

ON EXCHANGE IDEALS

HUANYIN CHEN

ABSTRACT. In this paper, we investigate exchange ideals and get some new characterization of exchange rings. It is shown that an ideal I of a ring R is an exchange ideal if and only if so is $QM_2(I)$. Also we observe that every exchange ideal can be characterized by exchange elements.

1. Introduction

A ring R is an exchange ring if for every right R -module A and two decompositions $A = M \oplus N = \bigoplus_{i \in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A'_i \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i \in I} A'_i)$ (see [2]). It is well known that a ring R is an exchange ring if and only if for any $x \in R$ there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$. Following Ara (cf. [1]), we say that an ideal I of a ring R is an exchange ideal if for any $x \in I$ there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$. We observe that every exchange ring can be characterized by exchange ideals.

Let I be an ideal of a ring R . Define

$$QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in R \right\},$$
$$QM_2(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in I \right\}.$$

It is easy to verify that $QM_2(I)$ is an ideal of $QM_2(R)$. Define

$$Q^T M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = c + d, a, b, c, d \in R \right\},$$
$$Q^T M_2(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = c + d, a, b, c, d \in I \right\}.$$

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Then $Q^T M_2(I)$ is an ideal of $Q^T M_2(R)$. We observe that an ideal I is an exchange ideal if and only if so is $QM_2(I)$, if and only if so is $Q^T M_2(I)$.

We say that $x \in R$ is an exchange element in case there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$. Every idempotent in a ring is an exchange element. But the converse is not true. We prove that every exchange ring can be characterized by exchange elements.

Throughout the paper, every ring is associative with the identity. We always use $J(R)$ to denote the Jacobson radical of R .

2. Exchange rings

According to [1, Theorem 2.2], R is an exchange ring if and only if R/I is an exchange ring, I is an exchange ideal and idempotents lift modulo I . Now we extend this result and give several new characterizations of exchange rings by exchange ideals.

THEOREM 2.1. *Let I be an exchange ideal of a ring R . Then the following are equivalent:*

- (1) R is an exchange ring,
- (2) For any $a \in R$, there exists an idempotent $e \in R$ such that $e \in Ra + I$ such that $1 - e \in R(1 - a) + I$.

Proof. (1) \Rightarrow (2) is clear by [9, Proposition 1.1].

(2) \Rightarrow (1) For any $x + I \in R/I$, there exists an idempotent $e \in R$ such that $e \in Rx + I$ such that $1 - e \in R(1 - x) + I$. Hence we have an idempotent $e + I \in (R/I)(x + I)$ such that $(1 + I) - (e + I) \in (R/I)((1 + I) - (x + I))$. By [9, Proposition 1.1], R/I is an exchange ring.

Given $x - x^2 \in I$, then we have an idempotent $e \in R$ such that $e \in Rx + I$ such that $1 - e \in R(1 - x) + I$. So $e - x = e(1 - x) - (1 - e)x \in R(x - x^2) + I \subseteq I$. That is, every idempotent lifts modulo I . According to [1, Theorem 2.2], R is an exchange ring. \square

COROLLARY 2.2. *The following are equivalent:*

- (1) R is an exchange ring,
- (2) For any $a \in R$, there exists an idempotent $e \in R$ such that $e \in Ra + J(R)$ such that $1 - e \in R(1 - a) + J(R)$.

Proof. Since $J(R)$ is an exchange ideal of R , we complete the proof by Theorem 2.1. \square

COROLLARY 2.3. *The following are equivalent:*

- (1) R is an exchange ring,
- (2) For any $a \in R$, there exist an idempotent $e \in Ra + J(R)$ and a $c \in R$ such that $(1 - e) - c(1 - a) \in J(R)$,
- (3) For any $a \in R$, there exists an idempotent $e \in Ra + J(R)$ such that $R = Re + R(1 - a)$.

Proof. (1) \Rightarrow (2) is clear by [9, Proposition 1.1].

(2) \Rightarrow (3) For any $x \in R$, we have an idempotent $e \in Rx + J(R)$ and a $c \in R$ such that $(1 - e) - c(1 - x) \in J(R)$. So $e + c(1 - x) = 1 + r$ for a $r \in J(R)$. Clearly, $1 + r \in U(R)$; hence, $(1 + r)^{-1}e + (1 + r)^{-1}c(1 - x) = 1$. This means that $R = Re + R(1 - x)$.

