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# WEAK DIMENSION AND CHAIN-WEAK DIMENSION OF ORDERED SETS

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Abstract. In this paper, we define the weak dimension and the chain-weak dimension of an ordered set by using weak orders and chain-weak orders, respectively, as realizers. First, we prove that if $P$ is not a weak order, then the weak dimension of $P$ is the same as the dimension of $P$. Next, we determine the chain-weak dimension of the product of $k$-element chains. Finally, we prove some properties of chain-weak dimension which hold for dimension.

## 1. Introduction

Let $P$ be an ordered set. Another ordered set $Q$ on the same underlying set as $P$ is called an *extension* of $P$ if $x \leq y$ in $Q$ whenever $x \leq y$ in $P$. Furthermore, we say that an extension is *linear* if it is a linear order. A family $\Re$ of extensions of $P$ is called a *realizer* of $P$ if $x \leq y$ in $P$ whenever $x \leq y$ in $L$ for all $L \in \Re$. The *dimension* of $P$, denoted by $\dim(P)$, is the least cardinality of a family $\Re$ of linear extensions of $P$ such that $\Re$ is a realizer of $P$. Equivalently, the dimension of an ordered set $P$ is the least cardinality of chains whose product embeds $P$.

If an extension of an ordered set $P$ is an interval order [tree], then it is called an *interval extension* [*tree extension*] of $P$. Replacing linear extensions in the definition of dimension by interval extensions, the *interval dimension* of an ordered set was first introduced by Trotter and many authors studied this problem([3], [4], [8]). Similarly, Behrendt[2] defined *tree dimension* by using tree extensions.

In this paper, we define the weak dimension and the chain-weak dimension of ordered sets by using weak orders and chain-weak orders, respectively, as realizers. First, we prove that if $P$ is not a weak order, then

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the weak dimension of $P$ is the same as the dimension of $P$. Next, we determine the chain-weak dimension of the product of $k$-element chains. Finally, we prove some properties of chain-weak dimension which hold similarly for dimension. It is presumed throughout that every ordered set is finite.

## 2. Weak dimension

Let $P$ and $Q$ be two disjoint ordered sets. The *disjoint sum* $P+Q$ of $P$ and $Q$ is the ordered set on $P \cup Q$ such that $x < y$ if and only if $x < y$ in $P$ or $x < y$ in $Q$. The *linear sum* $P \oplus Q$ of $P$ and $Q$ is obtained from $P+Q$ by adding the new relations $x < y$ for all $x \in P$ and $y \in Q$. An ordered set $P$ is called a *weak order* if $P$ can be represented as the linear sum of antichains, that is, $P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where $A_i$ is an antichain for $i = 1, 2, \cdots, n$, and we call each $A_i$ a *level* of $P$. We see that every weak order has dimension at most two and its realizer consisting of linear extensions can be easily obtained. Now it is natural to consider a new parameter to describe ordered sets by weak orders. For an ordered set $P$, an extension of $P$ is said to be *weak* if it is a weak order and then the *weak dimension* of $P$, denoted by $\mathrm{wdim}(P)$, is defined to be the least integer $t$ for which there exist $t$ weak extensions $W_1, W_2, \cdots, W_t$ which form a realizer of $P$,

If $x, y$ are incomparable in $P$, we denote $x \parallel y$ and let $\mathrm{Inc}(P) = \{(x,y) \mid x \parallel y \text{ in } P\}$. We say that an extension $L$ of $P$ (not necessarily a linear extension) *reverses* $(x,y) \in \mathrm{Inc}(P)$ if $y < x$ in $L$ or $x \parallel y$ in $L$ and that a family $\mathfrak{R}$ of extensions of $P$ *reverses* a set $S \subseteq \mathrm{Inc}(P)$ if each $(x,y) \in S$ is reversed by at least one $L \in \mathfrak{R}$. Observe that a family $\mathfrak{R}$ of extensions of $P$ is a realizer if and only if $\mathfrak{R}$ reverses $\mathrm{Inc}(P)$.

THEOREM 2.1. *If $P$ is not a weak order, then* $\mathrm{dim}(P) = \mathrm{wdim}(P)$.

