REAL HYPERSURFACES OF THE JACOBI
OPERATOR WITH RESPECT TO THE STRUCTURE
VECTOR FIELD IN A COMPLEX SPACE FORM

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Abstract. We study a real hypersurface $M$ satisfying $L_\xi S = 0$
and $R_\xi S = SR_\xi$ in a complex hyperbolic space $H_n \mathbb{C}$, where $S$ is the
Ricci tensor of type $(1,1)$ on $M$, $L_\xi$ and $R_\xi$ denotes the operator
of the Lie derivative and the Jacobi operator with respect to the
structure vector field $\xi$ respectively.

0. Introduction

A Kaehlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form. As is well known a complete and
simply connected complex space form is a complex projective space $P_n \mathbb{C}$,
a complex Euclidean space $\mathbb{C}_n$, or a complex hyperbolic space $H_n \mathbb{C}$
according as $c > 0$, $c = 0$ or $c < 0$.

In his study[18] of real hypersurfaces of $P_n \mathbb{C}$, Takagi showed that all
homogeneous real hypersurfaces could be divided into six types and in
[3] Cecil-Ryan and Kimura[9] proved that they are realized as the tubes
of constant radius over Kaehlerian submanifolds. Namely he proved the
following

Theorem T. [18] Let $M$ be a homogeneous real hyperspace of $P_n \mathbb{C}$.
Then $M$ is a tube of radius $r$ over one of the following Kaehlerian sub-
manifolds:

(A) a hyperplane $P_{n-1} \mathbb{C}$, where $0 < r < \frac{\pi}{2}$,
(B) a totally geodesic $P_k \mathbb{C}(1 \leq k \leq n - 2)$, where $0 < r < \frac{\pi}{2}$,
(C) a complex quadric $Q_{n-1}$, where $0 < r < \frac{\pi}{4}$,
(C) \( P_1 \mathbb{C} \times P_{(n-1)/2} \mathbb{C} \), where \( 0 < r < \frac{\pi}{4} \) and \( n \geq 5 \) is odd,
(D) a complex Grassmann \( G_{2,5} \mathbb{C} \), where \( 0 < r < \frac{\pi}{5} \) and \( n = 9 \),
(E) a Hermitian symmetric space \( SO(10)/U(5) \), where \( 0 < r < \frac{\pi}{4} \) and \( n = 15 \).

In what follows the induced almost contact metric structure of a real hypersurface in a complex space form is denoted by \((\phi, g, \xi, \eta)\). The structure operator \( \xi \) is said to be principal if \( A_\xi = \alpha \xi \), where \( A \) is the shape operator in the direction of the unit normal and \( \alpha = \eta(A\xi) \). We denote by \( \nabla \) and \( S \), the Levi-Civita connection with respect to the Riemannian metric tensor \( g \) and the Ricci tensor of type \((1,1)\) on the real hypersurface respectively. Theorem T is generalized by many authors ([1], [6], [9], [10], [11], [12], [16] etc.) One of them, Maeda asserts the following theorem:

**Theorem M.** [13] Let \( M \) be a real hypersurface with constant mean curvature in \( P_n \mathbb{C} \) \((n \geq 3)\) on which \( \xi \) is a principal curvature vector and the focal map \( \phi_r \) has constant rank on \( M \). If \( \nabla_\xi S = 0 \), then \( M \) is locally congruent to one of \( A_1 \), \( A_2 \), \( B \), \( C \), \( D \), and \( E \).

On the other hand, real hypersurfaces of \( H_n \mathbb{C} \) have been also investigated by many geometers ([2], [14], [15], [16] etc.) from different points of view. In particular, Berndt proved the following:

**Theorem B.** [2] Let \( M \) be a real hypersurface of \( H_n \mathbb{C} \). Then \( M \) has constant principal curvatures and \( \xi \) is principal if and only if \( M \) is locally congruent to one of the following:

(A\textsubscript{0}) a self-tube, that is, a horosphere,
(B) a geodesic hypersphere or a tube over a hyperplane \( H_{n-1} \mathbb{C} \),
(C) a tube over a totally geodesic \( H_k \mathbb{C} \) \((1 \leq k \leq n-2)\),
(D) a tube over a totally real hyperbolic space \( H_n \mathbb{R} \).

For a real hypersurface of \( H_n \mathbb{C} \), Ki, Kim and Lee proved the following

**Theorem KM.** [7] Let \( M \) be a real hypersurface of \( H_n \mathbb{C} \). If the structure vector \( \xi \) is principal and if \( \nabla_\xi S = 0 \), then \( M \) is locally congruent one of \( A_0 \), \( A_1 \) or \( A_2 \).

