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A NOTE ON THE LEFSCHETZ FIXED POINT THEOREM FOR ADMISSIBLE SPACES

RAVI P. AGARWAL AND DONAL O'REGAN

ABSTRACT. The Lefschetz fixed point theorem is extended to compact continuous maps defined on an admissible subset of a Hausdorff topological space.

1. Introduction

In this paper we present new Lefschetz fixed point theorems for single valued continuous compact maps $f : X \to X$ where X is an admissible (to be defined later) subset of a Hausdorff topological space. Our definition of admissibility will include NES(compact) spaces so our results improve those in the literature; see [2] and the references therein.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \backslash N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if dim $\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

DEFINITION 1.1. Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f the *Leray endomorphism* if (i) all f_q are admissible and (ii) almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q Tr(f_q).$$

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Let H be the singular homology functor (with coefficients in the field K) from the category of topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional singular homology group of X. For a continuous map $f : X \to Y$, $H(f)$ is the induced linear map $f_\star = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$.

DEFINITION 1.2. A continuous map $f : X \to X$ is called a *Lefschetz map* provided $f_\star : H(X) \to H(X)$ is a Leray endomorphism. For such f we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_\star)$. We know if f and g are homotopic ($f \sim g$) and if f is a Lefschetz map, then g is a Lefschetz map with $\Lambda(g) = \Lambda(f)$.

DEFINITION 1.3. A space X is said to be a *Lefschetz space* provided any continuous map $f : X \to X$ is a Lefschetz map and $\Lambda(f) \neq 0$ implies f has a fixed point.

By a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is a *neighborhood extension space* for Q (written $Y \in NES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and for any continuous function $f_0 : K \to Y$, there exists a continuous extension $f : U \to Y$ of f_0 over a neighborhood U of K in X.

The following result was established in [2].

THEOREM 1.1. *Every NES(compact) is a Lefschetz space.*

2. Fixed point theory

We begin this section with a simple extension of Theorem 1.1. Let X be a subset of a Hausdorff topological space. Then X is said to be Borsuk NES(compact) if X is dominated by a NES(compact) space Y i.e. there exists a subset Y of a Hausdorff topological space with $Y \in NES$(compact), and continuous maps $r : Y \to X$, $s : X \to Y$ with $r\,s = 1_X$.

THEOREM 2.1. *Let X be a subset of a Hausdorff topological space and assume X is Borsuk NES(compact). Then X is a Lefschetz space.*

Proof. Let $f : X \to X$ be a continuous compact map. We know there exists Y, r and s as described above. Notice $s\,f\,r : Y \to Y$ is a continuous compact map. From Theorem 1.1 we know $\Lambda(s\,f\,r)$ is well defined. Also [3, Lemma 3] (or [2, Example 3.4]) guarantees that $\Lambda(f)$

is well defined and $\Lambda(f) = \Lambda(s\,f\,r)$. Next assume $\Lambda(f) \neq 0$. Then $\Lambda(s\,f\,r) \neq 0$ so Theorem 1.1 guarantees that there exists $x \in Y$ with $x = s\,f\,r(x)$. Let $w = r(x)$ and notice $x = s\,f(w)$ so $w = r\,s\,f(w)$. This together with $r\,s = 1_X$ yields $w = f(w)$, so the proof is complete. □

For our next result we assume X is a subset of a Hausdorff topological vector space E. We say X is NES admissible if for every compact subset K of X and every neighborhood V of zero there exists a continuous function $h_V : K \to X$ such that

(i) $x - h_V(x) \in V$ for all $x \in K$;
(ii) $h_V(K)$ is contained in a subset C of X with $C \in NES(\text{compact})$;
(iii) h_V and $i : K \hookrightarrow X$ are homotopic.