*Proof.* First, we prove a special case in the following claim.

CLAIM. If $\mathrm{wdim}(P) = 2$, then $\mathrm{dim}(P) = 2$.

Let $W_1$ and $W_2$ be weak extensions which form a realizer of $P$. Then

$$W_1 = A_1 \oplus A_2 \oplus \cdots \oplus A_m, \quad W_2 = B_1 \oplus B_2 \oplus \cdots \oplus B_n,$$

where $A_i$, $B_j$ are antichains for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. We will construct linear extensions $L_1$ and $L_2$ of $P$ from $W_1$ and $W_2$ such that $\{L_1, L_2\}$ is a realizer of $P$. For each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, let $C_{ij}$ be any linear extension of $A_i \cap B_j$. Let $C_i = C_{in} \oplus$

$\cdots \oplus C_{i2} \oplus C_{i1}$. Then $C_i$ is a linear extension of $A_i$ and so $L_1 = C_1 \oplus C_2 \oplus \cdots \oplus C_m$ is a linear extension of $P$. Similarly, for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, let $D_{ij}$ be the dual of $C_{ij}$ and $D_j = D_{mj} \oplus \cdots \oplus D_{2j} \oplus D_{1j}$. Then $L_2 = D_1 \oplus D_2 \oplus \cdots \oplus D_n$ is a linear extension of $P$.

Now we show that $\{L_1, L_2\}$ is a realizer of $P$. It is enough to show that, for any $(x, y) \in \mathrm{Inc}(P)$, $y < x$ in $L_1$ or $y < x$ in $L_2$. To prove this, we have the following four cases to consider.

CASE 1. $y < x$ in $W_1$ or $y < x$ in $W_2$.

We easily see that $y < x$ in $L_1$ or $y < x$ in $L_2$ from the construction of $L_1$ and $L_2$.

CASE 2. $x \parallel y$ in $W_1$ and $x < y$ in $W_2$.

Clearly, $x$ and $y$ are contained in the same level of $W_1$ and we denote this level by $A_r$. Let $x \in B_s$ and $y \in B_t$ for some $s, t = 1, 2, \cdots, n$. Then $s < t$, whence $y < x$ in $C_r$. Hence $y < x$ in $L_1$.

CASE 3. $x < y$ in $W_1$ and $x \parallel y$ in $W_2$.

It is similar to Case 2.

CASE 4. $x \parallel y$ in $W_1$ and $x \parallel y$ in $W_2$.

In this case, we have $x, y \in A_s$ and $x, y \in B_t$ for some $s$ and $t$. Hence $x, y \in C_{st} \cap D_{st}$. Since $D_{st}$ is the dual of $C_{st}$, we have $y < x$ in $L_1$ or $y < x$ in $L_2$.

To complete the proof, let $\mathrm{wdim}(P) = t > 2$. Then there are weak orders $W_1, W_2, \cdots, W_t$ such that $\{W_1, W_2, \cdots, W_t\}$ is a realizer of $P$. We will construct a linear extension $L_i$ of $P$ from each $W_i$, $i = 1, 2, \cdots, t$. Let $W_i = A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{in_i}$, where $A_{ij}$ is an antichain for $j = 1, 2, \cdots, n_i$. We first obtain $L_1$ and $L_2$ from $W_1$ and $W_2$ as in the above claim. We then define the new order $A'_{3j}$ on $A_{3j}$ ($j = 1, 2, \cdots, n_3$) which is induced by the dual of $L_2$. Then $L_3 = A'_{31} \oplus A'_{32} \oplus \cdots \oplus A'_{3n_3}$ is a linear extension of $P$. Continuing the method, we obtain a linear extension $L_i$ from $L_{i-1}$ for $4 \leq i \leq t$.

