* for $\Gamma$ is a point $\hat{x} \in X = \Pi_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in cl\, A_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. In particular, when $I = \{1, \cdots, n\}$, we may call $\Gamma$ an *n-person game.*

In mathematical models of economies, the value $A_i(x)$ of constraint multimap is usually consumer's budget set, and the value $P_i(x)$ of preference multimap is usually consumer's preferences for commodities. In the theory of game, the value $A_i(x)$ is player's possible actions or strategies, and the value $P_i(x)$ is player's preferences over strategy vectors. Hence the equilibrium choice or action means the best choice of maximal preference under his/her budget restriction for every consumer simultaneously, or the strategy vector in which no player, acting alone, can benefit from changing his/her strategy choice.

The following continuous selection theorem due to Michael[8] is essential in proving our main result :

LEMMA 1. *Let $X$ be a non-empty paracompact Hausdorff topological space and $Y$ be a Banach space. Let $T : X \to 2^Y$ be a lower semicontinuous multimap such that each $T(x)$ is non-empty closed convex in $Y$. Then $T$ has a continuous selection, i.e. there exists a continuous map $f : X \to Y$ such that $f(x) \in T(x)$ for each $x \in X$.*

Now we shall prove the following

LEMMA 2. *Let $X$ be a non-empty compact convex subset of a Banach space, and let $T : X \to 2^X$ be a lower semicontinuous multimap such that $x \notin \overline{co}\,T(x)$ for each $x \in X$. Then there exists a maximal element $\bar{x} \in X$ for $T$, i.e., $T(\bar{x}) = \emptyset$.*

*Proof.* Suppose the contrary, then $T(x)$ is non-empty for each $x \in X$. Consider the multimap $\overline{co}\,T : X \to 2^X$ defined by

$$(\overline{co}\,T)(x) := cl\left(co\,T(x)\right), \text{ for each } x \in X.$$

Then, by Propositions 7.3.3 and 7.3.17 in Klein-Thompson [6], $\overline{co}\ T$ is also lower semicontinuous in $X$ and each $\overline{co}\ T(x)$ is a non-empty closed convex subset of $X$. By Lemma 1, there exists a continuous map $f : X \to X$ such that $f(x) \in \overline{co}\ T(x)$ for each $x \in X$. Then, by Schauder's fixed point theorem, $f$ has a fixed point $\hat{x} \in X$ such that $\hat{x} = f(\hat{x}) \in \overline{co}\ T(\hat{x})$, which contradicts the assumption. Therefore, $T$ has a maximal element. □

REMARK. Lemma 2 is different from the previous many maximal element existence theorems as we mentioned before. In fact, in Lemma 2, we do not require the upper semicontinuity assumption on $T$ nor the open lower section assumption on $T$, but we do need the continuous selection property for $T$ so that $X$ is assumed to be a compact convex subset of a Banach space.

We can restate Lemma 2 as the following fixed point theorem.

COROLLARY 1. *Let $X$ be a non-empty compact convex subset of a Banach space, and let $T : X \to 2^X$ be a lower semicontinuous multimap such that $T(x)$ is non-empty for each $x \in X$. Then there exists a point $\bar{x} \in X$ such that $\bar{x} \in \overline{co}\, T(\bar{x})$.*

## 3. Existence of equilibrium for generalized games

We begin with the following new equilibrium existence theorem :

THEOREM 1. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a generalized game where $I$ be a finite set of agents such that for each $i \in I$,*

(1) *$X_i$ is a non-empty compact convex subset of a Banach space, and denote $X = \Pi_{i \in I} X_i$ ;*
(2) *the multimap $A_i : X \to 2^{X_i}$ is lower semicontinuous in $X$ such that $A_i(x)$ is a non-empty convex subset of $X_i$ for each $x \in X$;*
(3) *the multimap $A_i \cap P_i$ is lower semicontinuous in $W_i$ such that $(A_i \cap P_i)(x)$ is (possibly empty) convex for each $x \in X$ ;*
(4) *the set $W_i := \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\}$ is (possibly empty) closed in $X$;*
(5) *for each $x \in W_i$, $x_i \notin cl\, P_i(x)$.*

*Then there exists an equilibrium point $\bar{x} \in X$ for $\Gamma$, i.e., for each $i \in I$,*

$$\bar{x}_i \in cl\, A_i(\bar{x}) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset.$$

