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TENSOR PRODUCTS OF LOG-HYPONORMAL OPERATORS

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ABSTRACT. The tensor product $A \otimes B$ of Hilbert space operators A and B will be shown to be log-hyponormal if and only if A and B are log-hyponormal. Some general comments about generalized hyponormality are also made.

1. Log hyponormality

Assume that $\mathcal{J} \subseteq \mathbb{R}$ is an interval and that $f : \mathcal{J} \longrightarrow \mathbb{R}$ is an operator monotone function. Consider a bounded linear operator T acting on a complex Hilbert space such that the spectrum of $|T|$ is contained in $\mathcal{J}$, where $|T|$ denotes $(T^*T)^{1/2}$, the positive square root of the operator T^*T. The operator T is said to be *f-hyponormal* if

$$f(TT^*) \leq f(T^*T).$$

That is, T is f-hyponormal if $f(T^*T) - f(TT^*)$ is a positive operator.

The most commonly occurring cases of f-hyponormality occur with the following two operator monotone functions:

(i) $\mathcal{J} = \mathbb{R}_0^+$ and $f(t) = t^p$ for a fixed $p \in (0, 1]$, which is the notion of p-hyponormality and which in the case $p = 1$ is simply the classical meaning of hyponormality;

(ii) $\mathcal{J} = \mathbb{R}^+$ and $f(t) = \log t$, which yields the notion of log-hyponormality.

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It is known from [3] and [7] that if A and B are operators acting on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively, then $A \otimes B$ is a p-hyponormal operator acting on the Hilbert space $\mathfrak{H} \otimes \mathfrak{K}$ if and only if A and B are p-hyponormal. The main purpose of this note is to point out that a similar result holds true for log-hyponormality. This is not immediately obvious because the log-function is not separable (that is, the log-function is not a function f for which $f(st) = f(s)f(t)$.)

Because log-hyponormality requires the *chaotic order* $\log H \geq \log K$ amongst invertible positive operators H and K, for our purposes it is easier to work with two known alternate expressions of this partial order.

LEMMA A. ([4, Theorem 3], [5, Theorem B]) *If H and K are invertible positive operators, then*

(1) $\log H \geq \log K$ *if and only if* $H^p \geq \left(H^{\frac{p}{2}} K^p H^{\frac{p}{2}}\right)^{\frac{1}{2}}$, *for every* $p \in \mathbb{R}_0^+$, *and*
(2) $\log H \geq \log K$ *if and only if for each* $\delta \in (0,1]$ *there exist* $\alpha, \varepsilon \in \mathbb{R}^+$ *such that*

$$(e^\delta H)^\alpha \geq K^\alpha + \varepsilon 1 .$$

THEOREM 1. *If A and B are invertible operators, then $A \otimes B$ is log-hyponormal if and only if A and B are log-hyponormal.*

Proof. Assume that A and B act on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively. We begin with two observations. The first is that $(A \otimes B)^*(A \otimes B) = (A^*A) \otimes (B^*B)$ and so, by the uniqueness of positive square roots, $|A \otimes B|^r = |A|^r \otimes |B|^r$ for any positive rational number r. From the density of the rationals in the reals, we obtain $|A \otimes B|^p = |A|^p \otimes |B|^p$ for every positive real number p. The second observation is that if $H_1 \geq H_2$ and $K_1 \geq K_2$, then $H_1 \otimes K_1 \geq H_2 \otimes K_2$ (see, for example, Proposition 2.2 of [7]).

Assume now that A and B are log-hyponormal operators. By Lemma A.1,

$$(A^*A)^p \geq \left((A^*A)^{\frac{p}{2}} (AA^*)^p (A^*A)^{\frac{p}{2}}\right)^{\frac{1}{2}}$$

and

$$(B^*B)^p \geq \left((B^*B)^{\frac{p}{2}} (BB^*)^p (B^*B)^{\frac{p}{2}}\right)^{\frac{1}{2}},$$

for all $p \in \mathbb{R}^+$. Therefore,

$$
\begin{aligned}
&((A\otimes B)^*(A\otimes B))^p\\
&= (A^*A)^p\otimes(B^*B)^p\\
&\geq \left((A^*A)^{\frac{p}{2}}(AA^*)^p(A^*A)^{\frac{p}{2}}\right)^{\frac{1}{2}}\otimes\left((B^*B)^{\frac{p}{2}}(BB^*)^p(B^*B)^{\frac{p}{2}}\right)^{\frac{1}{2}}\\
&= \left[((A\otimes B)^*(A\otimes B))^{\frac{p}{2}}\,((A\otimes B)(A\otimes B)^*)^p\,((A\otimes B)^*(A\otimes B))^{\frac{p}{2}}\right]^{\frac{1}{2}},
\end{aligned}
$$

which implies $A\otimes B$ is log-hyponormal.