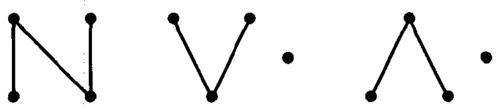
Now we show that $\{L_1, , L_2, \cdots, L_t\}$ is a realizer of $P$. Let $(x, y) \in \mathrm{Inc}(P)$. Then $y < x$ or $x \parallel y$ in some $W_i$. Let $k = \min\{\, i \mid y < x \text{ or } x \parallel y \text{ in } W_i \,\}$. Such a $k$ exists, since otherwise $\{W_1, W_2, \cdots, W_t\}$ is not a realizer of $P$. If $k \leq 2$ then we know that $y < x$ in $L_1$ or $L_2$. If $k \geq 3$, then $x < y$ in $L_{k-1}$ and $y < x$ in $L_k$ from the construction of $L_k$. Hence, $\{\, L_1, L_2, \cdots, L_t \,\}$ is a realizer of $P$ and so $\dim(P) \leq \mathrm{wdim}(P)$. Since $\dim(P) \geq \mathrm{wdim}(P)$, we conclude that $\dim(P) = \mathrm{wdim}(P)$. □

Theorem 2.1 shows that in order to determine the dimension of an ordered set, we may take its weak extensions instead of linear extensions if it is not a weak order. In general, a weak extension reverses more incomparable pairs than a linear extension does. Hence weak extensions may be more effective than linear extensions to determine the dimension of an ordered set.

## 3. Chain-weak dimension

In this section, we introduce a new parameter of an ordered set which generalizes the concept of weak dimension. An ordered set $P$ is called a *chain-weak order* if it can be represented by a linear sum of disjoint sum of chains, that is, $P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, where $A_i$ is a disjoint sum of chains for each $i = 1, 2, \cdots, n$, and again we call each $A_i$ a *level* of $P$. We see that every chain-weak order has dimension at most two and its linear extensions for a realizer can be easily obtained. An extension of an ordered set $P$ is said to be *chain-weak* if it is a chain-weak order. Then the *chain-weak dimension* of $P$, denoted by $\mathrm{cwdim}(P)$, is defined to be the least integer $t$ for which there exist $t$ chain-weak extensions $W_1, W_2, \cdots, W_t$ of $P$ which form a realizer of $P$.

PROPOSITION 3.1. *Let $P$ be an ordered set. Then $P$ is a chain-weak order if and only if $P$ does not contain any of the following ordered sets as an induced suborder.*



(a) (b) (c)

FIGURE 1

*Proof.* Let $P$ be a chain-weak order, that is, $P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $A_i$ is a disjoint sum of chains for each $i = 1, 2, \cdots, n$. Suppose that $P$ contains one of (a), (b), and (c) as an induced suborder. If $x \parallel y$ in $P$ then $x$ and $y$ belong to the same level. Hence one of (a), (b), and (c) must be contained in the same level. But each of (a), (b), and (c) cannot be contained in a disjoint sum of chains as an induced suborder.

Conversely, suppose that $P$ be an ordered set that does not contain any one of (a), (b), (c) as an induced suborder. To prove that $P$ is a chain-weak order, we use an induction on $|P|$. It is well known that $P$ does not contains (a) as an induced suborder if and only if $P$ is series-parallel, that is, $P$ is constructed from singletons by using only $+$ and $\oplus$. Hence $P = P' + P''$ or $P = P' \oplus P''$, where $P'$ and $P''$ are nonempty. By induction hypothesis, $P'$ and $P''$ are chain-weak orders. If $P = P' \oplus P''$, then it is clearly a chain-weak order. Suppose that $P = P' + P''$ and $P'$ contains one of the following ordered sets as an induced suborder:

Then $P = P' + P''$ contains (b) or (c) as an induced suborder. Thus $P'$ is a disjoint sum of chains. Similarly, $P''$ is also a disjoint sum of chains. Hence $P = P' + P''$ is a chain-weak order. □

For an ordered set $P$, we call $(x, y) \in \mathrm{Inc}(P)$ a *critical pair* in $P$ if $z < x$ implies $z < y$ and $y < w$ implies $x < w$ in $P$. Then we denote the set of all critical pairs by $\mathrm{Crit}(P)$. In [6], Rabinovitch and Rival proved that a family $\mathfrak{R}$ of linear extensions of $P$ is a realizer of $P$ if and only if $\mathfrak{R}$ reverses $\mathrm{Crit}(P)$. By using the same method, we prove the following lemma.

