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REAL HYPERSURFACES OF THE JACOBI OPERATOR WITH RESPECT TO THE STRUCTURE VECTOR FIELD IN A COMPLEX SPACE FORM

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ABSTRACT. We study a real hypersurface M satisfying $L_\xi S = 0$ and $R_\xi S = SR_\xi$ in a complex hyperbolic space $H_n\mathbb{C}$, where S is the Ricci tensor of type (1,1) on M, L_ξ and R_ξ denotes the operator of the Lie derivative and the Jacobi operator with respect to the structure vector field ξ respectively.

0. Introduction

A Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form. As is well known a complete and simply connected complex space form is a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$, or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0, c = 0$ or $c < 0$.

In his study[18] of real hypersurfaces of $P_n\mathbb{C}$, Takagi showed that all homogeneous real hypersurfaces could be divided into six types and in [3] Cecil-Ryan and Kimura[9] proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds. Namely he proved the following

THEOREM T. [18] *Let M be a homogeneous real hyperspace of $P_n\mathbb{C}$. Then M is a tube of radius r over one of the following Kaehlerian submanifolds:*

(A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*
(A_2) *a totally geodesic $P_k\mathbb{C}(1 \le k \le n-2)$, where $0 < r < \frac{\pi}{2}$,*
(B) *a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*

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(C) $P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, *where* $0 < r < \frac{\pi}{4}$ *and* $n(\geq 5)$ *is odd,*
(D) *a complex Grassmann* $G_{2,5}C$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 9$,
(E) *a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0 < r < \frac{\pi}{4}$ *and* $n = 15$.

In what follows the induced almost contact metric structure of a real hypersurface in a complex space form is denoted by (ϕ, g, ξ, η). The structure vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal and $\alpha = \eta(A\xi)$. We denote by ∇ and S, the Levi-Civita connection with respect to the Riemannian metric tensor g and the Ricci tensor of type (1,1) on the real hypersurface respectively. Theorem T is generalized by many authors ([1], [6], [9], [10], [11], [12], [16] etc.) One of them, Maeda asserts the following theorem :

THEOREM M. [13] *Let* M *be a real hypersurface with constant mean curvature in* $P_n\mathbb{C}$ $(n \geq 3)$ *on which* ξ *is a principal curvature vector and the focal map* ϕ_r *has constant rank on* M. *If* $\nabla_\xi S = 0$, *then* M *is locally congruent to one of* A_1, A_2, B, C, D, *and* E.

On the other hand, real hypersurfaces of $H_n\mathbb{C}$ have been also investigated by many geometers ([2], [14], [15], [16] etc.) from different points of view. In particular, Berndt proved the following:

THEOREM B. [2] *Let* M *be a real hypersurface of* $H_n\mathbb{C}$. *Then* M *has constant principal curvatures and* ξ *is principal if and only if* M *is locally congruent to one of the following:*

(A_0) *a self-tube, that is, a horosphere,*
(A_1) *a geodesic hypersphere or a tube over a hyperplane* $H_{n-1}\mathbb{C}$,
(A_2) *a tube over a totally geodesic* $H_k\mathbb{C}(1 \leq k \leq n-2)$,
(B) *a tube over a totally real hyperbolic space* $H_n\mathbb{R}$.

For a real hypersurface of $H_n\mathbb{C}$, Ki, Kim and Lee proved the following

THEOREM KM. [7] *Let* M *be a real hypersurface of* $H_n\mathbb{C}$. *If the structure vector* ξ *is principal and if* $\nabla_\xi S = 0$, *then* M *is locally congruent one of* A_0, A_1 *or* A_2.

Denoting by R the curvature tensor of a real hypersurface, we define the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector ξ. Then R_ξ is a self-adjoint endomorphism on the tangent space of the real hypersurface.

In this paper we study a real hypersurface of a complex hyperbolic space $H_n\mathbb{C}$ which satisfies $L_\xi S = 0$ and $R_\xi S = SR_\xi$, where L_ξ denotes the operator of the Lie derivative with respect to ξ.

All manifolds in the present paper are assumed to be connected and of class C^∞.