Conversely, assume that $A\otimes B$ is log-hyponormal. We aim to show that A and B are log-hyponormal. Without loss of generality, it is enough to show that A is log-hyponormal. That is, by Lemma A.2, we aim to show that for each $\delta\in(0,1]$ there are $\alpha,\varepsilon\in\mathbb{R}^+$ such that $(e^\delta A^*A)^\alpha\geq(AA^*)^\alpha+\varepsilon 1$.

Thus, let $\delta\in(0,1]$ be arbitrary. Because $A\otimes B$ is log-hyponormal, there are (by Lemma A.2) $\alpha,\varepsilon_0\in\mathbb{R}^+$ such that

$$(1.1)\qquad (e^\delta\, A^*A\otimes B^*B)^\alpha-(AA^*\otimes BB^*)^\alpha\geq\varepsilon_0 1\,.$$

Let $\mathfrak{A}$ denote the unital C*-algebra generated by B and let $S(\mathfrak{A})$ be the state space of $\mathfrak{A}$, a convex and weak*-compact set. By continuous functional calculus, $(B^*B)^\alpha,(BB^*)^\alpha\in\mathfrak{A}$. The spectral equation $\sigma(ST)\cup\{0\}=\sigma(TS)\cup\{0\}$ (for arbitrary S and T) coupled with the invertibility of B demonstrate that $(B^*B)^\alpha$ and $(BB^*)^\alpha$ have the same spectrum. Hence, in passing the spectral radius, we have that $\|\,(B^*B)^\alpha\,\|=\|\,(BB^*)^\alpha\,\|$.

We claim that there is a state φ on $\mathfrak{A}$ for which $\varphi((B^*B)^\alpha)=\varphi((BB^*)^\alpha)$. Because $(B^*B)^\alpha-(BB^*)^\alpha$ is hermitian, the range of the map $S(\mathfrak{A})\to\mathbb{R}$ that sends each state φ to $\varphi((B^*B)^\alpha-(BB^*)^\alpha)$ is a closed interval $\mathcal{I}$. If $0\in\mathcal{I}$, then clearly the claim is proved. Assume, therefore, that $0\notin\mathcal{I}$. Without loss of generality we may assume that $\mathcal{I}$ contains only positive real numbers; let λ be the least element of $\mathcal{I}$. Then, for every unit vector $\xi\in\mathfrak{H}$, $\langle(B^*B)^\alpha\xi,\xi\rangle\leq\lambda+\langle(BB^*)^\alpha\xi,\xi\rangle$, implying that $\|\,(B^*B)^\alpha\,\|<\|\,(BB^*)^\alpha\,\|$, which is a contradiction. Thus, there is some state φ on $\mathfrak{A}$ for which $\varphi((B^*B)^\alpha)=\varphi((BB^*)^\alpha)$. Moreover, $\varphi((B^*B)^\alpha)$ is in the convex hull of the spectrum of the invertible positive operator $(B^*B)^\alpha$; hence, if $\beta=\varphi((B^*B)^\alpha)=\varphi((BB^*)^\alpha)$, then $\beta>0$.

Each unit vector $\xi \in \mathfrak{H}$ induces a state on $\mathcal{B}(\mathfrak{H}) \otimes \mathcal{B}(\mathfrak{K})$ denoted by $\omega_\xi \otimes \varphi$ and whose value on elementary tensors $X \otimes Y$ is $\langle X\xi, \xi\rangle\, \varphi(Y)$. Thus, from inequality (1.1), we have that

$$\begin{aligned}\varepsilon_0 &\leq \omega_\xi \otimes \varphi\left((e^\delta A^*A \otimes B^*B)^\alpha - (AA^* \otimes BB^*)^\alpha\right)\\ &= \langle (e^\delta A^*A)^\alpha \xi, \xi\rangle\, \varphi\left((B^*B)^\alpha\right) - \langle (AA^*)^\alpha \xi, \xi\rangle\, \varphi\left((BB^*)^\alpha\right)\\ &= \langle \left[(e^\delta A^*A)^\alpha - (AA^*)^\alpha\right] \xi, \xi\rangle\, \beta\,,\end{aligned}$$

which implies that $\varepsilon 1 \leq (e^\delta A^*A)^\alpha - (AA^*)^\alpha$, where $\varepsilon = \varepsilon_0/\beta$. □

Theorem 1 suggests the following open questions of interest.