LEMMA 3.2. *Let $P$ be an ordered set and $\mathfrak{R}$ be a family of chain-weak extensions of $P$. Then $\mathfrak{R}$ is a realizer of $P$ if and only if $\mathfrak{R}$ reverses* $\mathrm{Crit}(P)$.

*Proof.* It is clear that if $\mathfrak{R}$ is a realizer of $P$ then $\mathfrak{R}$ reverses $\mathrm{Crit}(P)$. Conversely, suppose that $\mathfrak{R}$ reverses $\mathrm{Crit}(P)$. Let $(u, v) \in \mathrm{Inc}(P)$. We show that $v < u$ or $u \parallel v$ in some $W \in \mathfrak{R}$. Let $u'$ be a maximal element with the property that $u' \leq u$ and $u' \not\leq v$ and similarly let $v'$ be a minimal element with the property that $v \leq v'$ and $u' \not\leq v'$. Then $(u', v') \in \mathrm{Crit}(P)$ and so $v' < u'$ or $u' \parallel v'$ in some $W \in \mathfrak{R}$. If $u < v$ in $W$, then $u' \leq u < v \leq v'$ in $W$, which is a contradiction. Hence, $v < u$ or $u \parallel v$ in $W$. □

Now we determine the chain-weak dimension of some familiar ordered sets. First, we find an upper bound of the chain-weak dimension of generalized crowns. Trotter[7] defined the *generalized crown* $S_n^k$, for integers $n \geq 3$ and $k \geq 0$, to be the bipartite ordered set with $\min(S_n^k) = \{a_1, a_2, \cdots, a_{n+k}\}$, $\max(S_n^k) = \{b_1, b_2, \cdots, b_{n+k}\}$. For $i = 1, 2, \cdots, n+k$, $b_i$ and $a_j$ are incomparable in $S_n^k$ for $j = i, i+1, \cdots, i+k$ and $b_i > a_j$ in $S_n^k$ for $j = i+k+1, i+k+2, \cdots, i+k+n-1$. Here, subscripts are to be interpreted modulo $n+k$. For example, see Figure 2 for $S_4^2$.

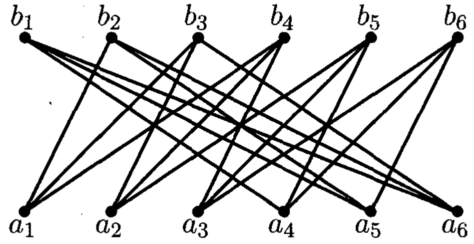



FIGURE 2. $S_4^2$

Let $A = \min(S_n^k)$ and $B = \max(S_n^k)$. Observe that $(a, b) \in \mathrm{Crit}(S_n^k)$ if and only if $a \in A$, $b \in B$ and $a \parallel b$ in $S_n^k$. We consider the two chain-weak extensions $W_1$ and $W_2$ of the generalized crowns $S_n^k$ in Figure 3. Let $A(b_i) = \{a \in A \mid a \parallel b_i \text{ in } S_n^k\}$. Then we can easily show that for $a \in A(b_i)$ and $i = n+k, 1, 2, \cdots, k+3$,

$$a \parallel b_i \text{ or } b_i < a \text{ in } W_1 \cap W_2.$$

Let $\mathrm{Crit}(b_i) = \{(a, b_i) \in A \times B \mid a \parallel b_i \text{ in } S_n^k\}$. Then $W_1$ and $W_2$ reverse $\mathrm{Crit}(b_i)$ for $i = n+k, 1, 2, \cdots, k+3$. Since $|\max(S_n^k)| = n+k$, we have the following proposition.

PROPOSITION 3.3. *For the generalized crown $S_n^k$,*

$$\mathrm{cwdim}(S_n^k) \leq \left\lceil \frac{2(n+k)}{k+4} \right\rceil.$$

For $k = 0$, $S_n^0 = S_n$ is the $n$-dimensional standard ordered set and so $\mathrm{cwdim}(S_n) \leq \lceil \frac{n}{2} \rceil$ by Proposition 3.3. Since $\mathrm{cwdim}(P) \geq \left\lceil \frac{\dim(P)}{2} \right\rceil$ for any ordered set $P$, we have the following.