Denoting by \( R \) the curvature tensor of a real hypersurface, we define the Jacobi operator \( R_\xi = R(\cdot, \xi)\xi \) with respect to the structure vector \( \xi \). Then \( R_\xi \) is a self-adjoint endomorphism on the tangent space of the real hypersurface.
In this paper we study a real hypersurface of a complex hyperbolic space \( H_n \mathbb{C} \) which satisfies \( L_\xi S = 0 \) and \( R_\xi S = SR_\xi \), where \( L_\xi \) denotes the operator of the Lie derivative with respect to \( \xi \).

All manifolds in the present paper are assumed to be connected and of class \( C^\infty \).

1. Preliminaries

Let \( \tilde{M} \) be a Kaehlerian manifold of real dimension \( 2n \) equipped with an almost complex structure \( J \) and a Hermitian metric tensor \( G \). Then for any vector fields \( X \) and \( Y \) an \( \tilde{M} \), we have

\[
J^2 = -X, G(JX, JY) = G(X, Y), \quad \nabla J = 0,
\]

where \( \nabla \) denotes the Riemannian connection of \( \tilde{M} \).

Let \( M \) be a real \((2n-1)\)-dimensional hypersurface of \( \tilde{M} \) covered by a system of coordinate neighborhoods \( \{ U; x^i \} \) and isometrically immersed in \( \tilde{M} \) by the immersion \( i : M \to \tilde{M} \). When the argument is local, \( M \) need not be distinguished from \( i(M) \) itself. Throughout this paper the indices \( i, j, k, \cdots \) run from 1 to \( 2n-1 \) and the summation convention will be used. We represent the immersion \( i \) locally by

\[
y^A = y^A(x^h), \quad (A = 1, \cdots, 2n-1, 2n)
\]

and put \( B_j = (B_j^A) \), \( (\partial_j = \partial/\partial x^j) \) then \( B_j \) are \((2n-1)\)-linearly independent local tangent vectors of \( M \). A unit normal \( C \) to \( M \) may then be chosen. The induced Riemannian metric \( g \) with components \( g_{ji} \) on \( M \) is given by \( g_{ji} = G_{BA} B_j^B B_i^A \) because the immersion is isometric.

For the unit normal \( C \) to \( M \), the transformations of \( B_i \) and \( C \) by \( J \) are respectively represented in each coordinate neighborhood as follows:

\[
JB_i = \phi_i^h B_h + \xi_i C, \quad JC = -\xi_i^i B_i,
\]

where we have put \( \phi_{ji} = G(JB_j, B_i) \) and \( \xi_i = G(JB_i, C) \), \( \xi_i^h \) being components of a vector fields \( \xi \) associated with \( \xi_i \) and \( \phi_{ji} = \phi_j^r g_{ri} \). By the properties of the almost Hermitian structure \( J \), it is clear that \( \phi_{ji} \) is skew-symmetric. A tensor field of type \((1,1)\) with components \( \phi_i^h \) will be denoted by \( \phi \). By properties of the almost complex structure \( J \), the following relations are then given.

\[
\phi_i^r \phi_r^h = -\delta_i^h + \xi_i \xi_j^h, \quad \xi_r^r \phi_r^h = 0, \quad \xi_r \phi_i^r = 0, \quad \xi_i \xi_i^i = 1,
\]
that is, the aggregate \((\phi, g, \xi)\) defines an almost contact metric structure.  

Denoting by \(\nabla_j\) the operator of van der Waerden-Bortolotti covariant differentiation with respect to \(g\) and \(G\), equations of Gauss and Weingarten for \(M\) are respectively given by

\[
\nabla_j B_i = A_{ji} C, \quad \nabla_j C = -A_j^r B_r,
\]

where \(A = (A_j^h)\), which is related by \(A_{ji} = A_j^r g_{ri}\) is the shape operator in direction \(C\). By means of above equations the covariant derivatives of the structure tensors are yielded:

\[
(1.1) \quad \nabla_j \phi_i^h = A_j^h \xi_i - A_{ji} \xi^h, \quad \nabla_j \xi_i = -A_{jr} \phi_i^r.
\]

If the ambient space \(\tilde{M}\) is a Kaehlerian manifold of constant holomorphic sectional curvature \(c\), which is called a complex space form and denoted \(M_n(c)\), then equations of Gauss and Codazzi are respectively given by

\[
(1.2) \quad R_{kjih} = \frac{c}{4} \left( g_{kh} g_{ji} - g_{jh} g_{ki} + \phi_{kh} \phi_{ji} - \phi_{jh} \phi_{ki} - 2 \phi_{kj} \phi_{ih} \right) + A_{kh} A_{ji} - A_{jh} A_{ki},
\]

\[
(1.3) \quad \nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4} \left( \xi_k \phi_{ji} - \xi_j \phi_{ki} - 2 \xi_i \phi_{kj} \right),
\]

where \(R_{kjih}\) are components of the Riemann-Christoffel curvature tensor \(R\) of \(M\).