QUESTION 1. *Which intervals $\mathcal{J} \subseteq \mathbb{R}_0^+$ and operator monotone functions $f : \mathcal{J} \to \mathbb{R}$ have the property that $A \otimes B$ is f-hyponormal if and only if A and B are f-hyponormal?*

QUESTION 2. *If $\mathcal{J} \subseteq \mathbb{R}_0^+$ contains the spectrum of $|A|$, and if A is f-hyponormal for every operator monotone function $f : \mathcal{J} \to \mathbb{R}$, then is A necessarily hyponormal? (The converse is trivially true.)*

2. Tensor products involving other notions of hyponormality

There are many generalizations of hyponormality, and most (or all?) appear to behave well when passing to tensor products. See the work of Duggal[3] and Hou[6] for specific examples. Below, we mention two additional examples (namely, w-hyponormality and p-quasihyponormality) that do not seem to be addressed by the present literature on operator tensor products.

Let $T = V|T|$ denote the polar decomposition of an operator T and set $\tilde{T} = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}}$. The operator T is said (see [1], [2]) to be

(i) *w-hyponormal* if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$, and

(ii) *iw-hyponormal* if T is invertible and $|T| \geq \left(|T|^{\frac{1}{2}} V |T|^{\frac{1}{2}} V^* |T|^{\frac{1}{2}}\right)^{\frac{1}{2}}$.

Every p-hyponormal operator is w-hyponormal[1]; every iw-hyponormal operator is w-hyponormal; and every invertible w-hyponormal operator is iw-hyponormal.

THEOREM 2. *A nonzero operator $A \otimes B$ is w-hyponormal if and only if A and B are w-hyponormal.*

Proof. Below we shall use the fact that the function $T \to \tilde{T}$ has the property $\widetilde{A \otimes B} = \tilde{A} \otimes \tilde{B}$.

It is clear that $\left|\widetilde{A \otimes B}\right| \geq |A \otimes B| \geq \left|(\widetilde{A \otimes B})^*\right|$ if and only if

$$\begin{aligned} &\left(|\tilde{A}| - |A|\right) \otimes |\tilde{B}| + |A| \otimes \left(|\tilde{B}| - |B|\right) \geq 0 \quad \text{and} \\ &\left(|A| - |\tilde{A}^*|\right) \otimes |B| + |\tilde{A}^*| \otimes \left(|B| - |\tilde{B}^*|\right) \geq 0, \end{aligned} \tag{2.1}$$

or, equivalently, if and only if

$$\begin{aligned} &\left(|\tilde{A}| - |A|\right) \otimes |B| + |\tilde{A}| \otimes \left(|\tilde{B}| - |B|\right) \geq 0 \quad \text{and} \\ &\left(|A| - |\tilde{A}^*|\right) \otimes |\tilde{B}^*| + |A| \otimes \left(|B| - |\tilde{B}^*|\right) \geq 0. \end{aligned} \tag{2.2}$$

So, the sufficiency is clear.

To prove the necessity, suppose that $A \otimes B$ is w-hyponormal. Then

$$|\tilde{A}| \otimes |\tilde{B}| \geq |A| \otimes |B| .$$

Therefore, by Proposition 2.2 of [7], there exists a $c \in \mathbb{R}^+$ such that

$$c\,|\tilde{A}| \geq |A| \quad \text{and} \quad c^{-1}|\tilde{B}| \geq |B|.$$

Consequently,

$$\|A\| = \|\,|A|\,\| \leq c\,\|\,|\tilde{A}|\,\| = c\,\|\tilde{A}\| \leq c\,\|A\|$$

and

$$\|B\| = \|\,|B|\,\| \leq c^{-1}\|\,|\tilde{B}|\,\| = c^{-1}\|\tilde{B}\| \leq c^{-1}\|B\|.$$

