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n-WEAK AMENABILITY AND STRONG DOUBLE LIMIT PROPERTY

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ABSTRACT. Let $\mathcal{A}$ be a Banach algebra, we say that $\mathcal{A}$ has the strongly double limit property (SDLP) if for each bounded net (a_α) in $\mathcal{A}$ and each bounded net (a^*_β) in $\mathcal{A}^*$, $\lim_\alpha \lim_\beta \langle a_\alpha, a^*_\beta \rangle = \lim_\beta \lim_\alpha \langle a_\alpha, a^*_\beta \rangle$ whenever both iterated limits exist. In this paper among other results we show that if $\mathcal{A}$ has the SDLP and $\mathcal{A}^{**}$ is $(n-2)$-weakly amenable, then $\mathcal{A}$ is n-weakly amenable. In particular, it is shown that if $\mathcal{A}^{**}$ is weakly amenable and $\mathcal{A}$ has the SDLP, then $\mathcal{A}$ is weakly amenable.

1. Introduction.

Let $\mathcal{A}$ be a Banach algebra, X be a Banach $\mathcal{A}$-bimodule. Then X^* is a Banach $\mathcal{A}$-bimodule under the actions:

$$\langle x, ax^* \rangle = \langle xa, x^* \rangle,$$
$$\langle x, x^*a \rangle = \langle ax, x^* \rangle, \ (a \in \mathcal{A}, x \in X, x^* \in X^*).$$

A derivation $D : \mathcal{A} \longrightarrow X$ is a (bounded) linear map such that

$$D(ab) = D(a)b + aD(b), \ (a, b \in \mathcal{A}).$$

For each $x \in X$, $\delta_x(a) = ax - xa$ is a derivation, such derivation is called inner. The first cohomology group $H^1(\mathcal{A}, X)$ is the quotient of the space of derivations by the inner derivations; $\mathcal{A}$ is called contractible if, for every Banach $\mathcal{A}$-bimodule X, $H^1(\mathcal{A}, X) = \{0\}$, amenable if, for every Banach $\mathcal{A}$-bimodule X, $H^1(\mathcal{A}, X^*) = \{0\}$ (this definition was introduced by B.E. Johnson in [4]), n-weakly amenable if, $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$ (this definition generalizes that introduced by Bade, Curtis and Dales in [1]) and weakly amenable if $\mathcal{A}$ is 1-weakly amenable (this definition was introduced by Dales, Ghahramani and Gronbaek in [2]).

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We shall also consider the second dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ as a Banach algebra. Where, two products are defined on $\mathcal{A}^{**}$ as follows. Let $a \in \mathcal{A}, a^* \in \mathcal{A}^*$ and $F, G \in \mathcal{A}^{**}$, then Fa^* and a^*F are defined in $\mathcal{A}^*$ by the formulae

$$\langle a, Fa^* \rangle = \langle a^*a, F \rangle\,,\ \langle a, a^*F \rangle = \langle aa^*, F \rangle$$

and $F \Box G$ and $F \triangle G$ are defined in $\mathcal{A}^{**}$ by the formulae

$$\langle a^*, F \Box G \rangle = \langle Ga^*, F \rangle\,,\ \langle a^*, F \triangle G \rangle = \langle a^*F, G \rangle\,.$$

Then $\mathcal{A}^{**}$ is a Banach algebra with respect to the either of the products $\Box$ and $\triangle$; these products are called the first and second Arens products on $\mathcal{A}^{**}$, respectively. The algebra $\mathcal{A}$ is defined to be Arens regular if the two products $\Box$ and $\triangle$ coincide in $\mathcal{A}^{**}$. For the general theory of Arens products, see [3] and [5], for example. Let $\mathcal{A}$ be a Banach algebra and $P_n : \mathcal{A}^{(n)} \longrightarrow \mathcal{A}^{(n+2)}$ be the natural embedding where

$$\langle \phi_{n+1}, P_n(\phi_n) \rangle = \langle \phi_n, \phi_{n+1} \rangle\,,\ \ (\phi_n \in \mathcal{A}^{(n)}, \phi_{n+1} \in \mathcal{A}^{(n+1)})$$