1. Preliminaries

Let $\tilde{M}$ be a Kaehlerian manifold of real dimension $2n$ equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y an $\tilde{M}$, we have

$$J^2 = -X, G(JX, JY) = G(X, Y), \ \ \tilde{\nabla}J = 0,$$

where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

Let M be a real $(2n-1)$-dimensional hypersurface of $\tilde{M}$ covered by a system of coordinate neighborhoods $\{U; x^h\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. When the argument is local, M need not be distinguished from $i(M)$ itself. Throughout this paper the indices $i, j, k, \cdots$ run form 1 to $2n-1$ and the summation convention will be used. We represent the immersion i locally by

$$y^A = y^A(x^h), \ (A = 1, \cdots, 2n-1, 2n)$$

and put $B_j = (B_j{}^A)$, $(\partial_j = \partial/\partial_x{}^j)$ then B_j are $(2n-1)$-linearly independent local tangent vectors of M. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji} = G_{BA}B_j{}^B B_i{}^A$ because the immersion is isometric.

For the unit normal C to M, the transformations of B_i and C by J are respectively represented in each coordinate neighborhood as follows:

$$JB_i = \phi_i{}^h B_h + \xi_i C, \ \ JC = -\xi^i B_i,$$

where we have put $\phi_{ji} = G(JB_j, B_i)$ and $\xi_i = G(JB_i, C)$, ξ^h being components of a vector fields ξ associated with ξ_i and $\phi_{ji} = \phi_j{}^r g_{ri}$. By the properties of the almost Hermitian structure J, it is clear that ϕ_{ji} is skew-symmetric. A tensor field of type (1,1) with components $\phi_i{}^h$ will be denoted by ϕ. By properties of the almost complex structure J, the following relations are then given.

$$\phi_i{}^r \phi_r{}^h = -\delta_i{}^h + \xi_i \xi^h, \ \ \ \xi^r \phi_r{}^h = 0, \ \ \ \xi_r \phi_i{}^r = 0, \ \ \ \xi_i \xi^i = 1,$$

that is, the aggregate (ϕ, g, ξ) defines an almost contact metric structure.

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g and G, equations of Gauss and Weingarten for M are respectively given by

$$\nabla_j B_i = A_{ji} C, \quad \nabla_j C = -A_j{}^r B_r,$$

where $A = (A_j{}^h)$, which is related by $A_{ji} = A_j{}^r g_{ri}$ is the shape operator in direction C. By means of above equations the covariant derivatives of the structure tensors are yielded:

$$(1.1) \qquad \nabla_j \phi_i{}^h = A_j{}^h \xi_i - A_{ji} \xi^h, \quad \nabla_j \xi_i = -A_{jr} \phi_i{}^r.$$

If the ambient space $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature c, which is called a complex space form and denoted $M_n(c)$, then equations of Gauss and Codazzi are respectively given by

$$(1.2) \qquad \begin{aligned} R_{kjih} = {} & \frac{c}{4}(g_{kh} g_{ji} - g_{jh} g_{ki} + \phi_{kh} \phi_{ji} - \phi_{jh} \phi_{ki} - 2\phi_{kj} \phi_{ih}) \\ & + A_{kh} A_{ji} - A_{jh} A_{ki}, \end{aligned}$$

$$(1.3) \qquad \nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}),$$

where R_{kjih} are components of the Riemann-Christoffel curvature tensor R of M.

From (1.2) the Ricci tensor S of type (1,1) with components $S_j{}^h$ is verified that

$$(1.4) \qquad S_j{}^h = \frac{c}{4}\{(2n+1)\delta_j{}^h - 3\xi_j \xi^h\} + h A_j{}^h - A_j{}^r A_r{}^h,$$

where $h = TrA$. Hereafter, to write our formulas in convention form, we denote by $A_{ji}^2 = A_{jr} A_i{}^r, \alpha = A_{ji} \xi^j \xi^i, \beta = A_{ji}^2 \xi^j \xi^i$, and $\gamma = A_{ji}^3 \xi^j \xi^i$.

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector ξ. Because of properties of the almost contact metric structure and the second equation of (1.1), we have

$$(1.5) \qquad U^r \nabla_j \xi_r = A_{jr}^2 \xi^r - \alpha A_{jr} \xi^r,$$

$$\phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j, \tag{1.6}$$

which shows that $g(U,U) = \beta - \alpha^2$.