Thus, $c = 1$ and

$$|\tilde{A}| \geq |A| \quad \text{and} \quad |\tilde{B}| \geq |B|. \tag{2.3}$$

Now we just need to show that $|A| \geq |\tilde{A}^*|$ and $|B| \geq |\tilde{B}^*|$. Let $\xi \in \mathfrak{H}$ and $\eta \in \mathfrak{K}$ be arbitrary. Then, from (2.1) and (2.2), we have

$$\left\langle \left(|A| - |\tilde{A}^*|\right) \xi, \xi \right\rangle \langle |B|\eta, \eta\rangle + \left\langle |\tilde{A}^*|\xi, \xi \right\rangle \left\langle \left(|B| - |\tilde{B}^*|\right) \eta, \eta \right\rangle \geq 0 \tag{2.4}$$

and

$$(2.5)\quad \left\langle \left(|A| - |\tilde{A}^*|\right)\xi, \xi\right\rangle \left\langle |\tilde{B}^*|\eta, \eta\right\rangle + \langle |A|\xi, \xi\rangle \left\langle (|B| - |\tilde{B}^*|)\eta, \eta\right\rangle \geq 0\,.$$

Suppose that $|A| - |\tilde{A}^*|$ is not a positive operator. Then there is a $\xi_0 \in \mathfrak{H}$ such that

$$\left\langle \left(|A| - |\tilde{A}^*|\right)\xi_0, \xi_0\right\rangle = \alpha < 0 \quad \text{and} \quad \left\langle |\tilde{A}^*|\xi_0, \xi_0\right\rangle = \beta > 0.$$

From (2.4) we get

$$(\alpha + \beta)\|\,|B|\eta\,\| \geq \beta\,\|\,|\tilde{B}^*|\eta\,\|\,.$$

That is,

$$(\alpha + \beta)\|\,|B|\,\| \geq \beta\,\|\,|\tilde{B}^*|\,\|\,.$$

Since, by (2.3), $|\tilde{B}| \geq |B|$, we have also

$$(\alpha + \beta)\,\|B\| = (\alpha + \beta)\,\|\,|B|\,\| \geq \beta\,\|\,|\tilde{B}^*|\,\| = \beta\,\|\,|\tilde{B}|\,\| \geq \beta\,\|B\|\,.$$

This is a contradiction. Hence, $|A| \geq |\tilde{A}^*|$. A similar argument shows, by using (2.5), that $|B| \geq |\tilde{B}^*|$. □

COROLLARY. *If A and B are invertible operators, then $A \otimes B$ is iw-hyponormal if and only if A and B are iw-hyponormal.*

If $p \in (0, 1]$, then an operator T is *p-quasihyponormal* if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$. Again, one has the desired result about tensor products: Theorem 3 below, which extends a result of Hou [6].

THEOREM 3. *A nonzero operator $A \otimes B$ is p-quasihyponormal if and only if A and B are p-quasihyponormal.*

Proof. It is clear that $A \otimes B$ is p-quasihyponormal if and only if

$$(3.1)\quad A^*\left(|A|^{2p} - |A^*|^{2p}\right)A \otimes B^*|B|^{2p}B + A^*|A^*|^{2p}A \otimes B^*\left(|B|^{2p} - |B^*|^{2p}\right)B$$

is a positive operator. Therefore, if A and B are p-quasihyponormal, then the operator in (3.1) is a sum of positive operators and is, hence, positive. This proves sufficiency.

To prove necessity, let $\xi \in \mathfrak{H}$ and $\eta \in \mathfrak{K}$ be arbitrary. Then

$$\begin{aligned} &\langle A^* \left(|A|^{2p} - |A^*|^{2p}\right) A\xi, \xi\rangle\langle B^*|B|^{2p}B\eta, \eta\rangle \\ &+ \langle A^*|A^*|^{2p}A\xi, \xi\rangle\langle B^* \left(|B|^{2p} - |B^*|^{2p}\right) B\eta, \eta\rangle \geq 0 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\langle A^* \left(|A|^{2p} - |A^*|^{2p}\right) A\xi, \xi\rangle\langle B^*|B^*|^{2p}B\eta, \eta\rangle \\ &+ \langle A^*|A|^{2p}A\xi, \xi\rangle \langle B^* \left(|B|^{2p} - |B^*|^{2p}\right) B\eta, \eta\rangle \geq 0\,. \end{aligned} \tag{3.3}$$