COROLLARY 3.4. *For an integer $n \geq 3$,*

$$\mathrm{cwdim}(S_n) = \left\lceil \frac{n}{2} \right\rceil.$$

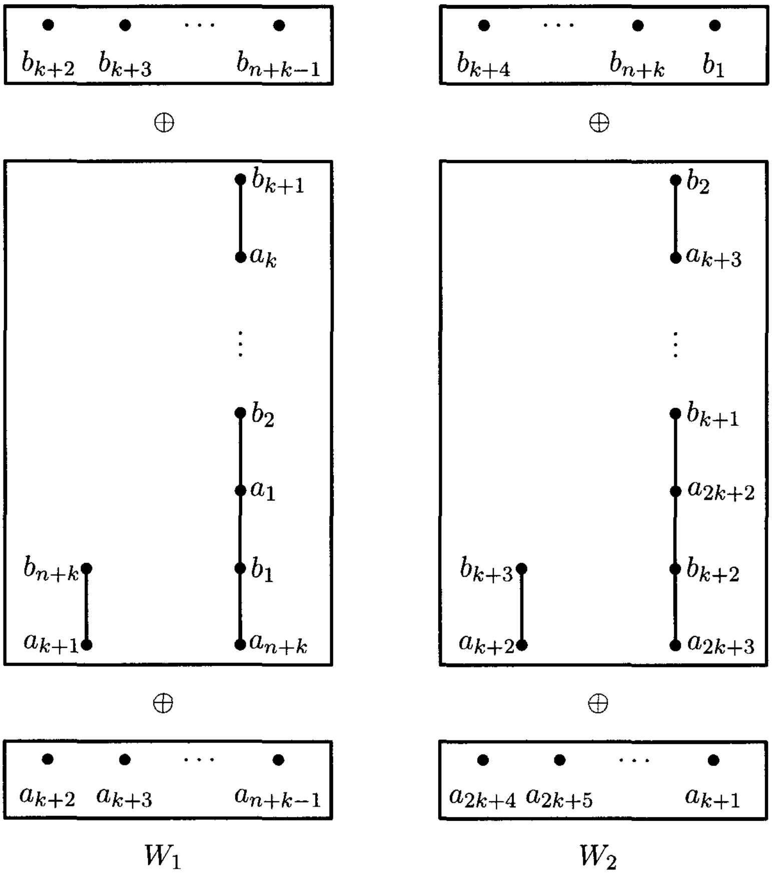



FIGURE 3

For $k \geq 1$ and $n = 3$ or 4, we know that $1 < \left\lceil \frac{2(n+k)}{k+4} \right\rceil < 3$. Hence $\text{cwdim}(S_n^k) \leq 2$ by Proposition 3.3. Since $S_n^k$ is not a chain-weak order, we have the following.

COROLLARY 3.5. *For $k \geq 1$ and $n = 3$ or 4,*

$$\text{cwdim}(S_n^k) = 2.$$

Next, we shall determine the chain-weak dimension of $\mathbf{k}^n$, the $n$-fold product of $k$-element chains. To do this, we define $S_n^*$ for $n \geq 3$ with the ground set $X = \{a_i, a_i', b_i, b_i' \mid i = 1, 2, \cdots, n\}$ whose order is the transitive closure of the following order relations:

(1) $\{a_i, b_i \mid n = 1, 2, \cdots, n\}$ is the $n$-dimensional standard ordered set.

(2) $a'_i < a_i$ and $b_i < b'_i$, for $i = 1, 2, \cdots, n$.

(3) $a'_i < b'_i$, for $i = 1, 2, \cdots, n$.

For example, the following is a diagram for $S_5^*$.