From (1.2) the Ricci tensor \(S\) of type (1,1) with components \(S_j^h\) is verified that

\[
(1.4) \quad S_j^h = \frac{c}{4} \left\{ (2n + 1) \delta_j^h - 3 \xi_j \xi^h \right\} + h A_j^h - A_j^r A_r^h,
\]

where \(h = Tr A\). Hereafter, to write our formulas in convention form, we denote by \(A_j^2 = A_{jr} A_i^r\), \(\alpha = A_{ji} \xi_j \xi^i\), \(\beta = A_{ji} \xi^j \xi^i\), and \(\gamma = A_{ji} \xi^j \xi^i\).

If we put \(U = \nabla_\xi \xi\), then \(U\) is orthogonal to the structure vector \(\xi\). Because of properties of the almost contact metric structure and the second equation of (1.1), we have

\[
(1.5) \quad U^r \nabla_j \xi_r = A_{jr}^2 \xi^r - \alpha A_{jr} \xi^r,
\]
(1.6) \[ \phi_j U^r = A_{jr} \xi^r - \alpha \xi_j, \]

which shows that \( g(U, U) = \beta - \alpha^2. \)

Differentiating (1.6) covariantly along \( M \) and using (1.1), we find

(1.7) \[ \xi_j (A_{kr} U^r + \alpha_k) + \phi_j^r \nabla_k U_r = \xi^r \nabla_k A_{jr} - A_{jr} A_{ks} \phi^{rs} + \alpha A_{kr} \phi_j^r, \]

which unable us to obtain

(1.8) \[ (\nabla_k A_{rs}) \xi^r \xi^s = 2 A_{kr} U^r + \alpha_k, \]

where \( \alpha_k = \partial_k \alpha. \)

Now, we put

(1.9) \[ A \xi = \alpha \xi + \mu W, \]

where \( \mu \) is a function on \( M \), and \( W \) is a unit vector orthogonal to \( \xi. \) Then we have \( \mu^2 = \beta - \alpha^2 \) and \( \phi U = -\mu W. \) Hence it is, using (1.1), seen that

(1.10) \[ \mu \xi^r \nabla_j W_r = A_{jr} U^r \]

because \( \xi \) and \( W \) are mutually orthogonal.

2. Real hypersurfaces satisfying \( L_\xi S = 0 \)

Let \( M \) be a real hypersurface of a complex space form \( M_n(c), c \neq 0. \) By definition, the Lie derivative of the Ricci tensor \( S \) with respect to the structure vector \( \xi \) is given by

\[ L_\xi S^h_j = \xi^r \nabla_r S^h_j + (\nabla_j \xi^r) S^h_r - (\nabla_r \xi^h) S^r_j, \]

or using the second equation of (1.1),

\[ L_\xi S^h_j = \xi^r \nabla_r S^h_j + A_{rt} \phi^{ht} S^r_j - A_{jt} \phi^{rt} S^h_r. \]

In what follows we assume that the Ricci tensor \( S \) satisfies \( L_\xi S = 0, \) that is,

(2.1) \[ \xi^r \nabla_r S_{ji} = A_{jt} \phi_r^t S_{ij}^r - A_{rt} \phi_i^t S_{jr}^r, \]
which shows that

\[(2.2) \quad (A_{j\theta} \phi^t_r + A_{\theta \theta} \phi^t_j) S^\gamma_i = (A_{i\theta} \phi^t_r + A_{\theta \theta} \phi^t_i) S^\gamma_j.\]

From (1.4) we get

\[(2.3) \quad S_j r \xi^r = \frac{c}{2} (n - 1) \xi_j + h A_j r \xi^r - A_{j \theta}^2 \xi^r,\]

\[(2.4) \quad S_{j \theta} \phi^r_i + S_{\theta \theta} \phi^r_j = h (A_{j \theta} \phi^r_i + A_{\theta \theta} \phi^r_j) - A_{j \theta}^2 \phi^r_i - A_{\theta \theta}^2 \phi^r_j.\]