(3) \Rightarrow (1) For any $x \in R$, there exist an idempotent $e \in Rx + J(R)$ such that $R = Re + R(1 - x)$. So we have $r, s \in R$ such that $re + s(1 - x) = 1$. Let $f = e + (1 - e)re$. Then we check that $f = f^2 \in Rx + J(R)$. Furthermore, we have $1 - f = (1 - e)(1 - re) = (1 - e)s(1 - x) \in R(1 - x)$, as required. \square

THEOREM 2.4. *Let I be an exchange ideal of a ring R . Then the following are equivalent:*

- (1) R is an exchange ring,
- (2) For any $a \in R$, there exists an idempotent $e \in R$ such that $e - a \in R(a - a^2) + I$.

Proof. (1) \Rightarrow (2) is clear by [9, Proposition 1.1].

(2) \Rightarrow (1) For any $x \in R$, there exists an idempotent $e \in R$ such that $e - x \in R(x - x^2) + I$. Assume now that $e - x = r(x - x^2) + s$ for $r \in R, s \in I$. Then we have $e = (1 + r(1 - x))x + s \in Rx + I$ such that $1 - e = (1 - rx)(1 - x) - s \in R(1 - x) + I$. According to Theorem 2.1, R is an exchange ring. \square

COROLLARY 2.5. *Let I be an exchange ideal of a ring R . If for any $a \in R$ there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a \equiv e + u \pmod{I}$. Then R is an exchange ring.*

Proof. Given any $x \in R$, we have an idempotent $e \in R$ and a unit $u \in R$ such that $x \equiv e + u \pmod{I}$. Clearly, we have $u(x - u^{-1}(1 - e)u) \equiv x^2 - x \pmod{I}$. Set $f = u^{-1}(1 - e)u$. Then $f = f^2 \in R$. In addition, we get $x - f \in R(x - x^2) + I$. Consequently, R is an exchange ring by Theorem 2.4. \square

We end this section by asking three problems:

- (1) If for any $x \in I$ there exists an idempotent $e \in R$ such that $e \in xR + J(R)$ and $1 - e \in (1 - x)R + J(R)$, is I an exchange ideal?

- (2) If for any $x \in I$ there exists an idempotent $e \in R$ such that $eR + J(R) = xR + J(R)$, is I an exchange ideal of R ?
- (3) If there exists an idempotent $e \in R$ such that eIe and $(1-e)I(1-e)$ are exchange ideals of eR and $(1-e)R(1-e)$ respectively, is I an exchange ideal of R ?

3. Extensions

In [8], Hong et. al. proved that every triangular ring over an exchange ring is also an exchange ring. Now we extend this result and get several necessary and sufficient conditions under which an ideal of a ring is an exchange ideal.

THEOREM 3.1. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) I is an exchange ideal of R ,
 (2) $QM_2(I)$ is an exchange ideal of $QM_2(R)$.

Proof. (1) \Rightarrow (2) Let $TM_2(R)$ denote the ring of all 2×2 lower triangular matrices over R , and let $TM_2(I)$ denote the ideal of all 2×2 lower triangular matrices over I . We construct a map $\psi : QM_2(R) \rightarrow TM_2(R)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+c & 0 \\ c & d-c \end{pmatrix}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$. For any $\begin{pmatrix} x & 0 \\ z & y \end{pmatrix} \in TM_2(R)$, we have $\psi \left(\begin{pmatrix} x-z & x-y-z \\ z & y+z \end{pmatrix} \right) = \begin{pmatrix} x & 0 \\ z & y \end{pmatrix}$. Thus it is easy to verify that ψ is a ring isomorphism. Also we get $\psi|_{QM_2(I)} : QM_2(I) \cong TM_2(I)$. Thus it suffices to prove that $TM_2(I)$ is an exchange ideal of $TM_2(R)$.

Given any idempotent matrix $\begin{pmatrix} e & 0 \\ ** & f \end{pmatrix} \in TM_2(I)$, we have $e = e^2, f = f^2 \in I$. In addition, we get $\begin{pmatrix} e & 0 \\ ** & f \end{pmatrix} TM_2(R) \begin{pmatrix} e & 0 \\ ** & f \end{pmatrix} \cong \begin{pmatrix} eRe & 0 \\ ** & fRf \end{pmatrix}$. Since eRe and fRf are both exchange ring, by [9, Corollary 2.6], $\begin{pmatrix} eRe & 0 \\ ** & fRf \end{pmatrix}$ is an exchange ring. Therefore we conclude that $TM_2(I)$ is an exchange ideal of $TM_2(R)$, as required.