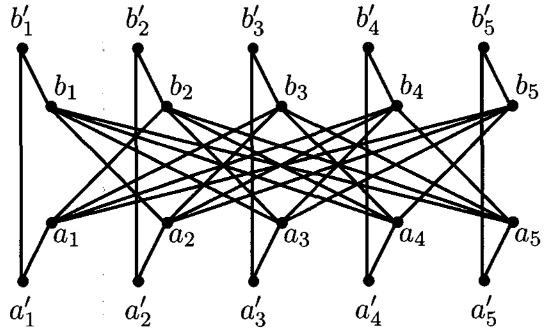



FIGURE 4. $S_5^*$

LEMMA 3.6. *For an integer $n \geq 3$,*

$$\mathrm{cwdim}(S_n^*) = n.$$

*Proof.* To begin with, we divide the set of all critical pairs of $S_n^*$ into two subsets:

$$A = \{(a_i, b'_i) \mid i = 1, \cdots, n\}, \ B = \{(a'_i, b_i) \mid i = 1, \cdots, n\}.$$

Let $W$ be any chain-weak extension of $S_n^*$. Then we will show that if $W$ reverses two critical pairs in $A$, then $W$ cannot reverse any critical pairs in $B$ and vice versa.

We may assume that $W$ reverses two critical pairs, $(a_1, b'_1)$, $(a_2, b'_2)$ in $A$. Now we have the four cases to consider.

CASE 1. $b'_1 < a_1$ and $b'_2 < a_2$ in $W$.

Since $a_2 < b'_1$ and $a_1 < b'_2$ in $S_n^*$, we have $a_2 < b'_2$ in $W$, but it is a contradiction.

CASE 2. $a_1 \parallel b'_1$ and $b'_2 < a_2$ in $W$.

Then $a_1$ and $b'_1$ lie in the same level $L$ of $W$. Since $a_1 < b'_2$ in $S_n^*$, $b'_2$ lies in the level $L$ or above the level $L$. If $b'_2$ lies above $L$, then $a_2$ lies above $L$ because $b'_2 < a_2$ in $W$. Since $a_2 < b'_1$ in $S_n^*$, $b'_1$ lies above $L$, which is a contradiction. Hence $b'_2 \in L$. For $i = 1, 2, \cdots, n$, $a'_i$ lies below the level $L$, because $a'_i < b'_1$ and $a'_i < b'_2$ in $S^*_n$. Since $b'_2 < a_2$ in $W$, $a_1$ and $a_2$ lie in $L$ or above $L$. Hence $b_i$ lies in $L$ or above $L$. Therefore, $W$ cannot reverse any critical pairs in $B$

CASE 3. $b'_1 < a_1$ and $a_2 \parallel b'_2$ in $W$.

This case can be treated by a similar method to Case 2.

CASE 4. $a_1 \parallel b'_1$ and $a_2 \parallel b'_2$ in $W$.

In this case, $a_1$ and $b'_1$ lie in one level, and $a_2$ and $b'_2$ may lie in another level. But, since $a_1 \prec b_2 \prec b'_2$ and $a_2 \prec b_1 \prec b'_1$ in $S^*_n$, all these six elements lie in the same level $L$ of $W$. Then, for $i = 1, 2, \cdots, n$, $a'_i$ lies below the level $L$ because $a'_i < b'_1$ and $a'_i < b'_2$ in $S^*_n$. For $j = 3, 4, \cdots n$, $b_j$ lies above the level $L$ because $a_1 < b_j$ and $a_2 < b_j$ in $S^*_n$. Hence $W$ cannot reverse any critical pair of $B$.

Similarly, if $W$ reverses two critical pairs in $B$ then $W$ cannot reverse any critical pairs in $A$. Consequently, $W$ reverses at most two critical pairs of $S^*_n$. Hence, $\text{cwdim}(S^*_n) \geq n$. Since $\text{cwdim}(S^*_n) \leq \dim(S^*_n) = n$, we finally conclude that $\text{cwdim}(S^*_n) = n$. □