Differentiating (1.6) covariantly along M and using (1.1), we find

$$\xi_j(A_{kr}U^r + \alpha_k) + \phi_j{}^r\nabla_k U_r = \xi^r\nabla_k A_{jr} - A_{jr}A_{ks}\phi^{rs} + \alpha A_{kr}\phi_j{}^r, \tag{1.7}$$

which unable us to obtain

$$(\nabla_k A_{rs})\xi^r\xi^s = 2A_{kr}U^r + \alpha_k, \tag{1.8}$$

where $\alpha_k = \partial_k\alpha$.

Now, we put

$$A\xi = \alpha\xi + \mu W, \tag{1.9}$$

where μ is a function on M, and W is a unit vector orthogonal to ξ. Then we have $\mu^2 = \beta - \alpha^2$ and $\phi U = -\mu W$. Hence it is, using (1.1), seen that

$$\mu\xi^r\nabla_j W_r = A_{jr}U^r \tag{1.10}$$

because ξ and W are mutually orthogonal.

2. Real hypersurfaces satisfying $L_\xi S = 0$

Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0$. By definition, the Lie derivative of the Ricci tensor S with respect to the structure vector ξ is given by

$$L_\xi S_j{}^h = \xi^r\nabla_r S_j{}^h + (\nabla_j\xi^r)S_r{}^h - (\nabla_r\xi^h)S_j{}^r,$$

or using the second equation of (1.1),

$$L_\xi S_j{}^h = \xi^r\nabla_r S_j{}^h + A_{rt}\phi^{ht}S_j{}^r - A_{jt}\phi^{rt}S_r{}^h.$$

In what follows we assume that the Ricci tensor S satisfies $L_\xi S = 0$, that is,

$$\xi^r\nabla_r S_{ji} = A_{jt}\phi_r{}^t S_i{}^r - A_{rt}\phi_i{}^t S_j{}^r, \tag{2.1}$$

which shows that

$$(A_{jt}\phi_r{}^t + A_{rt}\phi_j{}^t)S_i{}^r = (A_{it}\phi_r{}^t + A_{rt}\phi_i{}^t)S_j{}^r. \tag{2.2}$$

From (1.4) we get

$$S_{jr}\xi^r = \frac{c}{2}(n-1)\xi_j + hA_{jr}\xi^r - A_{jr}{}^2\xi^r, \tag{2.3}$$

$$S_{jr}\phi_i{}^r + S_{ir}\phi_j{}^r = h(A_{jr}\phi_i{}^r + A_{ir}\phi_j{}^r) - A_{jr}{}^2\phi_i{}^r - A_{ir}{}^2\phi_j{}^r. \tag{2.4}$$

Because of (2.2) and (2.4), it follows that

$$(S_{jr}\phi_i{}^r + S_{ir}\phi_j{}^r)(A_t{}^j\phi^{it} + A_t{}^i\phi^{jt}) = 0.$$

Hence, by applying $A_t{}^j\phi^{it}$ to (2.2) and making use of (2.3) we obtain (see [11])

$$||S\phi - \phi S||^2 + \frac{3}{2}c||U||^2 = 0. \tag{2.5}$$

Therefore if $c > 0$, then we have $S\phi = \phi S$ and $U = 0$, and consequently α is locally constant on M ([8]). Using the fact that $A\xi = \alpha\xi$, it is clear that $SA = AS$. Hence (2.1) is reduced to $\nabla_\xi S = 0$.

Now, suppose that $g(S\xi, \xi) = const.$ Then by (2.3) we have $g(S\xi, \xi) = \frac{c}{2}(n-1) - \alpha^2 + h\alpha$ by virtue of $\beta - \alpha^2 = 0$.