where $\mathcal{A}^{(0)} = \mathcal{A}$ and $\mathcal{A}^{(n)}$ is the n-th dual of $\mathcal{A}$. We shall require the following standard properties of the Arens products. Suppose (a_α) and (b_β) are nets in $\mathcal{A}$ with $P_0(a_\alpha) \longrightarrow F$ and $P_0(b_\beta) \longrightarrow G$ in $(\mathcal{A}^{**}, \sigma)$, where $\sigma = \sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ is the weak* topology on $\mathcal{A}^{**}$. Then $F \Box G = \lim_\alpha \lim_\beta P_0(a_\alpha b_\beta)$ and $F \triangle G = \lim_\beta \lim_\alpha P_0(a_\alpha b_\beta)$ in $(\mathcal{A}^{**}, \sigma)$.

DEFINITION 1.1. Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ has the strongly double limit property (SDLP) if for each bounded net (a_α) in $\mathcal{A}$ and each bounded net (a^*_β) in $\mathcal{A}^*$, $\lim_\alpha \lim_\beta \langle a_\alpha, a^*_\beta \rangle = \lim_\beta \lim_\alpha \langle a_\alpha, a^*_\beta \rangle$ whenever both iterated limits exist.

LEMMA 1.2. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ has the SDLP if and only if $P_0^{**} = P_2$.*

Proof. Let $\mathcal{A}$ has the SDLP, $F \in \mathcal{A}^{**}$, $\phi_3 \in A^{(3)}$ and let (a_α) in $\mathcal{A}$ and (a^*_β) in $\mathcal{A}^*$ such that $\|a_\alpha\| \leq \|F\|$, $P_0(a_\alpha) \longrightarrow F$ and $\|a^*_\beta\| \leq \|\phi_3\|$, $P_1(a^*_\beta) \longrightarrow \phi_3$. Then

$$\begin{aligned}
\langle \phi_3, P_2(F) \rangle &= \langle F, \phi_3 \rangle \\
&= \lim_\beta \left\langle F, P_1(A^*_\beta) \right\rangle \\
&= \lim_\beta \left\langle a^*_\beta, F \right\rangle \\
&= \lim_\beta \lim_\alpha \left\langle a_\alpha, a^*_\beta \right\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \phi_3, P_0^{**}(F)\rangle &= \langle P_0^*(\phi_3), F\rangle \\ &= \lim_\alpha \langle a_\alpha, P_0^*(\phi_3)\rangle \\ &= \lim_\alpha \lim_\beta \left\langle a_\alpha, a_\beta^*\right\rangle .\end{aligned}$$

Since $\mathcal{A}$ has the SDLP, $P_0^{**}(F) = P_2(F)$, therefore $P_0^{**} = P_2$.

Conversely, let $P_0^{**} = P_2$, (a_α) in $\mathcal{A}$ and (a_β^*) in $\mathcal{A}^*$ be bounded nets such that $\lim_\alpha \lim_\beta \langle a_\alpha, a_\beta^*\rangle$ and $\lim_\beta \lim_\alpha \langle a_\alpha, a_\beta^*\rangle$ exist. We choose a subnet $(P_0(a_\gamma))$ of $(P_0(a_\alpha))$ and $F \in \mathcal{A}^{**}$ such that $P(a_\gamma) \longrightarrow F$ in $(\mathcal{A}^{**}, \omega^*)$ and let $(P_1(a_\delta^*))$ be the subnet of $(P_1(a_\beta^*))$ and $\phi_3 \in \mathcal{A}^{(3)}$ such that $P_1(a_\delta^*) \longrightarrow \phi_3$ in $(\mathcal{A}^{(3)}, \omega^*)$. Then we have:

$$\begin{aligned}\lim_\alpha \lim_\beta \left\langle a_\alpha, a_\beta^*\right\rangle &= \lim_\gamma \lim_\delta \langle a_\gamma, a_\delta^*\rangle \\ &= \lim_\gamma \lim_\delta \langle P_0(a_\gamma), P_1(a_\delta^*)\rangle \\ &= \lim_\gamma \langle P_0(a_\gamma), \phi_3\rangle \\ &= \lim_\gamma \langle a_\gamma, P_0^*(\phi_3)\rangle \\ &= \lim_\gamma \langle P_0^*(\phi_3), P_0(a_\gamma)\rangle \\ &= \langle P_0^*(\phi_3), F\rangle = \langle \phi_3, P_0^{**}(F)\rangle\end{aligned}$$