If A is not p-quasihyponormal, then there exists a vector $\xi_0 \in \mathfrak{H}$ such that

$$\langle A^* \left(|A|^{2p} - |A^*|^{2p}\right) A\xi_0, \xi_0\rangle = \alpha < 0$$

and

$$\langle A^*|A^*|^{2p}A\xi_0, \xi_0\rangle = \beta > 0\,.$$

From (3.2) we have

$$(\alpha + \beta)\langle B^*|B|^{2p}B\eta, \eta\rangle \geq \beta\langle B^*|B^*|^{2p}B\eta, \eta\rangle\,. \tag{3.4}$$

Recall the Hölder-McCarthy inequalities: For $H \geq 0$ and $\nu \in \mathfrak{H}$,

(1) $\langle H\nu, \nu\rangle \leq \|\nu\|^{2(1-\frac{1}{q})} \langle H^q\nu, \nu\rangle^{\frac{1}{q}}$ if $q \geq 1$,

(2) $\langle H\nu, \nu\rangle \geq \|\nu\|^{2(1-\frac{1}{q})} \langle H^q\nu, \nu\rangle^{\frac{1}{q}}$ if $0 < q \leq 1$.

Therefore, in using $q = 1/p$ and $H = (B^*B)^p$, we obtain

$$\begin{aligned} \left\langle B^*|B|^{2p}B\eta, \eta\right\rangle &= \langle (B^*B)^pB\eta, B\eta\rangle \\ &\leq \langle (B^*B)B\eta, B\eta\rangle^p\, ||B\eta||^{2(1-p)} \quad \text{(by(1))} \\ &= ||B^2\eta||^{2p}||B\eta||^{2(1-p)} \quad \text{for all } \eta\,. \end{aligned}$$

With $q = (p+1)^{-1}$ and $H = (B^*B)^{p+1}$, we have

$$\begin{aligned} \left\langle B^*|B^*|^{2p}B\eta, \eta\right\rangle &= \left\langle (B^*B)^{p+1}\eta, \eta\right\rangle \\ &\geq \langle (B^*B)\eta, \eta\rangle^{p+1}\, ||\eta||^{-2p} \quad \text{(by (2))} \\ &= ||B\eta||^{2(p+1)}||\eta||^{-2p} \quad \text{for all } \eta\,. \end{aligned}$$

Thus,

$$(\alpha + \beta)||B^2\eta||^{2p}||B\eta||^{2(1-p)} \geq \beta||B\eta||^{2(p+1)}||\eta||^{-2p} \quad \text{for all } \eta \text{ by (3.4).}$$

Hence, we have

$$\beta||B\eta||^{4p} \leq (\alpha+\beta)||B^2\eta||^{2p}||\eta||^{2p} \quad \text{for all } \eta, \tag{3.5}$$

and (replacing η by $B\eta$) we have

$$\beta||B^2\eta||^{4p} \leq (\alpha+\beta)||B^3\eta||^{2p}\ ||B\eta||^{2p} \quad \text{for all } \eta. \tag{3.6}$$

Now let $B = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}$ on $\overline{\operatorname{ran}(B)} \oplus \ker(B^*)$. Then B is p-quasihyponormal by (3.4) and B_1 is p-hyponormal (hence it is normaloid) by [8, Lemma 1]. By (3.5) we have

$$\beta||B_1\zeta||^{4p} \leq (\alpha+\beta)||B_1^2\zeta||^{2p}||\zeta||^{2p} \quad \text{for all } \zeta \in \overline{\operatorname{ran}(B)},$$

so we have

$$\beta||B_1||^{4p} \leq (\alpha+\beta)||B_1^2||^{2p} = (\alpha+\beta)||B_1||^{4p} \quad \text{(since } B_1 \text{ is normaloid).}$$

This implies that $B_1 = 0$. Since $B^2\eta = B_1B\eta = 0$ for all η, $B^2 = 0$ and $B = 0$ by (3.5). This contradicts the assumption $B \neq 0$. Hence A must be p-quasihyponormal. A similar argument shows that B is also p-quasihyponormal. □

Surely other results are possible. However, it would be most interesting to find a *meta-theorem* that captures situations as diverse as those described in Theorems 2 and 3.

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