*Proof.* For each $i \in I$, we first define a multimap $\phi_i : X \to 2^{X_i}$ by

$$\phi_i(x) = \begin{cases} cl\, A_i(x) & \text{if} \quad x \notin W_i, \\ cl\, (A_i \cap P_i)(x) & \text{if} \quad x \in W_i. \end{cases}$$

Then for each $x \in X$, $\phi_i(x)$ is a non-empty closed convex subset of $X_i$. Since $W_i$ is closed, $\phi_i$ is a lower semicontinuous multimap in $X$. In fact, for every closed subset $V$ of $X_i$, we have

$$\begin{aligned} U &:= \{x \in X \mid \phi_i(x) \subset V\} \\ &= \{x \in W_i \mid \phi_i(x) \subset V\} \cup \{x \in X \setminus W_i \mid \phi_i(x) \subset V\} \\ &= \{x \in W_i \mid cl\, (A_i \cap P_i)(x) \subset V\} \cup \{x \in X \setminus W_i \mid cl\, A_i(x) \subset V\} \\ &= \{x \in W_i \mid cl\, (A_i \cap P_i)(x) \subset V\} \cup \{x \in X \mid cl\, A_i(x) \subset V\}. \end{aligned}$$

Since $W_i$ is closed and $cl\, (A_i \cap P_i)$ is lower semicontinuous in $W_i$, the set $\{x \in W_i \mid cl\, (A_i \cap P_i)(x) \subset V\}$ is closed in $X$. Since $cl\, A_i$ is lower semicontinuous in $X$, the set $\{x \in X \mid cl\, A_i(x) \subset V\}$ is closed in $X$, and hence we have $U$ is closed in $X$. Therefore, $\phi_i$ is lower semicontinuous in $X$.

We now define a multimap $\Phi : X \to 2^X$, by

$$\Phi(x) := \Pi_{i \in I}\ \phi_i(x) \quad \text{for each } x \in X.$$

Then, by Theorem 7.3.12 in Klein-Thompson[6], $\Phi$ is also lower semicontinuous such that each $\Phi(x)$ is non-empty closed convex. Note that $X = \Pi_{i \in I} X_i$ is a non-empty compact convex subset of a Banach space. Therefore, by applying Corollary 1 to $\Phi$, there exists a point $\bar{x} \in \Phi(\bar{x})$, i.e., for each $i \in I$, $\bar{x}_i \in \phi_i(\bar{x})$. If $\bar{x} \in W_i$ for some $i \in I$, then

$$\bar{x}_i \in \phi_i(\bar{x}) = cl\,(A_i \cap P_i)(\bar{x}) \subset cl\, P_i(\bar{x}),$$

which contradicts the assumption (5). Therefore, for each $i \in I$, $\bar{x} \notin W_i$, i.e., $\bar{x}_i \in \phi_i(\bar{x}) = cl\, A_i(\bar{x})$ and $(A_i \cap P_i)(\bar{x}) = \emptyset$. This completes the proof. □

REMARKS. (i) Theorem 1 is different from the previous many equilibrium existence theorems as we mentioned before. In fact, in Theorem 1, we do not require the upper semicontinuity of $A_i$ nor the open lower section assumption on $A_i \cap P_i$, but we only need the weaker lower semicontinuity assumptions on $A_i$ and $A_i \cap P_i$.

(ii) As remarked before, since the set $W_i$ is closed, $A_i \cap P_i$ is not needed to be lower semicontinuous in $X$. If $A_i \cap P_i$ is lower semicontinuous in $X$, then we have $W_i = \{x \in X \mid (A_i \cap P_i)(x) \neq \emptyset\} = \{x \in X \mid (A_i \cap P_i)(x) \cap X_i \neq \emptyset\}$ is an open subset of $X$. In this case, since $X$ is connected, the set $W_i$ might be empty-set or the whole space $X$.

(iii) In case the product space $X = \Pi_{i \in I} X_i$ is a non-empty compact convex subset of a Banach space (e.g., $X = \Pi_{n \in \mathbb{N}}[0, \frac{1}{n}]$ is a Hilbert cube), then we can apply Lemma 2 so that the index set $I$ is possibly infinite without affecting the conclusion.