and similarly

$$\begin{aligned}\lim_\beta \lim_\alpha \left\langle a_\alpha, a_\beta^*\right\rangle &= \lim_\delta \lim_\gamma \langle a_\gamma, a_\delta^*\rangle \\ &= \lim_\delta \lim_\gamma \langle a_\delta^*, P_0(a_\gamma)\rangle \\ &= \lim_\delta \langle a_\delta^*, F\rangle \\ &= \lim_\delta \langle F, P_1(a_\delta^*)\rangle \\ &= \langle F, \phi_3\rangle = \langle \phi_3, P_2(F)\rangle .\end{aligned}$$

Therefore, $\lim_\alpha \lim_\beta \langle a_\alpha, a_\beta^*\rangle = \lim_\beta \lim_\alpha \langle a_\alpha, a_\beta^*\rangle$, so $\mathcal{A}$ has the SDLP. □

LEMMA 1.3. *Let $\mathcal{A}$ be a Banach algebra and P_2 be ω^*-ω^* continuous. Then $\mathcal{A}$ has the SDLP. In particular, if $\mathcal{A}$ is reflexive, then $\mathcal{A}$ has the SDLP.*

Proof. It is easy to see that P_0^{**} is ω^*-ω^* continuous and $P_0^{**} \circ P_0 = P_2 \circ P_0$. Let $F \in \mathcal{A}^{**}$ and (a_α) be a net in $\mathcal{A}$ such that $P_0(a_\alpha) \longrightarrow F$

in $(\mathcal{A}^{**}, w^*)$. Since P_2 is ω^*-ω^* continuous, so $P_2(P_0(a_\alpha)) \longrightarrow P_2(F)$ in $(\mathcal{A}^{(4)}, w^*)$, similarly since P_0^{**} is ω^*-ω^* continuous, so $P_0^{**}(P_0(a_\alpha)) \longrightarrow P_0^{**}(F)$. Therefore, $P_2(F) = P_0^{**}(F)$ and by lemma 1.1. $\mathcal{A}$ has the SDLP.

Let $F_\alpha \longrightarrow F$ in $(\mathcal{A}^{**}, w^*)$ and $\phi_3 \in \mathcal{A}^{(3)}$ and $\mathcal{A}^*$ is reflexive, so there is $a^* \in \mathcal{A}^*$ such that $P_1(a^*) = \phi_3$.

$$\begin{aligned} \langle \phi_3, P_2(F_\alpha)\rangle &= \langle P_1(a^*), P_2(F_\alpha)\rangle \\ &= \langle a^*, F_\alpha\rangle \longrightarrow \langle a^*, F\rangle \\ &= \langle \phi_3, P_2(F)\rangle . \end{aligned}$$

Therefore, $P_2(F_\alpha) \longrightarrow P_2(F)$ in $(\mathcal{A}^{(4)}, w^*)$, so P_2 is ω^*-ω^* continuous and therefore, $\mathcal{A}$ has the SDLP. □

It is easy to check that the converse of the lemma 1.3. is true, i.e. if $\mathcal{A}$ has the SDLP, then P_2 is ω^*-ω^* continuous, so we have the following proposition.

PROPOSITION 1.4. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

1) $\mathcal{A}$ has the SDLP;

2) $P_0^{**} = P_2$;

3) P_2 is ω^*-ω^* continuous.