According to Theorem M, we have

THEOREM 2.1. *Let M be a real hypersurface satisfying $L_\xi S = 0$ in a complex projective space $P_n\mathbb{C}(\geq 3)$. If $g(S\xi, \xi) = const.$, then M is locally congruent to one of A_1, A_2, B, C, D, and E provided that $g(A\xi, \xi) \neq 0$.*

For a real hypersurface of a complex hyperbolic space $H_n\mathbb{C}$, it is known that

THEOREM K. [5] *Let M be a real hypersurface of $H_n\mathbb{C}$. If it satisfies $L_\xi S = 0$ and $S\xi = \sigma\xi$ for some function σ on M, then ξ is principal.*

3. Jacobi operators of real hypersurfaces

Let M be a real hypersurface satisfying $L_\xi S = 0$ in a complex hyperbolic space $H_n\mathbb{C}$. We define the Jacobi operator field $R_X = R(\cdot, X)X$ with respect to a unit vector field X. Then from (1.2) we have

$$(R_\xi)_{ji} = \frac{c}{4}(g_{ji} - \xi_j\xi_i) + \alpha A_{ji} - (A_{jr}\xi^r)(A_{is}\xi^s).$$

Suppose that $R_\xi S = SR_\xi$. Then we have

$$\begin{aligned}&(A_{jr}^{\ 3}\xi^r)(A_{is}\xi^s) - (A_{ir}^{\ 3}\xi^r)(A_{js}\xi^s)\\ &= (A_{jr}^{\ 2}\xi^r)(hA_{is}\xi^s - \frac{c}{4}\xi_i) - (A_{ir}^{\ 2}\xi^r)(hA_{js}\xi^s - \frac{c}{4}\xi_j)\\ &+ \frac{c}{4}h(\xi_i A_{jr}\xi^r - \xi_j A_{ir}\xi^r),\end{aligned}$$

which implies that

$$\alpha A_{jr}^{\ 3}\xi^r = (\alpha h - \frac{c}{4})A_{jr}^{\ 2}\xi^r + (\gamma - \beta h + \frac{c}{4}h)A_{jr}\xi^r + \frac{c}{4}(\beta - h\alpha)\xi_j. \tag{3.1}$$

Combining the last two equations, it follows that

$$(A_{jr}^{\ 2}\xi^r)(A_{is}\xi^s - \alpha\xi_i) - (A_{ir}^{\ 2}\xi^r)(A_{js}\xi^s - \alpha\xi_j) = \beta(\xi_j A_{jr}\xi^r - \xi_i A_{jr}\xi^r).$$

Multiplying $A_s^{\ j}\xi^s$ to the last equation and summing for j, we find

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi.$$

If $\mu \neq 0$, then we have

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{3.2}$$

where we have put $\mu^2\rho = \gamma - \beta\alpha$, $\mu^2(\beta - \rho\alpha) = \beta^2 - \alpha\gamma$, which shows that

$$A^3\xi = (\rho^2 + \beta - \rho\alpha)A\xi + \rho(\beta - \rho\alpha)\xi.$$

Thus (3.1) implies that $(h - \rho)(\beta - \rho\alpha - \frac{c}{4}) = 0$ because of $\mu \neq 0$. Therefore we have

$$\mu(h - \rho)(\beta - \rho\alpha - \frac{c}{4}) = 0 \tag{3.3}$$

on M.

Let Ω_0 be a set of points in M such that $\mu(\beta-\rho\alpha-\frac{c}{4})\neq 0$. Then we have $h=\rho$ on Ω_0. Thus (3.2) turns out to be $A^2\xi=hA\xi+(\beta-h\alpha)\xi$ and hence $S\xi=\sigma\xi$ on Ω_0 because of (2.3), where we have put $\sigma=\frac{c}{2}(n-1)+h\alpha-\beta$. Owing to Theorem K, it is seen that $A\xi=\alpha\xi$, a contradiction. Therefore (3.3) is reduces to

$$\mu(\beta-\rho\alpha-\frac{c}{4})=0 \tag{3.4}$$

on M.

In the following we assume that $\mu\neq 0$ on M, namely, ξ is not a principal curvature vector and we put $\Omega=\{p\in M:\mu(p)\neq 0\}$. Then Ω is an open subset of M. From now on we discuss our arguments on Ω.