THEOREM 2.2. *Let E be a Hausdorff topological vector space and let $X \subseteq E$ be NES admissible. Then X is a Lefschetz space.*

Proof. Let $f : X \to X$ be a continuous compact map. Next let $\mathcal{N}$ be a fundamental system of neighborhoods of the origin 0 in E and $V \in \mathcal{N}$. Let $K = \overline{f(X)}$. Now there exists a continuous function $h_V : K \to X$ and a $C \subseteq X$ with $C \in NES(\text{compact})$, $x - h_V(x) \in V$ for all $x \in K$, $h_V(K) \subseteq C$ and $h_V \sim i$. Notice also that $h_V\,f : X \to X$ is a continuous compact map with $h_V\,f \sim f$. Let $g_V = h_V\,f|_C$ so $g_V : C \to C$ is a continuous compact map. From Theorem 1.1 we know that g_V is a Lefschetz map so in particular $\Lambda(g_V)$ is well defined. Also [2, Lemma 3.2 (see Example 3.3)] implies that $h_V\,f : X \to X$ is a Lefschetz map with $\Lambda(h_V\,f) = \Lambda(g_V)$. Now since $h_V\,f \sim f$ we have that $f : X \to X$ is a Lefschetz map with $\Lambda(f) = \Lambda(h_V\,f)$.

Next assume $\Lambda(f) \neq 0$. Then $\Lambda(h_V\,f) \neq 0$ so Theorem 1.1 guarantees that there exists $x_V \in C$ with $x_V = h_V(f(x_V))$. Now since $y_V = f(x_V) \in K$ then from (i) above we have $y_V - h_V(y_V) \in V$ so $y_V - x_V \in V$. Now since $K = \overline{F(X)}$ is compact we may assume without loss of generality that there exists x with $y_V \to x$. Also since $y_V - x_V \in V$ we have $x_V \to x$. This together with $y_V = f(x_V)$ and the continuity of f implies $x = f(x)$, and the proof is complete. □

REMARK 2.1. A similar result could be obtained if $C \in NES$ (compact) in (ii) above is replaced by C is Borsuk $NES(\text{compact})$; we only need replace Theorem 1.1 with Theorem 2.1 in the proof of Theorem 2.2.

Let X be a subset of a Hausdorff topological space. Then X is said to be Borsuk NES admissible if X is dominated by a NES admissible space Y i.e., there exists a Hausdorff topological vector space E, a $Y \subseteq E$

which is NES admissible, and continuous maps $r : Y \to X$, $s : X \to Y$ with $r\,s = 1_X$.

Essentially the same reasoning as in Theorem 2.1 establishes the following result.

THEOREM 2.3. *Let X be a subset of a Hausdorff topological space and assume X is Borsuk NES admissible. Then X is a Lefschetz space.*

Let X be a subset of a Hausdorff topological vector space E. Let V be a neighborhood of the origin 0 in E. X is said to be NES admissible V-dominated if there exists a NES admissible space X_V and two continuous functions $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V\, s_V(x) \in V$ for all $x \in X$ and also that $r_V\, s_V$ and $i : X \to X$ are homotopic. X is said to be almost NES admissible dominated if X is NES admissible V-dominated for every neighborhood V of the origin 0 in E.

THEOREM 2.4. *Let X be a subset of a Hausdorff topological vector space E. Also assume X is almost NES admissible dominated. Then X is a Lefschetz space.*

Proof. Let $f : X \to X$ be a continuous compact map and let $\mathcal{N}$ be a fundamental system of neighborhoods of the origin 0 in E and $V \in \mathcal{N}$. Let $K = \overline{f(X)}$. Now there exists a NES admissible space X_V and two continuous functions $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V\, s_V(x) \in V$ for all $x \in X$ and $r_V\, s_V \sim i$. Notice $s_V\, f\, r_V : X_V \to X_V$ is a continuous compact map and from Theorem 2.2 we know that $\Lambda(s_V\, f\, r_V)$ is well defined. Also [2, Lemma 3.2] guarantees that $\Lambda(f\, r_V\, s_V)$ is well defined and $\Lambda(f\, r_V\, s_V) = \Lambda(s_V\, f\, r_V)$. Also since $r_V\, s_V \sim i$ we have immediately that $f\, r_V\, s_V \sim f$. Thus f is a Lefschetz map and $\Lambda(f) = \Lambda(f\, r_V\, s_V) = \Lambda(s_V\, f\, r_V)$.