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}^{**}, \mathcal{A}^{(4)}, ..., \mathcal{A}^{(2n)}, ...$ are Banach algebras with respect to Arens products. Therefore, $\mathcal{A}^{(n)}$ is $\mathcal{A}$-bimodule, $\mathcal{A}^{**}$-bimodule,... and $\mathcal{A}^{(2m)}$-bimodule whenever $n, m \in \aleph \bigcup 0$ and $2m \leq n$. Hence for each $\phi_n \in \mathcal{A}^{(n)}$ and $a \in \mathcal{A}$ we have:

$a\phi_n = P_0(a)\phi_n = P_2(P_0(a))\phi_n = \cdots = P_{2m-2}(P_{2m-4}(\cdots(P_0(a))\cdots))\phi_n$

$\phi_n a = \phi_n P_0(a) = \phi_n P_2(P_0(a)) = \cdots = \phi_n P_{2m-2}(P_{2m-4}(\cdots(P_0(a))\cdots))$

Where $n, m \in \aleph \bigcup 0$ and $2m \leq n$.

THEOREM 1.5. *Let $\mathcal{A}$ be a Banach algebra with SDLP and $D : \mathcal{A} \longrightarrow \mathcal{A}^{(n)}$ $(n \in \aleph)$, be a derivation. Then $D^{**} : (\mathcal{A}^{**}, \Box) \longrightarrow (\mathcal{A}^{**})^{(n)}$ and $D^{**} : (\mathcal{A}^{**}, \triangle) \longrightarrow (\mathcal{A}^{**})^{(n)}$ are derivations.*

Proof. Let $F, G \in \mathcal{A}^{**}, a, b \in \mathcal{A}, \phi_{2k+1} \in \mathcal{A}^{(2k+1)}$ and $n = 2k$ for some $k \in \aleph$. (the proof for $n = 2k + 1$ for some $k \in \aleph$ is similar.)

$$\begin{aligned} & \langle b, D^*(\phi_{2k+1})a\rangle \\ = \ & \langle D(ab), \phi_{2k+1}\rangle \\ = \ & \langle D(a)b + aD(b), \phi_{2k+1}\rangle \\ = \ & \langle P_{2k-2}(P_{2k-4}(\cdots(P_0(b))\cdots)), \phi_{2k+1}D(a)\rangle + \langle b, D^*(\phi_{2k+1}a)\rangle \\ = \ & \left\langle b, P_0^*(P_2^*(\cdots(P_{2k-2}^*(\phi_{2k+1}D(a)))\cdots)) + D^*(\phi_{2k+1}a)\right\rangle . \end{aligned}$$

Hence

$$\begin{aligned}
&\langle a, GD^*(\phi_{2k+1})\rangle \\
= \;& \langle D^*(\phi_{2k+1})a, G\rangle \\
= \;& \big\langle P_0^*(P_2^*(\cdots(P_{2k-2}^*(\phi_{2k+1}D(a)))\cdots)) + D^*(\phi_{2k+1}a), G\big\rangle \\
= \;& \big\langle P_2^*(\cdots(P_{2k-2}^*(\phi_{2k+!}D(a)))\cdots), P_0^{**}(G)\big\rangle + \langle \phi_{2k+1}a, D^{**}(G)\rangle \\
= \;& \big\langle G, P_2^*(\cdots(P_{2k-2}^*(\phi_{2k+1}D(a)))\cdots)\big\rangle + \langle \phi_{2k+1}a, D^{**}(G)\rangle \\
= \;& \langle P_{2k-2}(\cdots(P_2(G))\cdots),, \phi_{2k+1}D(a)\rangle \\
&+ \langle P_{2k-2}(\cdots(P_0(a))\cdots), D^{**}(G)\phi_{2k+1}\rangle \\
= \;& \langle a, D^*(P_{2k-2}(\cdots(P_2(G))\cdots)) \\
&+ P_0^*(\cdots(P_{2k-2}^*(D^{**}(G)\phi_{2k+1}))\cdots)\big\rangle .
\end{aligned}$$