(2) \Rightarrow (1) As $QM_2(I)$ is an exchange ideal of $QM_2(R)$, we deduce that $TM_2(I)$ is an exchange ideal of $TM_2(R)$. Choose $e = \text{diag}(1, 0) \in TM_2(R)$. Then $eTM_2(I)e$ is an exchange ideal of $eTM_2(R)e$. Clearly,

$I \cong eTM_2(I)e$ and $R \cong eTM_2(R)e$. By using [1, Proposition 1.3], we prove that I is an exchange ideal of R , as asserted. \square

COROLLARY 3.2. *Let I be a π -regular ideal of a ring R . Then $QM_2(I)$ is an exchange ideal of $QM_2(R)$.*

Proof. Since every π -regular ideal is an exchange ideal, the result follows by Theorem 3.1. \square

It is well known that every exchange ring is left and right symmetric. We extend this fact as follows.

LEMMA 3.3. *Let I be an ideal of a ring R . Then I is an exchange ideal of R if and only if I° is an exchange ideal of the opposite ring R° .*

Proof. Suppose that I is an exchange ideal of R . Given any idempotent $e^\circ \in I^\circ$, then $e \in I$ is an idempotent. Hence eRe is an exchange ring. So $(eRe)^\circ$ is an exchange ring as well. That is, $e^\circ R^\circ e^\circ$ is an exchange ring. Therefore the proof is true. \square

THEOREM 3.4. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) I is an exchange ideal of R ,
- (2) $Q^T M_2(I)$ is an exchange ideal of $Q^T M_2(R)$.

Proof. Construct a map: $\psi : Q^T M_2(R) \rightarrow QM_2(R^\circ)$ given by

$$\psi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a^\circ & c^\circ \\ b^\circ & d^\circ \end{pmatrix}$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q^T M_2(R)$. Given $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in Q^T M_2(R)$, we have

$$\begin{aligned} \psi \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) &= \psi \left(\begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix} \right) \\ &= \begin{pmatrix} (a_1 a_2 + b_1 c_2)^\circ & (c_1 a_2 + d_1 c_2)^\circ \\ (a_1 b_2 + b_1 d_2)^\circ & (c_1 b_2 + d_1 d_2)^\circ \end{pmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi \left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \psi \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right) &= \begin{pmatrix} a_2^\circ & c_2^\circ \\ b_2^\circ & d_2^\circ \end{pmatrix} \begin{pmatrix} a_1^\circ & c_1^\circ \\ b_1^\circ & d_1^\circ \end{pmatrix} \\ &= \begin{pmatrix} a_2^\circ a_1^\circ + c_2^\circ b_1^\circ & a_2^\circ c_1^\circ + c_2^\circ d_1^\circ \\ b_2^\circ a_1^\circ + d_2^\circ b_1^\circ & b_2^\circ c_1^\circ + d_2^\circ d_1^\circ \end{pmatrix}. \end{aligned}$$

Thus ψ is an anti-isomorphism; hence, $Q^T M_2(R) \cong (QM_2(R^o))^o$. Likewise, we have $Q^T M_2(I) \cong (QM_2(I^o))^o$. Therefore we get the result by Lemma 3.3 and Theorem 3.1. \square

COROLLARY 3.5. *Let R be a ring. Then the following are equivalent:*

- (1) R is an exchange ring,
- (2) $QM_2(R)$ is an exchange ring,
- (3) $Q^T M_2(R)$ is an exchange ring.

Proof. It is clear by Theorem 3.1 and Theorem 3.4. \square

Let I be an ideal of a ring R , M a R - R -bimodule. The module extension of R by M is the ring $M(R)$ with the usual addition and multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$ for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. The module extension of I by M is the ideal $M(I)$ with the usual addition and multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$ for $r_1, r_2 \in I$ and $m_1, m_2 \in M$.

THEOREM 3.6. *Let I be an ideal of a ring R , M an R - R -bimodule. Then $M(I)$ is an exchange ideal of $M(R)$ if and only if so is I .*

Proof. Construct a map $\psi : R \rightarrow M(R)/J(M(R))$ given by $x \mapsto (x, 0) + J(M(R))$ for any $x \in R$. Given any $(x, m) + J(M(R)) \in M(R)/J(M(R))$, we have $(x, m) = (x, 0) + (0, m) + J(M(R)) = (x, 0) + J(M(R)) = \psi(x)$. Hence ψ is epimorphic. On the other hand, we claim that $\text{Ker } \psi = \{x \in R \mid (x, m) \in J(M(R))\} = J(R)$. Therefore $M(R)/J(M(R)) \cong R/J(R)$.