Now assume $\Lambda(f) \neq 0$. Then $\Lambda(s_V\, f\, r_V) \neq 0$ so Theorem 2.2 guarantees that there exists $x_V \in X_V$ with $x_V = s_V\, f\, r_V(x_V)$. Let $y_V = r_V(x_V)$ and notice $y_V = r_V\, s_V\, f(y_V)$. Since $x - r_V\, s_V(x) \in V$ for all $x \in X$ we have

$$f(y_V) - r_V\, s_V\, f(y_V) \in V.$$

Thus $f(y_V) - y_V \in V$. Let $w_V = f(y_V) \in K$. Now since $K = \overline{F(X)}$ is compact we may assume without loss of generality that there exists a x with $w_V \to x$. Also since $w_V - y_V \in V$ we have $y_V \to x$. These together with $w_V = f(y_V)$ and the continuity of f implies $x = f(x)$. □

Next we extend Theorem 2.2 to the case of Hausdorff topological spaces. First we gather together some well known preliminaries. For a subset K of a topological space X, we denote by $Cov_X(K)$ the set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $f : X \to X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α-fixed point of f if there exists a member $U \in \alpha$ such that $x \in U$ and $f(x) \in U$. Given two maps $f, g : X \to Y$ and $\alpha \in Cov(Y)$, f and g are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$, $f(x) \in U_x$ and $g(x) \in U_x$.

The following result can be found in [2, p.272].

THEOREM 2.5. *Let X be a topological space and $f : X \to X$ a continuous map. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X(\overline{f(X)})$ such that f has an α-fixed point for every $\alpha \in \theta$. Then f has a fixed point.*

REMARK 2.2. From Theorem 2.5 in proving the existence of fixed points in uniform spaces for continuous compact maps it suffices [1, p.298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [4, p.199] so such refinements form a cofinal family of open covers). For convenience in this paper we will apply Theorem 2.5 only when the space is uniform.

Let X be a subset of a Hausdorff topological space and let X be a uniform space. Then X is said to be Schauder NES admissible if for every compact subset K of X and every open covering $\alpha \in Cov_X(K)$ there exists a continuous function $\pi_\alpha : K \to X$ and a subset C of X with $C \in NES(\text{compact})$ and

(i) π_α and $i : K \hookrightarrow X$ are α-close;
(ii) $\pi_\alpha(K)$ is contained in C;
(iii) π_α and $i : K \hookrightarrow X$ are homotopic.

THEOREM 2.6. *Let X be a subset of a Hausdorff topological space and let X be a uniform space. Also suppose X is Schauder NES admissible. Then X is a Lefschetz space.*

Proof. Let $f : X \to X$ be a continuous compact map, $K = \overline{f(X)}$ and $\alpha \in Cov_X(K)$. Then there exists a continuous function $\pi_\alpha : K \to X$, a subset C of X with $C \in NES(\text{compact})$, $\pi_\alpha(K) \subseteq C$, π_α and $i : K \hookrightarrow X$ are α-close and $\pi_\alpha \sim i$. Let $f_\alpha = \pi_\alpha f$ and notice $f_\alpha : X \to X$ is a continuous compact map with $f_\alpha \sim f$. Let $g_\alpha = f_\alpha|_C$ and note $g_\alpha : C \to C$ is a continuous compact map. From Theorem 1.1 we know that g_α is a Lefschetz map and also from [2, Lemma 3.2] we have that $f_\alpha : X \to X$ is a Lefschetz map with $\Lambda(f_\alpha) = \Lambda(g_\alpha)$. Next since $f_\alpha \sim f$ we have that $f : X \to X$ is a Lefschetz map with $\Lambda(f) = \Lambda(f_\alpha)$.