Therefore

$$\begin{aligned}
&\langle \phi_{2k+1}, D^{**}(F\Box G)\rangle \\
= \;& \langle GD^*(\phi_{2k+1}), F\rangle \\
= \;& \langle D^*(P_{2k-2}(\cdots(P_2(G))\cdots)), F\rangle \\
&+ \big\langle P_0^*(\cdots(P_{2k-2}^*(D^{**}(G)\phi_{2k+1}))\cdots), F\big\rangle \\
= \;& \langle P_{2k-2}(\cdots(P_2(G))\cdots)\phi_{2k+1}, D^{**}(F)\rangle \\
&+ \big\langle P_2^*(\cdots(P_{2k-2}^*(D^{**}(G)\phi_{2k+1}))\cdots), P_0^{**}(F)\big\rangle \\
= \;& \langle \phi_{2k+1}, D^{**}(F)G\rangle \\
&+ \langle P_{2k-2}(\cdots(P_2(F))\cdots), D^{**}(G)\phi_{2k+1}\rangle \\
= \;& \langle \phi_{2k+1}, D^{**}(F)G + FD^{**}(G)\rangle .
\end{aligned}$$

Hence $D^{**} : (\mathcal{A}^{**}, \Box) \longrightarrow (\mathcal{A}^{**})^{(n)}$ is a derivation. Similarly $D^{**} : (\mathcal{A}^{**}, \triangle) \longrightarrow (\mathcal{A}^{**})^{(n)}$ is a derivation. $\Box$

COROLLARY 1.6. *Let $\mathcal{A}$ be a Banach algebra with the SDLP and $(\mathcal{A}^{**}, \Box)$ or $(\mathcal{A}^{**}, \triangle)$ be n-weakly amenable. Then $\mathcal{A}$ is n-weakly amenable.*

Dales, Ghahramani and Gronbeak in ([2, Corollary 1.10]) have shown that if $\mathcal{A}$ is an Arens regular Banach algebra and $H^1(\mathcal{A}^{**}, \mathcal{A}^{**}) = 0$, then $\mathcal{A}$ is 2-weakly amenable, we develop this result.

LEMMA 1.7. *Let $\mathcal{A}$ be a Banach algebra. Then $P_n : (\mathcal{A}^{**})^{(n-2)} \longrightarrow (\mathcal{A}^{**})^{(n)}$ is a $\mathcal{A}^{**}$-module homomorphism for each $n \geq 2$.*

Proof. Let $F \in \mathcal{A}^{**}, \phi_n \in (\mathcal{A}^{**})^{(n-2)}$ and $\phi_{n+1} \in (\mathcal{A}^{**})^{(n)}$. Then we have:

$$\begin{aligned} \langle \phi_{n+1}, P_n(\phi_n F)\rangle &= \langle \phi_n F, \phi_{n+1}\rangle \\ &= \langle \phi_n, F\phi_{n+1}\rangle \\ &= \langle F\phi_{n+1}, P_n(\phi_n)\rangle \\ &= \langle (\phi_{n+1}, P_n(\phi_n)F\rangle . \end{aligned}$$

Therefore, $P_n(\phi_n F) = P_n(\phi_n)F$ and similarly $P_n(F\phi_n) = FP_n(\phi_n)$. □

LEMMA 1.8. *Let $\mathcal{A}$ be an Arens regular Banach algebra. Then $P_1 : \mathcal{A}^* \longrightarrow (\mathcal{A}^{**})^*$ is a $\mathcal{A}^{**}$-module homomorphism; i.e., $P_1(a^*F) = P_1(a^*)F$ and $P_1(Fa^*) = FP_1(a^*)$ $(a^* \in \mathcal{A}^*, F \in \mathcal{A}^{**})$.*

Proof. Let $F, G \in \mathcal{A}^{**}$ and $a^* \in \mathcal{A}^*$. We have:

$$\begin{aligned} \langle G, P_1(a^*F)\rangle &= \langle a^*F, G\rangle \\ &= \langle a^*, F\Box G\rangle \\ &= \langle F\Box G, P_1(a^*)\rangle \\ &= \langle G, P_1(a^*)F\rangle . \end{aligned}$$

Then $P_1(a^*F) = P_1(a^*)F$ and similarly $P_1(Fa^*) = FP_1(a^*)$. □

THEOREM 1.9. *Let $\mathcal{A}$ be a Banach algebra, $n \geq 3$ be a integer, $D : \mathcal{A} \longrightarrow \mathcal{A}^{(n)}$ be a derivation and $\mathcal{A}$ has the SDLP. Then there exists a derivation $\tilde{D} : \mathcal{A}^{**} \longrightarrow (\mathcal{A}^{**})^{(n-2)}$ such that $\tilde{D}(\hat{a}) = D(a)$ $(a \in \mathcal{A})$.*