From (3.4) we have

$$\beta=\rho\alpha+\frac{c}{4}. \tag{3.5}$$

Thus (3.2) becomes

$$A^2\xi=\rho A\xi+\frac{c}{4}\xi. \tag{3.6}$$

From (1.9) and (3.6), we obtain

$$AW=\mu\xi+(\rho-\alpha)W \tag{3.7}$$

because of $\mu\neq 0$, which enable us to obtain

$$A^2W=\rho AW+\frac{c}{4}W. \tag{3.8}$$

Making use of (3.5), (3.6), and (3.7), the relationship (1.5) turns out to be

$$U^r\nabla_j\xi_r=\mu A_{jr}W^r. \tag{3.9}$$

Differentiating (3.7) covariantly along M, we find

$$\begin{aligned}&(\nabla_k A_{jr})W^r+A_{jr}\nabla_k W^r\\ &=\mu_k\xi_j+\mu\nabla_k\xi_j+(\rho_k-\alpha_k)W_j+(\rho-\alpha)\nabla_k W_j.\end{aligned} \tag{3.10}$$

Applying this by W^j and taking account of (1.10) and (3.7), we find

$$(\nabla_k A_{rs}) W^r W^s = -2A_{kr}U^r + \rho_k - \alpha_k \tag{3.11}$$

because ξ and W are mutually orthogonal. In the same way, we have from (3.10)

$$\mu(\nabla_k A_{rs}) W^r \xi^s = (\rho - 2\alpha) A_{kr} U^r + \mu\mu_k, \tag{3.12}$$

or using the Codazzi equation (1.3) and the fact that $\mu^2 = \beta - \alpha^2$,

$$\mu(\nabla_r A_{ks}) W^r \xi^s = (\rho - 2\alpha) A_{kr} U^r - \frac{c}{2} U_k + \frac{1}{2}\beta_k - \alpha\alpha_k. \tag{3.13}$$

Differentiating (3.6) covariantly and using (1.9), we find

$$\begin{aligned}(\alpha - \rho)(\nabla_k A_{jr})\xi^r + \mu(\nabla_k A_{jr}) W^r + A_{jr}(\nabla_k A_s{}^r)\xi^s \\ = \rho_k A_{jr}\xi^r - A_{jr}^2 \nabla_k \xi^r + \rho A_{jr} \nabla_k \xi^r + \frac{c}{4}\nabla_k \xi_j,\end{aligned} \tag{3.14}$$

from which, making use of (1.3), (1.8), and (3.12),

$$3A_{jr}^2 U^r - 2\rho A_{jr} U^r - \frac{c}{2} U_j = (\rho_t \xi^t) A_{jr}\xi^r - A_{jr}\alpha^r + \rho\alpha_j - \frac{1}{2}\beta_j. \tag{3.15}$$

If we take the skew-symmetric part of (3.14) and using (1.1) and (1.3), then we obtain

$$\begin{aligned}& A_{ks}(\nabla_j A_r{}^s)\xi^r - A_{js}(\nabla_k A_r{}^s)\xi^r + \rho_k A_{jr}\xi^r - \rho_j A_{kr}\xi^r \\ = & \frac{c}{4}(U_k \xi_j - U_j \xi_k) + \frac{c}{2}(\rho - \alpha)\phi_{kj} + A_{kr}^2 A_{js}\phi^{rs} - A_{jr}^2 A_{ks}\phi^{rs} \\ & + 2\rho A_{jr} A_{ks}\phi^{rs} + \frac{c}{4}(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r).\end{aligned}$$

Applying this by μW^j and taking account of (1.3), (1.8), (1.9), (3.7), (3.8), and (3.13), we find

$$\begin{aligned}(3\alpha - 2\rho) A_{jr}^2 U^r + (2\rho^2 - 2\rho\alpha + c) A_{jr} U^r + \frac{c}{4}(\alpha - \rho) U_j \\ = \mu A_{jr}\mu^r + (\alpha - \rho)\mu\mu_j + \mu^2(\rho_j - \alpha_j) - \mu(\rho_t W^t) A_{jr}\xi^r.\end{aligned} \tag{3.16}$$

On the other hand, since we have (3.6), the equation (2.3) is reduces to

$$S_{jr}\xi^r = \frac{c}{4}(2n - 3)\xi_j + (h - \rho) A_{jr}\xi^r. \tag{3.17}$$

Applying (2.2) by ξ^j and using (1.4), (3.6), and (3.17), we get

$$A_{jr}^{2}U^r = (2h-\rho)A_{jr}U^r + (\rho^2 - h\rho + c)U_j. \tag{3.18}$$

Because of (1.4), (3.7), and (3.8), it follows that

$$S_{jr}W^r = \mu(h-\rho)\xi_j + xW_j, \tag{3.19}$$

where we have put $x = \frac{c}{2}n + (\rho-\alpha)(h-\rho)$.