Next assume $\Lambda(f) \neq 0$. Then $\Lambda(f_\alpha) \neq 0$ so Theorem 1.1 guarantees that there exists $x \in C$ with $x = \pi_\alpha f(x)$. Since π_α and i are α-close there exists $U \in \alpha$ with $\pi_\alpha f(x) = x \in U$ and $f(x) \in U$. Thus f has an α-fixed point. The result now follows from Theorem 2.5 (with Remark 2.2). □

REMARK 2.3. As in Remark 2.1 it is possible to replace $C \in NES$ (compact) in (ii) above with C Borsuk NES(compact).

Let X be a subset of a Hausdorff topological space. Then X is said to be Borsuk Schauder NES admissible if X is dominated by a uniform space Y which is Schauder NES admissible i.e. there exists a uniform space Y which is Schauder NES admissible, and continuous maps $r : Y \to X$, $s : X \to Y$ with $r\,s = 1_X$.

Essentially the same reasoning as in Theorem 2.1 establishes the following result.

THEOREM 2.7. *Let X be a subset of a Hausdorff topological space and assume X is Borsuk Schauder NES admissible. Then X is a Lefschetz space.*

Let X be a Hausdorff topological space and let $\alpha \in Cov(X)$. X is said to be Schauder NES admissible α-dominated if there exists a Schauder NES admissible space X_α and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and $i : X \to X$ are α-close and also that $r_\alpha\, s_\alpha \sim i$. X is said to be almost Schauder NES admissible dominated if X is Schauder NES admissible α-dominated for every $\alpha \in Cov(X)$.

Our next result was motivated by ideas in [2].

THEOREM 2.8. *Let X be a uniform space and let X be almost Schauder NES admissible dominated. Then X is a Lefschetz space.*

Proof. Let $f : X \to X$ be a continuous compact map, $K = \overline{f(X)}$, and $\alpha \in Cov_X(K)$. Now there exists a Schauder NES admissible space X_α and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and $i : X \to X$ are α-close and also that $r_\alpha\, s_\alpha \sim i$. Notice $s_\alpha\, f\, r_\alpha : X_\alpha \to X_\alpha$ is a continuous compact map and from Theorem 2.6 we know that $\Lambda(s_\alpha\, f\, r_\alpha)$ is well defined. Also [2, Lemma 3.2] guarantees that $\Lambda(f\, r_\alpha\, s_\alpha)$ is well defined and $\Lambda(f\, r_\alpha\, s_\alpha) = \Lambda(s_\alpha\, f\, r_\alpha)$. Since $r_\alpha\, s_\alpha \sim i$ we have immediately that $f\, r_\alpha\, s_\alpha \sim f$. Thus f is a Lefschetz map and $\Lambda(f) = \Lambda(f\, r_\alpha\, s_\alpha) = \Lambda(s_\alpha\, f\, r_\alpha)$.

Now assume $\Lambda(f) \neq 0$. Then $\Lambda(s_\alpha\, f\, r_\alpha) \neq 0$ so Theorem 2.6 guarantees that there exists $x_\alpha \in X_\alpha$ with $x_\alpha = s_\alpha\, f\, r_\alpha(x_\alpha)$. Let $y_\alpha = r_\alpha(x_\alpha)$

and notice $y_\alpha = r_\alpha s_\alpha f(y_\alpha)$. Now since i and $r_\alpha s_\alpha$ are α-close there exists $U_\alpha \in \alpha$ with $f(y_\alpha) \in U_\alpha$ and $r_\alpha s_\alpha f(y_\alpha) \in U_\alpha$ i.e. $f(y_\alpha) \in U_\alpha$ and $y_\alpha \in U_\alpha$. In particular f has an α-fixed point. The result now follows from Theorem 2.5 (with Remark 2.2). □

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Ravi P. Agarwal, Department Of Mathematical Sciences, Florida Institute of Technology, Melbourne, Florida 32901, U.S.A
E-mail: agarwal@fit.edu

Donal O'Regan, Department Of Mathematics, National University of Ireland, Galway, Ireland
E-mail: donal.oregan@nui.galway.ie