Let $Q$ be a lattice. An element $x \in Q$ is *meet irreducible* if $x = y \wedge z$ implies that $x = y$ or $x = z$. Dually, an element $x \in Q$ is *join irreducible* if $x = y \vee z$ implies that $x = y$ or $x = z$. An element $x \in Q \setminus \{0, 1\}$ is *irreducible* if it is meet irreducible or join irreducible. We denote the set of irreducible elements of $Q$ by $\text{Irr}(Q)$. It is well known that $(x, y) \in \text{Crit}(Q)$ implies $\{x, y\} \subseteq \text{Irr}(Q)$ so that $Q$ has the same dimension as the induced suborder of $Q$ determined by $\text{Irr}(Q)$. Then it follows from Lemma 3.2 that $\text{cwdim}(Q) = \text{cwdim}(\text{Irr}(Q))$. Since $S_n = \text{Irr}(\mathbf{2}^n)$ and $S^*_n = \text{Irr}(\mathbf{3}^n)$, we obtain the following theorem from Corollary 3.4 and Lemma 3.6.

THEOREM 3.7. *For a natural number $k \geq 2$ and a natural number $n$,*

$$\text{cwdim}(\mathbf{k}^n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } \; k = 2 \\ n & \text{if } \; k \geq 3 \end{cases}$$

In the rest of this paper, we prove some simple properties of chain-weak dimension which hold for dimension. Recall that the dimension of an ordered set $P$ is also the least number $n$ such that $P$ is order-isomorphic to an induced suborder of the product of $n$ chains. In [2], Behrendt shows that a similar result holds for tree dimension. Here we show that it also holds for chain-weak dimension.

LEMMA 3.8. *Let $P$ be the product of a family $(P_i)_{i\in I}$ of chain-weak orders. Then* $\mathrm{cwdim}(P) \leq |I|$.

*Proof.* For each $i \in I$, let $Y_i := \prod_{j\in I\setminus\{i\}} P_j$ and let $Y_i'$ be a linear extension of $Y_i$. For $x \in P$ and $i \in I$, denote by $x^i$ the image of $x$ in $Y_i$ under the canonical projection $P \longrightarrow Y_i$ and by $x_i$ the image of $x$ in $P_i$ under the canonical projection $P \longrightarrow P_i$. Then we define an extension $Q_i$ of $P$ by $x \leq y$ in $Q_i \Leftrightarrow x_i < y_i$ in $P_i$, or $x_i = y_i$ and $x^i \leq y^i$ in $Y_i'$. Then $Q_i$ is the lexicographic sum of $Y_i'$ on $P_i$. Since $Y_i'$ is linear, $Q_i$ is a chain-weak order. If $x \leq y$ in $P$, then $x^i \leq y^i$ in $Y_i$ and $x_i \leq y_i$ in $P_i$ and hence $x^i \leq y^i$ in $Y_i'$ and $x_i \leq y_i$ in $P_i$, that is, $x \leq y$ in $Q_i$. Therefore, $Q_i$ is a chain-weak extension of $P$.

Let $(x, y) \in \mathrm{Inc}(P)$. Then, $x_i$ and $y_i$ are incomparable in $P_i$ for some $i \in I$, or there exist $i, j \in I$ such that $x_i < y_i$ in $P_i$ and $y_j < x_j$ in $P_j$. In the first case, by definition of $Q_i$, $x$ and $y$ are incomparable in $Q_i$. In the second case we have $x \leq y$ in $Q_i$ and $y \leq x$ in $Q_j$. Hence, $\{Q_i \mid i \in I\}$ is a realizer of $P$, that is, $\mathrm{cwdim}(P) \leq |I|$. □

PROPOSITION 3.9. *Let $P$ be an ordered set. Then the chain-weak dimension of $P$ is the least cardinal number $n$ such that $P$ is isomorphic to an induced suborder of the product of $n$ chain-weak orders.*

*Proof.* By Lemma 3.8, the product of $n$ chain-weak orders has chain-weak dimension of at most $n$. Now we show that every ordered set of chain-weak dimension $n$ can be embedded into a product of $n$ chain-weak orders. Let a family $(P_i)_{i\in I}$ of chain-weak orders be a realizer of $P$ with $|I| = n$. Let $Q = \prod_{i\in I} P_i$ and let $\varphi$ be the map from $P$ into $Q$ defined by $(\varphi(x))_i = x$ for all $i \in I$. Thus for $x, y \in P$, $\varphi(x) \leq \varphi(y)$ in $Q$ if and only if $x \leq y$ in $P_i$ for all $i \in I$ if and only if $x \leq y$ in $P$. Hence $\varphi$ is an embedding from $P$ into $Q$. □