If we transform μW^i to (2.2) and take account of (3.7) and (3.18) we also find

$$\{2(\rho-h)^2 + \frac{3}{4}c\}A_{jr}U^r = \{\rho(\rho-h)^2 - \frac{3}{4}c(\alpha+h-2\rho)\}U_j. \tag{3.20}$$

From (3.17) we have

$$\begin{aligned}(\nabla_k S_{jr})\xi^r + S_{jr}\nabla_k\xi^r = &\frac{c}{4}(2n-3)\nabla_k\xi_j + (h_k-\rho_k)A_{jr}\xi^r \\ &+ (h-\rho)(\nabla_k A_{jr})\xi^r + (h-\rho)A_{jr}\nabla_k\xi^r,\end{aligned}$$

which together with (1.4), (1.8), (2.1), and (3.17) gives

$$3(\rho-h)A_{jr}U^r - \rho(\rho-h)U_j = (h_t\xi^t - \rho_t\xi^t)A_{jr}\xi^r + (h-\rho)\alpha_j. \tag{3.21}$$

LEMMA 3.1. *$\alpha_t\xi^t = 0, \alpha_t W^t = 0, \rho_t\xi^t = 0, \rho_t W^t = 0$, and $h_t\xi^t = 0$ on Ω.*

Proof. Applying (3.21) by ξ^j or W^j and making use of (1.9), we obtain respectively

$$\begin{aligned}&\alpha(h_t\xi^t - \rho_t\xi^t) + (h-\rho)\alpha_t\xi^t = 0,\\ &\mu(h_t\xi^t - \rho_t\xi^t) + (h-\rho)\alpha_t W^t = 0,\end{aligned} \tag{3.22}$$

which enable us to obtain

$$\mu\alpha_t\xi^t = \alpha\alpha_t W^t. \tag{3.23}$$

By the way, combining (3.11) and (3.12), we have

$$\mu(\rho_t\xi^t - \alpha_t\xi^t) = \frac{1}{2}\beta_t W^t - \alpha\alpha_t W^t,$$

where we have used the fact that $\mu^2 = \beta - \alpha^2$. Thus, it follows that

$$\beta_t W^t = 2\mu\rho_t\xi^t. \tag{3.24}$$

Next, multiplying (3.15) with ξ^j and summing for j, and using (1.9) and (3.5), we find

$$2\mu\alpha_t W^t = \alpha\rho_t\xi^t + (\rho - 2\alpha)\alpha_t\xi^t,$$

which together with (3.23) yields

$$\alpha^2\rho_t\xi^t + (\rho\alpha - 2\beta)\alpha_t\xi^t = 0. \tag{3.25}$$

Because of (2.1) and (3.19), it is seen that

$$\xi^r(\nabla_r S_{ji})W^j W^i = 0. \tag{3.26}$$

Differentiating (3.19) covariantly, we find

$$(\nabla_k S_{jr})W^r + S_{jr}\nabla_k W^r = x_k W_j + x\nabla_k W_j + \{\mu(h-\rho)\}_k\xi_j + \mu(h-\rho)\nabla_k\xi_j.$$

If we apply this by $\xi^k W^j$ and take account of (1.10), (3.19), and (3.26), then we get $x_t\xi^t = 0$. By definition, it follows that

$$(h-\rho)(\rho_t\xi^t - \alpha_t\xi^t) + (\rho - \alpha)(h_t\xi^t - \rho_t\xi^t) = 0,$$

which together with (3.22) implies that

$$\alpha\rho_t\xi^t = \rho\alpha_t\xi^t$$

because $h - \rho \neq 0$ on Ω. From this and (3.25) we verify that $(\beta - \rho\alpha)\alpha_t\xi^t = 0$ and hence $\alpha_t\xi^t = 0$ by virtue of (3.5).