Because of (2.2) and (2.4), it follows that

\[(S_{j \theta} \phi^r_i + S_{\theta \theta} \phi^r_j)(A^{j \theta} \phi^{it} + A^{i \theta} \phi^{jt}) = 0.\]

Hence, by applying $A^{j \theta} \phi^{it}$ to (2.2) and making use of (2.3) we obtain (see [11])

\[(2.5) \quad ||S \phi - \phi S||^2 + \frac{3}{2} c ||U||^2 = 0.\]

Therefore if $c > 0$, then we have $S \phi = \phi S$ and $U = 0$, and consequently $\alpha$ is locally constant on $M$ ([8]). Using the fact that $A \xi = \alpha \xi$, it is clear that $SA = AS$. Hence (2.1) is reduced to $\nabla_\xi S = 0$.

Now, suppose that $g(S \xi, \xi) = \text{const}$. Then by (2.3) we have $g(S \xi, \xi) = \frac{c}{2} (n - 1) - \alpha^2 + h \alpha$ by virtue of $\beta = \alpha^2 = 0$.

According to Theorem M, we have

**Theorem 2.1.** Let $M$ be a real hypersurface satisfying $L_\xi S = 0$ in a complex projective space $P_n \mathbb{C} (\geq 3)$. If $g(S \xi, \xi) = \text{const}$, then $M$ is locally congruent to one of $A_1, A_2, B, C, D$, and $E$ provided that $g(A \xi, \xi) \neq 0$.

For a real hypersurface of a complex hyperbolic space $H_n \mathbb{C}$, it is known that

**Theorem K.** [5] Let $M$ be a real hypersurface of $H_n \mathbb{C}$. If it satisfies $L_\xi S = 0$ and $S \xi = \sigma \xi$ for some function $\sigma$ on $M$, then $\xi$ is principal.
3. Jacobi operators of real hypersurfaces

Let $M$ be a real hypersurface satisfying $L_{\xi}S = 0$ in a complex hyperbolic space $H_n\mathbb{C}$. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field $X$. Then from (1.2) we have

$$(R_{\xi})_{ji} = \frac{c}{4}(g_{ji} - \xi_j \xi_i) + \alpha A_{ji} - (A_{jr} \xi^r)(A_{is} \xi^s).$$

Suppose that $R_{\xi}S = SR_{\xi}$. Then we have

$$(A_{jr}^3 \xi^r)(A_{is} \xi^s) - (A_{ir}^3 \xi^r)(A_{js} \xi^s) = (A_{jr}^2 \xi^r)(hA_{is} \xi^s - \frac{c}{4} \xi_i) - (A_{ir}^2 \xi^r)(hA_{js} \xi^s - \frac{c}{4} \xi_j) + \frac{c}{4} h(\xi_i A_{jr} \xi^r - \xi_j A_{ir} \xi^r),$$

which implies that

$$(3.1) \quad \alpha A_{jr}^3 \xi^r = (\alpha h - \frac{c}{4}) A_{jr}^2 \xi^r + (\gamma - \beta h + \frac{c}{4} h) A_{jr} \xi^r + \frac{c}{4} (\beta - h\alpha) \xi_j.$$

Combining the last two equations, it follows that

$$(A_{jr}^2 \xi^r)(A_{is} \xi^s - \alpha \xi_i) - (A_{ir}^2 \xi^r)(A_{js} \xi^s - \alpha \xi_j) = \beta (\xi_j A_{jr} \xi^r - \xi_i A_{jr} \xi^r).$$

Multiplying $A^s \xi^s$ to the last equation and summing for $j$, we find

$$\mu^2 A^2 \xi = (\gamma - \beta \alpha) A \xi + (\beta^2 - \alpha \gamma) \xi.$$

If $\mu \neq 0$, then we have

$$(3.2) \quad A^2 \xi = \rho A \xi + (\beta - \rho \alpha) \xi,$$

where we have put $\mu^2 \rho = \gamma - \beta \alpha$, $\mu^2(\beta - \rho \alpha) = \beta^2 - \alpha \gamma$, which shows that

$$A^3 \xi = (\rho^2 + \beta - \rho \alpha) A \xi + \rho (\beta - \rho \alpha) \xi.$$ 

Thus (3.1) implies that $(h - \rho)(\beta - \rho \alpha - \frac{c}{4}) = 0$ because of $\mu \neq 0$. Therefore we have

$$(3.3) \quad \mu(h - \rho)(\beta - \rho \alpha - \frac{c}{4}) = 0$$
on \( M \).