Suppose that I is an exchange ideal of R . For any idempotent $(e, m) \in M(I)$, we have $e \in I$ is an idempotent. Hence $J((e, m)M(R)(e, m)) = (e, m)J(M(R))(e, m) = (e, m)M(J(R))(e, m)$. So we get

$$\begin{aligned} & (e, m)M(R)(e, m)/J((e, m)M(R)(e, m)) \\ & \cong (e, m)M(R)(e, m)/(e, m)J(M(R))(e, m) \\ & \cong \overline{(e, m)}M(R)/J(M(R))\overline{(e, m)} \\ & \cong \bar{e}R/J(R)\bar{e} \\ & \cong eRe/J(eRe). \end{aligned}$$

Clearly, eRe is an exchange ring; hence, so is $eRe/J(eRe)$. This means that $(e, m)M(R)(e, m)/J((e, m)M(R)(e, m))$ is an exchange ring. Given idempotent $(f, m) + J((e, m)M(R)(e, m)) \in (e, m)M(R)(e, m)/J((e, m)M(R)(e, m))$, then we see that $f - f^2 \in J(eRe)$. That is, $f + J(eRe)$ is an idempotent. Since eRe is an exchange ring, by [1, Theorem 2.2]

again, we can find some $g = g^2 \in eRe$ such that $f - g \in J(eRe)$. Clearly, $(g, 0) + J((e, m)M(R)(e, m)) = (f, m) + (0, -m) + (g - f, 0) + J((e, m)M(R)(e, m)) = (f, m) + J(M(R))$. Clearly, $(g, 0) = (g, 0)^2 \in M(R)$. So idempotents can be lifted modulo $J((e, m)M(R)(e, m))$. It follows by [1, Theorem 2.2] that $(e, m)M(R)(e, m)$ is an exchange ring, as required.

Conversely, we assume that $M(I)$ is an exchange ideal. Given any idempotent $e \in I$, the quotient $eRe/J(eRe) \cong M(eRe)/J(M(eRe))$ is also an exchange ring. Given any idempotent $e + J(R) \in R/J(R)$, we have $e - e^2 \in J(R)$. Hence $(e - e^2, 0) \in J(M(R))$, so $(e, 0) + J(M(R)) \in M(R)/J(M(R))$ is an idempotent. By [1, Theorem 2.2], we have $(f, m) = (f, m)^2 \in M(R)$ such that $(e, 0) + J(M(R)) = (f, m) + J(M(R))$. Therefore $e - f \in J(R)$ and $f = f^2 \in R$. Using [1, Theorem 2.2] again, we complete the proof. \square

COROLLARY 3.7. *Let R be a ring. Then R is an exchange ring if and only if so is $R(R)$.*

Proof. Choose $M = R$. We get the result by Theorem 3.6. \square

4. Exchange element

THEOREM 4.1. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) I is an exchange ideal,
- (2) For any $a \in I$, there exists an exchange element $x \in R$ such that $x \in Ra$ such that $1 - x \in R(1 - a)$.

Proof. (1) \Rightarrow (2) is obvious because every idempotent is an exchange element.

(2) \Rightarrow (1) For any $x \in I$, there exists an exchange element $y \in R$ such that $y \in Rx$ and $1 - y \in R(1 - x)$. As $y \in Rx \subseteq I$ is an exchange element, we have an idempotent $e \in R$ such that $e \in Ry$ and $1 - e \in R(1 - y)$. Therefore $e \in Rx$ and $1 - e \in R(1 - x)$, as required. \square

Recall that $x \in R$ is regular provided that there exists $y \in R$ such that $x = xyx$.

COROLLARY 4.2. *Let I be an ideal of a ring R . Then the following are equivalent:*

- (1) I is an exchange ideal,

- (2) For any $a \in I$, there exists a regular element $x \in R$ such that $x \in Ra$ such that $1 - x \in R(1 - a)$.

Proof. (1) \Rightarrow (2) is clear because every idempotent is a regular element.