Baker[1] showed that for an ordered set $P$, $\dim(P) = \dim(C(P))$, where $C(P)$ is the MacNeille completion of $P$, that is, the smallest lattice containing $P$ as an induced suborder. In [4], it was proved that a similar result holds for interval dimension. We have the following proposition for chain-weak dimension.

PROPOSITION 3.10. *The chain-weak dimension of an ordered set and its MacNeille completion are the same.*

*Proof.* Let $P$ be an ordered set of chain-weak dimension $k$. By Proposition 3.9, $P$ can be embedded into a product of $k$ chain-weak orders, $Q_1, Q_2, \cdots, Q_k$. For each $i = 1, 2, \cdots, k$, let $Q_i = L_{i1} \oplus L_{i2} \oplus \cdots \oplus L_{it_i}$, where $L_{ij}$ is a level of $Q_i$ $(j = 1, 2, \cdots, t_i)$. Let $Q'_i$ be the chain-weak order obtained from $Q_i$ by inserting elements $0, 1$ and $x_{ij}$ $(j = 1, 2, \cdots t_{i-1})$ such that $Q'_i = \{0\} \oplus L_{i1} \oplus \{x_{i1}\} \oplus L_{i2} \oplus \{x_{i2}\} \oplus \cdots \oplus \{x_{it_{i-1}}\} \oplus L_{it_i} \oplus \{1\}$. Then $Q'_i$ is a lattice and $P$ embeds into $\prod_{i=1}^k Q'_i$. Since $\prod_{i=1}^k Q'_i$ is a lattice and $C(P)$ is the least lattice containing $P$ as an induced suborder, $C(P)$ is embedded into $\prod_{i=1}^k Q'_i$. Hence, $\mathrm{cwdim}(C(P)) \le k$. □

Hiraguchi[5] showed that if $P$ is an ordered set with $|P| \ge 2$ and if $P \setminus \{x\}$ is an induced suborder of $P$ obtained by removing an element $x$ from $P$, then $\dim(P) \le \dim(P \setminus \{x\}) + 1$. Similarly, the following holds for chain-weak dimension.

PROPOSITION 3.11. *Let $P$ be an ordered set with $|P| \ge 2$ and let $x \in P$. Then*
$$\mathrm{cwdim}(P) \le \mathrm{cwdim}(P \setminus \{x\}) + 1.$$

*Proof.* Let $Q = P \setminus \{x\}$, $\mathrm{cwdim}(Q) = t$ and $\{W_1, W_2, \cdots, W_t\}$ be a family of chain-weak orders which realizes $Q$. Set $D(x) = \{y \in P \mid y < x$ in $P\}$ and $U(x) = \{y \in P \mid y > x$ in $P\}$. For each $i = 1, 2, \cdots, t$, we shall construct a chain-weak extension $W'_i$ of $P$ by adding $x$ to $W_i$. Let $L_i$ be the level of $W_i$ containing a maximal element of $D(x)$ in $W_i$. Then $L_i$ contains no element of $U(x)$. If $W'_i$ is obtained from $W_i$ by replacing $L_i$ by $L_i \oplus \{x\}$, then clearly $W'_i$ is a chain-weak extension of $P$. To find the remaining one for a realizer of $P$, let $I(x)$ be the set of all elements incomparable to $x$ in $P$ and let $L$, $M$ and $N$ be any linear extensions of $I(x)$, $D(x)$ and $U(x)$, respectively. Then $W'_{t+1} = M \oplus (L + \{x\}) \oplus N$ is a chain-weak extension of $P$. Now it is easy to check that $\{W'_1, W'_2, \cdots, W'_{t+1}\}$ is a chain-weak realizer of $P$, as desired. □

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