REMARK 1. We notice here that $\alpha \neq 0, \rho \neq 0$ or $\rho \neq \alpha$ on Ω because of (3.5) and $c < 0$.

From this fact it is seen that $\rho_t\xi^t = 0$ on Ω. If we take account of (3.22), (3.23), and (3.24), then we see respectively that $h_t\xi^t = 0, \alpha_t W^t = 0$ and $\beta_t W^t = 0$. From the last relation and (3.5) it is seen that $\rho_t W^t = 0$. This completes the proof. □

4. Real hypersurfaces satisfying $L_\xi S = 0$ and $R_\xi S = SR_\xi$

In the rest of this paper we shall suppose that M is a $(2n-1)$-dimensional real hypersurface in a complex hyperbolic space $H_n\mathbb{C}$ and that the Ricci tensor S satisfies $L_\xi S = 0$ and $R_\xi S = SR_\xi$ on M. Then (3.21) is reduces to

$$\nabla\alpha = \rho U - 3AU \tag{4.1}$$

because of Lemma 3.1 and the fact that $\rho - h \neq 0$ on Ω, which together with (3.15) and Lemma 3.1 gives

$$\alpha\nabla\rho = (\rho^2 + c)U - \rho AU. \tag{4.2}$$

From the last two equations, it follows that

$$\mu\nabla\mu = (\rho^2 - \rho\alpha + \frac{c}{2})U + (3\alpha - 2\rho)AU. \tag{4.3}$$

Substituting (4.1)–(4.3) into (3.16) and making use of Lemma 3.1, we find

$$(\rho - \alpha)AU = \{(\rho - \alpha)(\rho + 3\alpha) + c\}U.$$

Thus we have $AU = \lambda U$, where we have the function λ defined by

$$\lambda = \rho + 3\alpha + \frac{c}{\rho - \alpha} \tag{4.4}$$

because of Remark 1. Thus (4.1) and (4.2) are respectively reduces to

$$\nabla\alpha = (\rho - 3\lambda)U, \quad \nabla\rho = (3\alpha - \lambda - 2\rho)U \tag{4.5}$$

with the aid of Remark 1.

Since we have $AU = \lambda U$, (3.18) and (3.20) turn out respectively to be

$$(2\lambda - \rho)h = \lambda^2 + \rho\lambda - \rho^2 - c, \tag{4.6}$$

$$(2\lambda - \rho)(\rho - h)^2 = \frac{3}{4}c(2\rho - \alpha - h - \lambda). \tag{4.7}$$

On the other hand applying (2.2) by U^iW^j and using (1.4), (3.17), and (3.19), we find

$$\mu^2(\rho - h) = \{h\lambda - \lambda^2 - (h-\rho)(\rho-\alpha) + \frac{c}{4}\}(\lambda - \rho + \alpha),$$

which together with (4.6) implies that

$$\mu^2(\rho - h) = (\lambda - \rho + \alpha)\{(\rho - h)(\lambda - \alpha) - \frac{3}{4}c\}. \tag{4.8}$$

Using (3.5) and (4.4), it is seen that

$$(\rho - \alpha)(\lambda + \alpha - \rho) = 4\mu^2. \tag{4.9}$$

Thus $\alpha - \rho + \lambda$ does not vanish on Ω. Differentiation gives

$$(\rho - \alpha)\nabla\lambda = 8\mu\nabla\mu + (2\rho - 2\alpha - \lambda)(\nabla\rho - \nabla\alpha),$$

which connected with (3.5) gives

$$(\rho - \alpha)\nabla\lambda = (2\alpha + 2\rho - \lambda)\nabla\rho + (2\rho - 6\alpha + \lambda)\nabla\alpha.$$

Making use of (4.5), we have $(3\alpha - \lambda - 2\rho)\nabla\alpha = (\rho - 3\lambda)\nabla\rho$. Therefore the last equation turns out to be

$$(\rho - \alpha)\nabla\lambda = (2\alpha + 5\lambda)\nabla\rho - (\lambda + 2\rho)\nabla\alpha. \tag{4.10}$$