In Theorem 1, the condition (3) is weaker than the corresponding open lower section or open graph assumptions. In fact, we can give a simple example of a generalized game with finite number of agents which Theorem 6.1 in Yannelis-Prabhakar[10], Theorem in Shafer-Sonneinschein[9] or Theorem 4 in Ding-Tan[4] cannot be applied:

EXAMPLE 1. Let $\Gamma = (X_k, A_k, P_k)_{k \in I}$ be a generalized game where for each $k \in I = \{1, 2, \cdots, n\}$, let $X_k = [0, 1]$ be a compact convex choice set, $X := \Pi_{k\in I} X_k$ (we simply denote it by ${X_k}^n$), and the multimaps $A_k, P_k : X \to 2^{X_k}$ be defined as follows :

for each $x = (x_1, x_2, \cdots, x_n) \in X$,

$$A_k(x) := \begin{cases} [0, x_k^k] & \text{if } x \in X \text{ with } \forall x_i \in [0, \frac{1}{2}), \\ \{0\} & \text{otherwise;} \end{cases}$$
$$P_k(x) := \begin{cases} \{0\} & \text{if } x \in X \text{ with } \forall x_i \in \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right], \\ \emptyset & \text{otherwise.} \end{cases}$$

Then it is easy to see that $W_k = [\frac{1}{2^k}, \frac{1}{2^{k-1}}]^n$ is non-empty closed in $X$ for each $k \in I$. Also, for each $x \in W_k$, $x_k \notin cl\, P_k(x) = \{0\}$. And it is easy to see that $A_k$ is not upper semicontinuous at $(\frac{1}{2}, \cdots, \frac{1}{2})$, but lower semicontinuous in $X$. Also, each $A_k \cap P_k$ is constant in $W_k$, so it is lower semicontinuous in $W_k$. Therefore, all hypotheses of Theorem 1 are satisfied so that there exists an equilibrium point $\bar{x} = (0, 0, \cdots, 0) \in X$ such that $\bar{x}_k \in A_k(\bar{x})$ and $A_k(\bar{x}) \cap P_k(\bar{x}) = \emptyset$ for each $k \in I$. Here we note that each $P_k$ does not have open graph nor open lower sections, and $W_k$ is not open, so that Corollary 3 in Borglin-Keiding[3], Theorem 6.1 in Yannelis-Prabhakar[10], Theorem in Shafer-Sonneinschein[9] or Theorem 4 in Ding-Tan[4] can not be applied to this game.

The next simple example of 1-person game shows that the lower semicontinuity of $A_k$ is essential in Theorem 1:

$$A(x) := \begin{cases} \{1\} & \text{for each } x = \frac{1}{2}, \\ \{\frac{1}{2}\} & \text{for each } x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]; \end{cases}$$
$$P(x) := \{1 - x\} \quad \text{for each } x \in [0, 1].$$

Then it is easy to see that $A$ is not lower semicontinuous at $\frac{1}{2}$. We can see that $W = \emptyset$ is closed, and hence all assumptions, except the assumption (2), of Theorem 1 are satisfied. We know that $A(x) \cap P(x) = \emptyset$ for each $x \in [0, 1]$, and $A$ has no fixed point in $[0, 1]$. Therefore, there can not exist an equilibrium for this game, and so the lower semicontinuity assumption on $A_i$ is essential in Theorem 1.

Next, using Lemma 2, we prove another equilibrium existence theorem for an 1-person game :

THEOREM 2. *Let $\Gamma = (X, A, P)$ be an 1-person game such that*

(1) *$X$ is a non-empty compact convex subset of a Banach space ;*

(2) *the multimap* $A : X \to 2^X$ *is lower semicontinuous in* $X$ *such that* $A(x)$ *is a non-empty convex subset of* $X$ *for each* $x \in X$;
(3) *the multimap* $A \cap P$ *is lower semicontinuous in* $\mathcal{F}$ *such that* $(A \cap P)(x)$ *is (possibly empty) convex for each* $x \in X$ ;
(4) *the set* $\mathcal{F} := \{x \in X \mid x \in cl\, A(x)\}$ *is closed in* $X$;
(5) *for each* $x \in \mathcal{F}$, $x \notin cl\, P(x)$.