Let \( \Omega_0 \) be a set of points in \( M \) such that \( \mu(\beta - \rho \alpha - \frac{c}{4}) \neq 0 \). Then we have \( h = \rho \) on \( \Omega_0 \). Thus (3.2) turns out to be \( A^2 \xi = hA\xi + (\beta - h\alpha)\xi \) and hence \( S\xi = \sigma\xi \) on \( \Omega_0 \) because of (2.3), where we have put \( \sigma = \frac{\xi}{2}(n - 1) + h\alpha - \beta \). Owing to Theorem K, it is seen that \( A\xi = \alpha\xi \), a contradiction. Therefore (3.3) is reduces to

\[
\mu(\beta - \rho \alpha - \frac{c}{4}) = 0
\]

on \( M \).

In the following we assume that \( \mu \neq 0 \) on \( M \), namely, \( \xi \) is not a principal curvature vector and we put \( \Omega = \{ p \in M : \mu(p) \neq 0 \} \). Then \( \Omega \) is an open subset of \( M \). From now on we discuss our arguments on \( \Omega \).

From (3.4) we have

\[
\beta = \rho \alpha + \frac{c}{4}.
\]

Thus (3.2) becomes

\[
A^2 \xi = \rho A\xi + \frac{c}{4}\xi.
\]

From (1.9) and (3.6), we obtain

\[
AW = \mu \xi + (\rho - \alpha)W
\]

because of \( \mu \neq 0 \), which enable us to obtain

\[
A^2 W = \rho AW + \frac{c}{4}W.
\]

Making use of (3.5), (3.6), and (3.7), the relationship (1.5) turns out to be

\[
U^r \nabla_j \xi_r = \mu A_{jr} W^r.
\]

Differentiating (3.7) covariantly along \( M \), we find

\[
(\nabla_k A_{jr}) W^r + A_{jr} \nabla_k W^r
= \mu_k \xi_j + \mu \nabla_k \xi_j + (\rho_k - \alpha_k) W_j + (\rho - \alpha) \nabla_k W_j.
\]
Applying this by $W^j$ and taking account of (1.10) and (3.7), we find

$$\nabla_k A r s W^r W^s = -2 A_{kr} U^r + \rho_k - \alpha_k$$

(3.11)

because $\xi$ and $W$ are mutually orthogonal. In the same way, we have from (3.10)

$$\mu(\nabla_k A r s) W^r \xi^s = (\rho - 2\alpha) A_{kr} U^r + \mu \mu_k,$$

(3.12)

or using the Codazzi equation (1.3) and the fact that $\mu^2 = \beta - \alpha^2$,

$$\mu(\nabla_r A_{ks}) W^r \xi^s = (\rho - 2\alpha) A_{kr} U^r - \frac{c}{2} U_k + \frac{1}{2} \beta_k - \alpha \alpha_k.$$

(3.13)

Differentiating (3.6) covariantly and using (1.9), we find

$$(\alpha - \rho)(\nabla_k A_{jr}) \xi^r + \mu(\nabla_k A_{jr}) W^r + A_{jr}(\nabla_k A_s \xi^s)$$

(3.14)

$$= \rho_k A_{jr} \xi^r - A_{jr}^2 \nabla_k \xi^r + \rho A_{jr} \nabla_k \xi^r + \frac{c}{4} \nabla_k \xi_j,$$

from which, making use of (1.3), (1.8), and (3.12),

$$3 A_{jr}^2 U^r - 2 \rho A_{jr} U^r - \frac{c}{2} U_j = (\rho t \xi^t) A_{jr} \xi^r - A_{jr} \alpha^r + \rho \alpha_j - \frac{1}{2} \beta_j.$$

(3.15)

If we take the skew-symmetric part of (3.14) and using (1.1) and (1.3), then we obtain

$$A_{ks} (\nabla_j A_r \xi^s) - A_{js} (\nabla_k A_r \xi^s) \xi^r + \rho_k A_{jr} \xi^r - \rho_j A_{kr} \xi^r$$

$$= \frac{c}{4} (U_k \xi_j - U_j \xi_k) + \frac{c}{2} (\rho - \alpha) \phi_{kj} + A_{kr}^2 A_{js} \phi^{rs} - A_{jr}^2 A_{ks} \phi^{rs}$$

$$+ 2 \rho A_{jr} A_{ks} \phi^{rs} + \frac{c}{4} (A_{kr} \phi_j^r - A_{jr} \phi_k^r).$$

Applying this by $\mu W^j$ and taking account of (1.3), (1.8), (1.9), (3.7), (3.8), and (3.13), we find

$$\nabla_k A r s W^r + (\rho_2 - 2 \rho \alpha + c) A_{jr} U^r + \frac{c}{4} (\alpha - \rho) U_j$$