Proof. Let $\tilde{D} = P^*_{n-1}D^{**}$. Without loss of generality we may assume that $n = 2k$ for some $k \in \aleph$, therefore $\tilde{D} = P^*_{2k-1}D^{**}$.
Let $a, b \in \mathcal{A}, F, G \in \mathcal{A}^{**}, \phi_{2k-1} \in \mathcal{A}^{(2k-1)}$. Then we have:

$$\begin{aligned} &\langle b, D^*(P_{2k-1}(\phi_{2k-1}))a\rangle \\ = \;& \langle D(a)b + aD(b), P_{2k-1}(\phi_{2k-1})\rangle \\ = \;& \langle P_{2k-2}(\cdots(P_0(b))\cdots), P_{2k-1}(\phi_{2k-1})D(a)\rangle \\ &\quad + \langle b, D^*(P_{2k-1}(\phi_{2k-1})a)\rangle \\ = \;& \left\langle b, P_0^*(P_2^*(\cdots(P_{2k-2}^*(P_{2k-1}(\phi_{2k-1})D(a)))\cdots))\right\rangle \\ &\quad + \langle b, D^*(P_{2k-1}(\phi_{2k-1})a)\rangle . \end{aligned}$$

Therefore, $D^*(P_{2k-1}(\phi_{2k-1})a) = P_0^*(\cdots(P_{2k-2}^*(P_{2k-1}(\phi_{2k-1})D(a)))\cdots)+$ $D^*(P_{2k-1}(\phi_{2k-1})a)$. Hence

$$\begin{aligned}
&\langle a, GD^*(P_{2k-1}(\phi_{2k-1}))\rangle \\
= \ & \langle D^*(P_{2k-1}(\phi_{2k-1}))a, G\rangle \\
= \ & \left\langle P_2^*(\cdots(P_{2k-2}^*(P_{2k-1}(\phi_{2k-1})D(a)))\cdots), P_0^{**}(G)\right\rangle \\
& + \langle P_{2k-1}(\phi_{2k-1})a, D^{**}(G)\rangle \\
= \ & \langle P_{2k-2}(\cdots(P_2(G))\cdots), P_{2k-1}(\phi_{2k-1})D(a)\rangle \\
& + \langle P_{2k-1}(\phi_{2k-1})a, D^{**}(G)\rangle \\
= \ & \langle a, D^*(P_{2k-2}(\cdots(P_2(G))\cdots)P_{2k-1}(\phi_{2k-1}))\rangle \\
& + \left\langle a, P_0^*(P_2^*(\cdots(P_{2k-2}^*(D^{**}(G)P_{2k-1}(\phi_{2k-1}))))\cdots))\right\rangle.
\end{aligned}$$

Since P_{2k-2} is a $\mathcal{A}^{**}$-module homomorphism, so P_{2k-1}^* is a $\mathcal{A}^{**}$-module homomorphism and therefore, we have:

$$\begin{aligned}
&\left\langle \phi_{2k-1}, \tilde{D}(F\Box G)\right\rangle \\
= \ & \langle GD^*(P_{2k-1}(\phi_{2k-1})), F\rangle \\
= \ & \langle P_{2k-2}(\cdots(P_2(G))\cdots)P_{2k-1}(\phi_{2k-1}), D^{**}(F)\rangle \\
& + \langle P_{2k-2}(\cdots(P_2(F))\cdots), D^{**}(G)P_{2k-1}(\phi_{2k-1})\rangle \\
= \ & \langle P_{2k-1}(\phi_{2k-1}), D^{**}(F)G + FD^{**}(G)\rangle \\
= \ & \langle \phi_{2k-1}, P_{2k-1}^*D^* * (F)G + FP_{2k-1}^*(D^{**}(G))\rangle \\
= \ & \left\langle \phi_{2k-1}, \tilde{D}(F)G + F\tilde{D}(G)\right\rangle.
\end{aligned}$$