*Then there exists an equilibrium point* $\bar{x} \in X$ *for* $\Gamma$, *i.e.,*

$$\bar{x} \in cl\, A(\bar{x}) \quad \text{and} \quad A(\bar{x}) \cap P(\bar{x}) = \emptyset.$$

*Proof.* We first define a multimap $\phi : X \to 2^X$ by

$$\phi(x) = \begin{cases} cl\, A(x) & \text{if} \quad x \notin \mathcal{F}, \\ cl\,(A \cap P)(x) & \text{if} \quad x \in \mathcal{F}. \end{cases}$$

Then for each $x \in X$, $\phi(x)$ is a (possibly empty) closed convex subset of $X$. Since $\mathcal{F}$ is closed, by repeating the same argument in the proof of Theorem 1, $\phi$ is a lower semicontinuous multimap in $X$ such that each $\phi(x)$ is closed convex. By the assumption (5), we have $x \notin \phi(x)$ for each $x \in X$. Therefore, the whole assumption of Lemma 2 are satisfied so that there exists a point $\bar{x} \in X$ such that $\phi(\bar{x}) = \emptyset$. Since each $A(x)$ is non-empty, we conclude that $\bar{x} \in \mathcal{F}$ and $(A \cap P)(\bar{x}) = \emptyset$. Therefore, $\bar{x} \in X$ is the desired equilibrium for this game. This completes the proof. □

In Theorem 2, if $A$ is upper semicontinuous, then $cl\, A$ is also upper semicontinuous so that the fixed point set $\mathcal{F}$ for $cl\, A$ is closed. Hence, the condition (4) is automatically satisfied.

Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a generalized game. Then we may call an agent $j \in I$ is *decisive* for the game $\Gamma$ if $(A_j \cap P_j)(x) = \emptyset$ implies $(A_i \cap P_i)(x) = \emptyset$ for each $i \in I$. The concept of a decisive agent for the game $\Gamma$ is meaningful for an incomplete market having monopolistic agent or an oligopolistic market in a real economy. Also, this concept is automatically satisfied for an 1-person game.

If there exists some decisive agent, we can obtain an equilibrium existence theorem for generalized game as follows:

THEOREM 3. *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be a generalized game where $I$ be a finite set of agents such that for each $i \in I$,*

(1) *$X_i$ is a non-empty compact convex subset of a Banach space, and denote $X = \Pi_{i\in I} X_i$ ;*
(2) *the multimap $A_i : X \to 2^{X_i}$ is lower semicontinuous in $X$ such that $A_i(x)$ is a non-empty convex subset of $X_i$ for each $x \in X$;*
(3) *the set $\mathcal{F}_i := \{x \in X \mid x_i \in cl\, A_i(x)\}$ is closed in $X$.*

*Assume further that there exists some decisive agent $j \in I$ such that*

(4) *the multimap $A_j \cap P_j$ is lower semicontinuous in $\mathcal{F}$ such that $(A_j \cap P_j)(x)$ is (possibly empty) convex for each $x \in X$ ;*
(5) *for each $x \in \mathcal{F} := \bigcap_{i\in I} \mathcal{F}_i$, $x_j \notin P_j(x)$.*

*Then there exists an equilibrium point $\bar{x} \in X$ for $\Gamma$, i.e., for each $i \in I$,*

$$\bar{x}_i \in cl\, A_i(\bar{x}) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset.$$

*Proof.* Since each $cl\, A_i$ is lower semicontinuous, by Theorem 7.3.12 in Klein-Thompson[6], the multimap $A : X \to 2^X$, defined by $A(x) := \Pi_{i\in I} cl\, A_i(x)$ for each $x \in X$, is lower semicontinuous such that each $A(x)$ is non-empty closed convex subset of a compact convex set $X = \Pi_{i\in I} X_i$ in a Banach space. By Corollary 1, there exists a fixed point $\bar{x} \in X$ such that $\bar{x} \in A(\bar{x})$, i.e., $\bar{x}_i \in A_i(\bar{x})$ for each $i \in I$. This means that $\bar{x} \in \mathcal{F}$. Since $\mathcal{F}_i$ is closed for each $i \in I$, $\mathcal{F}$ is closed and non-empty.

For the decisive agent $j \in I$, we now define $\Phi : X \to 2^X$, by

$$\Phi(x) := \begin{cases} \Pi_{i\in I} cl\, A_i(x) & \text{if } \ x \notin \mathcal{F}, \\ \Pi_{i\in I\setminus\{j\}} cl\, A_i(x) \times cl\,(A_j \cap P_j)(x) & \text{if } \ x \in \mathcal{F}; \end{cases}$$

where the product is in order.