(2) \Rightarrow (1) Let $y \in I$ be regular. Then we have $z \in R$ such that $y = yzy$. Write $f = zy$ and $e = f + (1 - f)yf$. Then $e = e^2 \in Ry$. Furthermore, we get $1 - e = (1 - f)(1 - yf) = (1 - f)(1 - y) \in R(1 - f)$. Hence $y \in I$ is an exchange element. By Theorem 4.1, we complete the proof. \square

We say that $y \in I$ is π -regular in case there exist a $z \in R$ and a positive integer $n(y)$ such that $y^{n(y)} = y^{n(y)}zy^{n(y)}$. Furthermore, we claim that an ideal I of a ring R is an exchange ideal if and only if for any $x \in I$, there exists a π -regular $y \in Rx$ such that $1 - y \in R(1 - x)$.

LEMMA 4.3. Let I be an ideal of a ring R . Then the following are equivalent:

- (1) I is an exchange ideal,
- (2) For any $a \in I$, there exists an exchange element $x \in R$ such that $x - a \in R(a - a^2)$.

Proof. (1) \Rightarrow (2) is clear by [9, Proposition 1.1].

(2) \Rightarrow (1) For any $a \in R$, $x - a \in R(a - a^2)$ for an exchange element $x \in R$. Assume that $x - a = r(a - a^2)$ for $r \in R$. Then we have $x = (1 + r(1 - a))a \in Ra$ such that $1 - x = (1 - ra)(1 - a) \in R(1 - a)$. By Theorem 4.1, I is an exchange ideal. \square

THEOREM 4.4. Let I be an ideal of a ring R . Then the following are equivalent:

- (1) I is an exchange ideal,
- (2) For any left ideal J of R , if $a \in I$ and $a - a^2 \in J$, then there exists an exchange element $x \in R$ such that $x - a \in J$.

Proof. (1) \Rightarrow (2) is clear by Lemma 4.3.

(2) \Rightarrow (1) Let $a \in I$, and let $J = R(a - a^2)$. Clearly, $a - a^2 \in J$. So we have an exchange element $x \in R$ such that $x - a \in J$. That is, $x - a \in R(a - a^2)$. Using Lemma 4.3, we conclude that I is an exchange ideal. \square

COROLLARY 4.5. Let I be an ideal of a ring R . Then the following are equivalent:

- (1) I is an exchange ideal,

(2) For any left ideal J of R , if $a \in I$ and $a - a^2 \in J$, then there exists a regular element $x \in R$ such that $x - a \in J$.

Proof. (1) \Rightarrow (2) is obvious by Lemma 4.3.

(2) \Rightarrow (1) As every regular element in I is an exchange ideal, we get the result by Theorem 4.4. \square

It follows by Corollary 4.5, we deduce that a ring R is an exchange ring if and only if for any left ideal J of R , if $a - a^2 \in J$, then there exists a regular element $x \in R$ such that $x - a \in J$.

In [10, Corollary], Nicholson proved that if R is an exchange ring with stable rank one, then for any $a \in R$ there exist idempotents $e, f \in R$ such that (1) $e \in aR, 1 - e \in (1 - a)R$. (2) $f \in Ra, 1 - f \in R(1 - a)$. (3) $f = u^{-1}eu$ for some invertible $u \in R$. But he used the cancellation of modules in his proof. Following a new route, we obtain an analogue for general exchange rings.

LEMMA 4.6. Let I be an exchange ideal of a ring R . If $ax + b = 1$ with $a \in I, x, b \in R$, then $R(a + by)R = R$ for a $y \in R$.

Proof. Suppose that $ax + b = 1$ with $a \in I, x, b \in R$. Then $b \in 1 + I$. Since I is an exchange ideal, we have $e = e^2 \in R$ such that $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. Thus $1 - e = axt$. It is easy to check that $(1 - e)aR \subseteq (1 - e)R = (1 - e)axtR \subseteq (1 - e)aR$, and then $(1 - e)aR = (1 - e)R$. Hence $R = (1 - e)aR \oplus eR$, so there exist $u, v \in R$ such that $1 = (1 - e)au + ev$. We easily check that $(1 - e)((1 - e)a + e)u + e((1 - e)a + e)v = (1 - e)au + ev = 1$, so $R((1 - e)a + e)R = R$. Set $y = s(1 - a)$. Then $R(a + by)R = R(a + bs(1 - a))R = R(a + e(1 - a))R = R((1 - e)a + e)R = R$. \square

LEMMA 4.7. Let I be an exchange ideal of a ring R , and let $e, f \in I$ be idempotents. If $eR \cong fR$, then $eu = uf$ with $RuR = R$.