(3.16)

$$= \mu A_{jr} \mu^r + (\alpha - \rho) \mu \mu_j + \mu^2 (\rho_j - \alpha_j) - \mu (\rho_t W^t) A_{jr} \xi^r.$$

On the other hand, since we have (3.6), the equation (2.3) is reduced to

$$S_{jr} \xi^r = \frac{c}{4} (2n - 3) \xi_j + (h - \rho) A_{jr} \xi^r.$$

(3.17)
Applying (2.2) by $\xi^j$ and using (1.4), (3.6), and (3.17), we get

\begin{equation}
A_{jr}^2 U^r = (2h - \rho)A_{jr}^r U^r + (\rho^2 - h\rho + c)U_j.
\end{equation}

Because of (1.4), (3.7), and (3.8), it follows that

\begin{equation}
S_{jr}W^r = \mu(h - \rho)\xi_j + xW_j,
\end{equation}

where we have put $x = \frac{c}{2}n + (\rho - \alpha)(h - \rho)$.

If we transform $\mu W^t$ to (2.2) and take account of (3.7) and (3.18) we also find

\begin{equation}
\{2(\rho - h)^2 + \frac{3}{4}c\}A_{jr}^r U^r = \{\rho(\rho - h)^2 - \frac{3}{4}c(\alpha + h - 2\rho)\}U_j.
\end{equation}

From (3.17) we have

\[
(\nabla_k S_{jr})\xi^r + S_{jr}\nabla_k \xi^r = \frac{c}{4}(2n - 3)\nabla_k \xi_j + (h_k - \rho_k)A_{jr}^r \xi^r + (h - \rho)(\nabla_k A_{jr})\xi^r + (h - \rho)A_{jr}\nabla_k \xi^r,
\]

which together with (1.4), (1.8), (2.1), and (3.17) gives

\begin{equation}
3(\rho - h)A_{jr}^r U^r - \rho(\rho - h)U_j = (h_t \xi^t - \rho_t \xi^t)A_{jr}^r \xi^r + (h - \rho)\alpha_j.
\end{equation}

**Lemma 3.1.** $\alpha_t \xi^t = 0, \alpha_t W^t = 0, \rho_t \xi^t = 0, \rho_t W^t = 0$, and $h_t \xi^t = 0$ on $\Omega$.

**Proof.** Applying (3.21) by $\xi^j$ or $W^j$ and making use of (1.9), we obtain respectively

\begin{equation}
\alpha(h_t \xi^t - \rho_t \xi^t) + (h - \rho)\alpha_t \xi^t = 0,
\end{equation}

\begin{equation}
\mu(h_t \xi^t - \rho_t \xi^t) + (h - \rho)\alpha_t W^t = 0,
\end{equation}

which enable us to obtain

\begin{equation}
\mu \alpha_t \xi^t = \alpha \alpha_t W^t.
\end{equation}

By the way, combining (3.11) and (3.12), we have

\[
\mu(\rho_t \xi^t - \alpha_t \xi^t) = \frac{1}{2}\beta_t W^t - \alpha \alpha_t W^t,
\]
where we have used the fact that $\mu^2 = \beta - \alpha^2$. Thus, it follows that

\[(3.24) \quad \beta_t W^t = 2\mu \rho_t \xi^t.\]

Next, multiplying (3.15) with $\xi^j$ and summing for $j$, and using (1.9) and (3.5), we find

\[2\mu \alpha_t W^t = \alpha \rho_t \xi^t + (\rho - 2\alpha) \alpha_t \xi^t,\]

which together with (3.23) yields

\[(3.25) \quad \alpha^2 \rho_t \xi^t + (\rho \alpha - 2\beta) \alpha_t \xi^t = 0.\]

Because of (2.1) and (3.19), it is seen that

\[(3.26) \quad \xi^r (\nabla_r S_{ji}) W^j W^i = 0.\]

Differentiating (3.19) covariantly, we find

\[(\nabla_k S_{jr}) W^r + S_{jr} \nabla_k W^r = x_k W_j + x \nabla_k W_j + \{\mu(h - \rho)\} k \xi_j + \mu(h - \rho) \nabla_k \xi_j.\]

If we apply this by $\xi^k W^j$ and take account of (1.10), (3.19), and (3.26), then we get $x_t \xi^t = 0$. By definition, it follows that

\[(h - \rho) (\rho_t \xi^t - \alpha_t \xi^t) + (\rho - \alpha)(h_t \xi^t - \rho_t \xi^t) = 0,\]

which together with (3.22) implies that

\[\alpha \rho_t \xi^t = \rho \alpha_t \xi^t\]

because $h - \rho \neq 0$ on $\Omega$. From this and (3.25) we verify that $(\beta - \rho \alpha) \alpha_t \xi^t = 0$ and hence $\alpha_t \xi^t = 0$ by virtue of (3.5).