Then, by the assumption (4) and Theorem 7.3.12 in Klein-Thompson [6], the multimap $\Pi_{i\in I\setminus\{j\}} cl\, A_i \times cl\,(A_j \cap P_j)$ is also lower semicontinuous in $\mathcal{F}$. Since $\mathcal{F}$ is closed, by repeating the argument of the proof in Theorem 1, we can show that $\Phi$ is also lower semicontinuous in $X$ such that each $\Phi(x)$ is non-empty closed convex. By the assumption (5), it is clear that $x \notin \Phi(x)$ for each $x \in X$. Therefore, the whole assumption of Lemma 2 are satisfied so that there exists a point $\bar{x} \in X$ such that $\Phi(\bar{x}) = \emptyset$. Since $\Pi_{i\in I} cl\, A_i(\bar{x})$ and $A_i(\bar{x})$ are non-empty, we must have $\bar{x} \in \mathcal{F}$ and $(A_j \cap P_j)(\bar{x}) = \emptyset$. Since $\bar{x} \in \mathcal{F}$, $\bar{x} \in \mathcal{F}_i$ for each $i \in I$ so that $\bar{x}_i \in cl\, A_i(\bar{x})$ for each $i \in I$. Also, since $j$ is a decisive agent for this game, we have $(A_i \cap P_i)(\bar{x}) = \emptyset$ for each $i \in I$. Therefore, $\bar{x} \in X$ is the desired equilibrium for this game. This completes the proof. $\square$

As we remarked before, if we assume further that each $A_i$ is upper semicontinuous, then the assumption (3) is automatically satisfied. Moreover, if the product space $X = \Pi_{i\in I} X_i$ is a non-empty compact convex subset of a Banach space, then the index set $I$ is possibly infinite without affecting the conclusion.

Finally, we give an example of a generalized game with finite number of agents where Theorem 3 is applicable but the previous known results can not work :

EXAMPLE 2. Let $\Gamma = (X_k, A_k, P_k)_{k\in I}$ be a generalized game where for each $k \in I = \{1, 2, \cdots, n\}$, let $X_k = [0, 2]$ be a compact convex choice set, $X := \Pi_{k\in I} X_k$, and the multimaps $A_k, P_k : X \to 2^{X_k}$ be defined as follows : for each $x = (x_1, x_2, \cdots, x_n) \in X$,

$$A_k(x) := \left[\frac{1}{2}, \frac{k+1}{2k}\right];$$

$$P_k(x) := \begin{cases} \left\{\dfrac{k+2}{2k}\right\} & \text{if } x \in \Pi_{i\in I}\left[0, \dfrac{i+1}{2i}\right), \\ \left[0, \dfrac{k+2}{2k}\right] & \text{if } x \in \Pi_{i\in I}\left(\dfrac{i+1}{2i}, 2\right], \\ \emptyset & \text{otherwise.} \end{cases}$$

Then it is easy to show that the multimap $A_k$ is lower semicontinuous on $X$ such that each $A_k(x)$ is non-empty closed convex, and the fixed point set $\mathcal{F}(A)$ of $A = \Pi_{k\in I} A_k$ is equal to the set $\Pi_{k\in I}[\frac{1}{2}, \frac{k+1}{2k}]$, which is compact convex. Also it is easy to check that for each $k \in I$, $A_k \cap P_k$ is lower semicontinuous at every point $x \in X$. Note that for each $x \in \mathcal{F}$, $x_k \in [\frac{1}{2}, \frac{k+1}{2k}]$ so that $x_k \notin P_k(x)$ since $P_k(x)$ is either $\{\frac{k+2}{2k}\}$ or an empty-set. And we know that the preference multimap $P_k$ is decreasing and so the first agent is decisive in this game. In fact, if $(A_1 \cap P_1)(x) = \emptyset$, then $(A_k \cap P_k)(x) = \emptyset$ for each $k \in I$. Hence, the assumptions (3)–(5) of Theorem 3 are satisfied for the first index $1 \in I$. Therefore, all assumptions of Theorem 3 are satisfied so that we can obtain an equilibrium point $\hat{x} = (1, \frac{3}{4}, \frac{5}{6}, \cdots, \frac{n+1}{2n}) \in X$ such that $\hat{x}_k \in A_k(\hat{x})$ and $A_k(\hat{x}) \cap P_k(\hat{x}) = \emptyset$ for each $k \in I$.

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