If we take account of (4.8) and (4.9), then we obtain

$$(\rho - h)(4\lambda - 3\alpha - \rho) = 3c \tag{4.11}$$

because $\lambda + \alpha - \rho$ does not vanish on Ω, which together with (4.6) implies that

$$\lambda(\rho - \lambda)(4\lambda - 3\alpha - \rho) = c(2\lambda + 3\alpha - 2\rho). \tag{4.12}$$

Now, we prove

LEMMA 4.1. *Let M be a real hypersurface with $L_\xi S = 0$ and $R_\xi S = SR_\xi$ in $H_n\mathbb{C}$. If $g(S\xi,\xi) = const.$, then Ω is empty.*

Proof. By (3.17) we have

$$\alpha(h-\rho) + \frac{c}{4}(2n-3) = g(S\xi,\xi).$$

Thus we obtain $\alpha(h-\rho) = const.$ Hence if we differentiate (4.11), we find

$$(4\lambda-\rho)(\nabla\rho - \nabla h) + (\rho - h)(4\nabla\lambda - \nabla\rho) = 0,$$

which enable us to obtain

$$4\alpha\nabla\lambda = \alpha\nabla\rho + (4\lambda-\rho)\nabla\alpha$$

because $\rho - h$ does not zero on Ω. From this and (4.10) we verify that

$$4\rho\nabla\lambda = (9\alpha + 20\lambda)\nabla\rho - 9\rho\nabla\alpha. \tag{4.13}$$ □

On the other hand, from (4.11) we have $(\rho-h)(4\lambda-\rho) = 3a$, where we have put

$$\alpha(\rho-h) + c = a. \tag{4.14}$$

Hence we obtain

$$4\lambda = \rho + b\alpha, \tag{4.15}$$

where the constant b is defined by $(a-c)b = 3a$. From this and (4.13) it follows that

$$(9\alpha + 5b\alpha + 4\rho)\nabla\rho = (9\rho + b\rho)\nabla\alpha, \tag{4.16}$$

which together with (4.5) yields

$$(b+45)\rho^2 + (22b - 3b^2 + 33)\rho\alpha + (5b^2 - 51b - 108)\alpha^2 = 0. \tag{4.17}$$

Therefore ρ/α is a root of algebraic equation with constant coefficient and hence $\rho = \epsilon\alpha$ for some constant ϵ on Ω, which together with (4.16) gives

$$(b+\epsilon)\nabla\alpha = 0. \tag{4.18}$$

If $b+\epsilon=0$, then (4.17) implies that $(b+1)(b-3)(b+9)=0$ by virtue of Remark 1. By definition and equations (4.9) and (4.15), it is clear that $(b+1)(b-3)\neq 0$. Thus we have $b+9=0$, which together with (4.15) and (4.16) gives $\lambda\nabla\rho=0$. Since $\lambda\neq 0$ on Ω because of (4.9), it follows that $\nabla\rho=0$. From this and the second equation of (4.5) and (4.15) we see that $\nabla\alpha=0$. Consequently it is seen that $\alpha=const.$ Thus (4.5) implies that $\rho=3\lambda$ and $3\alpha-\lambda-2\rho=0$, and thus $7\lambda=3\alpha$ and $7\rho=9\alpha$. From these and (4.12) we get $36\alpha^2+7c=0$. We also, using (3.5) and (4.9), see that $54\alpha^2+49c=0$, which produces a contradiction. Therefore, it is seen that Ω is void. This completes the proof the lemma.

According to Lemma 4.1 and Theorem KM, we have

THEOREM 4.2. *Let M be a real hypersurface satisfying $L_\xi S=0$ and $R_\xi S=SR_\xi$ in a complex hyperbolic space $H_n\mathbb{C}$. If $g(S\xi,\xi)=const.$, then M is of type* A_0, A_1 *or* A_2, *where S denotes the Ricci tensor of type $(1,1)$ on M, and L_ξ the operator of the Lie derivative, R_ξ the Jacobi operator with respect to the structure vector ξ.*

Proof. Because of Lemma 4.1, (2.5) gives $S\phi=\phi S$. Further we also have from (1.4) $SA=AS$. Thus (2.1) turns out to be $\nabla_\xi S=0$. By Theorem KM, we arrive at the conclusions. □

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