Proof. Since $eR \cong fR$, we can find $a \in eRf$ and $b \in fRe$ such that $e = ab$ and $f = ba$. As $ab + (1 - ab) = 1$ with $a \in I$, it follows by Lemma 4.6 that $R(a + (1 - ab)y)R = R$ for a $y \in R$. Let $v = a + (1 - ab)y$. Then $b = bab = bvb$. Let $u = (1 - ab - vb)v(1 - ba - bv)$. Clearly, $(1 - ab - vb)^2 = 1 = (1 - ba - bv)^2$. Hence $RuR = RvR = R$. Furthermore, we have $eu = ab(1 - ab - vb)v(1 - ba - bv) = -abv(1 - ba - bv) = a = -(1 - ab - vb)vba = (1 - ab - vb)v(1 - ba - bv)ba = uf$. Therefore we complete the proof.

Very recently, Wang studied exchange rings by virtue of op-idempotents (see [S. Wang, On op-idempotents, Kyungpook Math. J., to appear]). Now we investigate exchange ideals by idempotents following a similar route. \square

THEOREM 4.8. *An ideal I of a ring R is an exchange ideal if and only if for any $a \in I$, there exist idempotent $e, f \in R$ such that*

- (1) $e \in aR, 1 - e \in (1 - a)R$,
- (2) $f \in Ra, 1 - f \in R(1 - a)$,
- (3) $euw = uvf$ with $RuR = RvR = R$.

Proof. One direction is obvious. Conversely, assume now that I is an exchange ideal of R . Let $a \in I$. Then we have an idempotent $e \in R$ such that $e \in aR, 1 - e \in (1 - a)R$. Set $g = 1 - e$ and $b = 1 - a$. Then $e \in aR$ and $g \in bR$. Suppose that $e = ar'$ and $g = bs'$. Set $r = r'e$ and $s = s'g$. Then $rar = r, sbs = s, rbs = 0$ and $sar = 0$. Let $r'' = 1 - sb + rb$ and $s'' = 1 - ra + sa$. Similarly to [10, Proposition], we get $r''ar'' = r'', s''bs'' = s''$ and $r''a + s''b = 1$. Let $f = r''a$. Then we have $f = f^2 \in Ra$ such that $1 - f = s''b \in R(1 - a)$.

Obviously, $s'bR \cong bs'R = gR$. In view of Lemma 4.7, we have $gu = us'b$ and $RuR = R$. Hence $eu = (1 - g)u = u(1 - s'b)$. On the other hand, $1 - s'b = ar''$; hence, we get $(1 - s'b)R \cong r''aR = fR$. By Lemma 4.7 again, we have $(1 - s'b)v = vf$ and $RvR = R$. Therefore $euw = u(1 - s'b)v = uvf$ and $RuR = RvR = R$. \square

COROLLARY 4.9. *An ideal I of a ring R is an exchange ideal if and only if for any $a \in I$, there exist exchange elements $x, y \in R$ such that*

- (1) $x \in aR, 1 - x \in (1 - a)R$,
- (2) $y \in Ra, 1 - y \in R(1 - a)$,
- (3) $xuv = vvy$ with $RuR = RvR = R$.

Proof. It is an immediate consequence of Theorem 4.8. \square

As an immediate consequence of Theorem 4.8, we deduce that a ring R is an exchange ring if and only if for any $a \in R$, there exist idempotents $e, f \in R$ such that (1) $e \in aR, 1 - e \in (1 - a)R$. (2) $f \in Ra, 1 - f \in R(1 - a)$. (3) $euw = uvf$ with $RuR = RvR = R$. We use R_q^{-1} to denote the set $\{u \in R \mid \exists v \in R \text{ such that } (1 - uv)R(1 - vu) = 0 = (1 - vu)R(1 - uv)\}$. Following Ara et. al. (see [3]), a ring R is a QB -ring if $aR + bR = R$ with $a, b \in R$ implies that $a + by \in R_q^{-1}$ for a $y \in R$. Similarly to Theorem 4.8, we extend [10, Corollary] and prove that if R is an exchange QB -ring then for any $a \in R$, there exist idempotents $e, f \in R$ such that (1)

$e \in aR, 1 - e \in (1 - a)R$. (2) $f \in Ra, 1 - f \in R(1 - a)$. (3) $euw = wvf$ for some $u, v \in R_q^{-1}$.

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DEPARTMENT OF MATHEMATICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA 321004,
P.R. CHINA

E-mail: chyzxl@hunnu.edu.cn