Remark 1. We notice here that $\alpha \neq 0, \rho \neq 0$ or $\rho \neq \alpha$ on $\Omega$ because of (3.5) and $c < 0$.

From this fact it is seen that $\rho_t \xi^t = 0$ on $\Omega$. If we take account of (3.22), (3.23), and (3.24), then we see respectively that $h_t \xi^t = 0$, $\alpha_t W^t = 0$ and $\beta_t W^t = 0$. From the last relation and (3.5) it is seen that $\rho_t W^t = 0$. This completes the proof. \qed
4. Real hypersurfaces satisfying $L_\xi S = 0$ and $R_\xi S = SR_\xi$

In the rest of this paper we shall suppose that $M$ is a $(2n - 1)$-dimensional real hypersurface in a complex hyperbolic space $H_nC$ and that the Ricci tensor $S$ satisfies $L_\xi S = 0$ and $R_\xi S = SR_\xi$ on $M$. Then (3.21) is reduces to

\begin{equation}
\nabla \alpha = \rho U - 3AU
\end{equation}

because of Lemma 3.1 and the fact that $\rho - h \neq 0$ on $\Omega$, which together with (3.15) and Lemma 3.1 gives

\begin{equation}
\alpha \nabla \rho = (\rho^2 + c)U - \rho AU.
\end{equation}

From the last two equations, it follows that

\begin{equation}
\mu \nabla \mu = (\rho^2 - \rho \alpha + \frac{c}{2})U + (3\alpha - 2\rho)AU.
\end{equation}

Substituting (4.1)–(4.3) into (3.16) and making use of Lemma 3.1, we find

\[(\rho - \alpha)AU = \{(\rho - \alpha)(\rho + 3\alpha) + c\}U.\]

Thus we have $AU = \lambda U$, where we have the function $\lambda$ defined by

\begin{equation}
\lambda = \rho + 3\alpha + \frac{c}{\rho - \alpha}
\end{equation}

because of Remark 1. Thus (4.1) and (4.2) are respectively reduces to

\begin{equation}
\nabla \alpha = (\rho - 3\lambda)U, \quad \nabla \rho = (3\alpha - \lambda - 2\rho)U
\end{equation}

with the aid of Remark 1.

Since we have $AU = \lambda U$, (3.18) and (3.20) turn out respectively to be

\begin{equation}
(2\lambda - \rho)h = \lambda^2 + \rho \lambda - \rho^2 - c,
\end{equation}

\begin{equation}
(2\lambda - \rho)(\rho - h)^2 = \frac{3}{4}c(2\rho - \alpha - h - \lambda).
\end{equation}
On the other hand applying (2.2) by $U^i W^j$ and using (1.4), (3.17), and (3.19), we find

$$\mu^2(\rho - h) = \{h\lambda - \lambda^2 - (h - \rho)(\rho - \alpha) + \frac{c}{4}\}(\lambda - \rho + \alpha),$$

which together with (4.6) implies that

$$(4.8) \quad \mu^2(\rho - h) = (\lambda - \rho + \alpha)((\rho - h)(\lambda - \alpha) - \frac{3}{4}c).$$

Using (3.5) and (4.4), it is seen that

$$(4.9) \quad (\rho - \alpha)(\lambda + \alpha - \rho) = 4\mu^2.$$

Thus $\alpha - \rho + \lambda$ does not vanish on $\Omega$. Differentiation gives

$$(\rho - \alpha)\nabla \lambda = 8\mu \nabla \mu + (2\rho - 2\alpha - \lambda)(\nabla \rho - \nabla \alpha),$$

which connected with (3.5) gives

$$(\rho - \alpha)\nabla \lambda = (2\alpha + 2\rho - \lambda)\nabla \rho + (2\rho - 6\alpha + \lambda)\nabla \alpha.$$

Making use of (4.5), we have $(3\alpha - \lambda - 2\rho)\nabla \alpha = (\rho - 3\lambda)\nabla \rho$. Therefore the last equation turns out to be

$$(4.10) \quad (\rho - \alpha)\nabla \lambda = (2\alpha + 5\lambda)\nabla \rho - (\lambda + 2\rho)\nabla \alpha.$$

If we take account of (4.8) and (4.9), then we obtain

$$(4.11) \quad (\rho - h)(4\lambda - 3\alpha - \rho) = 3c$$

because $\lambda + \alpha - \rho$ does not vanish on $\Omega$, which together with (4.6) implies that

$$(4.12) \quad \lambda(\rho - \lambda)(4\lambda - 3\alpha - \rho) = c(2\lambda + 3\alpha - 2\rho).$$

Now, we prove
Lemma 4.1. Let $M$ be a real hypersurface with $L_{\xi}S = 0$ and $R_{\xi}S = SR_{\xi}$ in $H_{n}\mathbb{C}$. If $g(S\xi, \xi) = \text{const.}$, then $\Omega$ is empty.

Proof. By (3.17) we have

$$\alpha(h - \rho) + \frac{c}{4}(2n - 3) = g(S\xi, \xi).$$

Thus we obtain $\alpha(h - \rho) = \text{const.}$ Hence if we differentiate (4.11), we find

$$(4\lambda - \rho)(\nabla\rho - \nabla h) + (\rho - h)(4\nabla\lambda - \nabla\rho) = 0,$$

which enable us to obtain

$$4\alpha\nabla\lambda = \alpha\nabla\rho + (4\lambda - \rho)\nabla\alpha$$

because $\rho - h$ does not zero on $\Omega$. From this and (4.10) we verify that

$$(4.13) \quad 4\rho\nabla\lambda = (9\alpha + 20\lambda)\nabla\rho - 9\rho\nabla\alpha. \quad \square$$

On the other hand, from (4.11) we have $(\rho - h)(4\lambda - \rho) = 3a$, where we have put

$$(4.14) \quad \alpha(\rho - h) + c = a.$$ 

Hence we obtain

$$(4.15) \quad 4\lambda = \rho + b\alpha,$$

where the constant $b$ is defined by $(a - c)b = 3a$. From this and (4.13) it follows that

$$(4.16) \quad (9\alpha + 5b\alpha + 4\rho)\nabla\rho = (9\rho + b\rho)\nabla\alpha,$$

which together with (4.5) yields

$$(4.17) \quad (b + 45)\rho^2 + (22b - 3b^2 + 33)\rho\alpha + (5b^2 - 51b - 108)\alpha^2 = 0.$$ 

Therefore $\rho/\alpha$ is a root of algebraic equation with constant coefficient and hence $\rho = \epsilon\alpha$ for some constant $\epsilon$ on $\Omega$, which together with (4.16) gives

$$(4.18) \quad (b + \epsilon)\nabla\alpha = 0.$$
If \( b + \epsilon = 0 \), then (4.17) implies that \((b + 1)(b - 3)(b + 9) = 0\) by virtue of Remark 1. By definition and equations (4.9) and (4.15), it is clear that \((b + 1)(b - 3) \neq 0\). Thus we have \(b + 9 = 0\), which together with (4.15) and (4.16) gives \(\lambda \nabla \rho = 0\). Since \(\lambda \neq 0\) on \(\Omega\) because of (4.9), it follows that \(\nabla \rho = 0\). From this and the second equation of (4.5) and (4.15) we see that \(\nabla \alpha = 0\). Consequently it is seen that \(\alpha = \text{const}\). Thus (4.5) implies that \(\rho = 3\lambda\) and \(3\alpha - \lambda - 2\rho = 0\), and thus \(7\lambda = 3\alpha\) and \(7\rho = 9\alpha\). From these and (4.12) we get \(36\alpha^2 + 7c = 0\). We also, using (3.5) and (4.9), see that \(54\alpha^2 + 49c = 0\), which produces a contradiction. Therefore, it is seen that \(\Omega\) isvoid. This completes the proof the lemma.

According to Lemma 4.1 and Theorem KM, we have

**Theorem 4.2.** Let \(M\) be a real hypersurface satisfying \(L_\xi S = 0\) and \(R_\xi S = SR_\xi\) in a complex hyperbolic space \(H_n\mathbb{C}\). If \(g(S_\xi, \xi) = \text{const.}\), then \(M\) is of type \(A_0\), \(A_1\) or \(A_2\), where \(S\) denotes the Ricci tensor of type \((1,1)\) on \(M\), and \(L_\xi\) the operator of the Lie derivative, \(R_\xi\) the Jacobi operator with respect to the structure vector \(\xi\).

**Proof.** Because of Lemma 4.1, (2.5) gives \(S\phi = \phi S\). Further we also have from (1.4) \(SA = AS\). Thus (2.1) turns out to be \(\nabla_\xi S = 0\). By Theorem KM, we arrive at the conclusions. \(\square\)

**References**


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