Therefore, $\tilde{D}$ is a derivation.
For each $a \in \mathcal{A}$ and $\phi_{n-1} \in \mathcal{A}^{(n-1)}$ we have:

$$\begin{aligned}
\left\langle \phi_{n-1}, \tilde{D}(\hat{a})\right\rangle &= \langle \phi_{n-1}, P_{n-1}^*(D^{**}(P_0(a)))\rangle \\
&= \langle D^*(P_{n-1}(\phi_{n-1})), P_0(a)\rangle \\
&= \langle a, D^*(P_{n-1}(\phi_{n-1}))\rangle \\
&= \langle \phi_{n-1}, D(a)\rangle .
\end{aligned}$$

Then $\tilde{D}(\hat{a}) = D(a)$ for each $a \in \mathcal{A}$. □

If $\mathcal{A}$ is Arens regular, then theorem 1.9. for $n = 2$ is correct [2, Corollary 1.10]. So, we have the following proposition.

Proposition 1.10. *Let $\mathcal{A}$ be a Banach algebra with the SDLP.*

i) *If $(\mathcal{A}^{**}, \Box)$ or $(\mathcal{A}^{**}, \triangle)$ is $(n-2)$-weakly amenable, then $\mathcal{A}$ is n-weakly amenable.*

ii) *If $\mathcal{A}$ is Arens regular and $H^1(\mathcal{A}^{**},\mathcal{A}^{**}) = 0$, then $\mathcal{A}$ is 2-weakly amenable.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^{(n)}$ be a derivation and $(\mathcal{A}^{**},\Box)$ is $(n-2)$-weakly amenable. In both i , ii there exists a derivation $\tilde{D} : (\mathcal{A}^{**},\Box) \longrightarrow (\mathcal{A}^{**})^{(n-2)}$ such that $\tilde{D}(\tilde{a}) = D(a), (a \in \mathcal{A})$. Since $(\mathcal{A}^{**},\Box)$ is $(n-2)$-weakly amenable, so there is $\phi_n \in (\mathcal{A}^{**})^{(n-2)}$ such that $\tilde{D} = \delta_{\phi_n}$. It is easy to see that $D = \delta_{\phi_n}$ and D is inner. □

PROPOSITION 1.11. *Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{A} = X^*$ for some Banach space X and product on $\mathcal{A}$ be separately ω^*-continuous and $\pi : X \longrightarrow X^{**} = \mathcal{A}^*$ be the natural embedding. Then $\pi^* : (\mathcal{A}^{**},\Box) \longrightarrow \mathcal{A}^*$ and $\pi^* : (\mathcal{A}^{**},\triangle) \longrightarrow \mathcal{A}^*$ are homomorphisms.*

Proof. It is easy to see that $\pi^*(P_0(a)) = a$, $(a \in \mathcal{A})$. Let $F, G \in \mathcal{A}^{**}$ and $(a_\alpha), (b_\beta)$ be bounded nets in $\mathcal{A}$ such that $P_0(a_\alpha) \longrightarrow F$ and $P_0(b_\beta) \longrightarrow G$ in $(\mathcal{A}^{**}, w^*)$. The mapping π^* is ω^*-ω^* continuous, so $a_\alpha \longrightarrow \pi^*(F)$ and $b_\beta \longrightarrow \pi^*(G)$ in $(\mathcal{A}, w^*)$. For each α we have:

$$\begin{aligned} w^* - \lim_\beta \pi^*(P_0(a_\alpha)\Box P_0(b_\beta)) &= w^* - \lim_\beta \pi^*(P_0(a_\alpha b_\beta)) \\ &= w^* - \lim_\beta a_\alpha b_\beta \\ &= a_\alpha G. \end{aligned}$$

On the other hand, $P_0(a_\alpha)\Box P_0(b_\beta) \longrightarrow P_0(a_\alpha)\Box G$ in $(\mathcal{A}^{**}, w^*)$, so $\pi^*(P_0(a_\alpha)\Box P_0(b_\beta)) \longrightarrow \pi^*(P_0(a_\alpha)\Box G)$. Therefore, $\pi^*(P_0(a_\alpha)\Box G) = a_\alpha \pi^*(G)$. $P_0(a_\alpha)\Box G \longrightarrow F\Box G$ in $(\mathcal{A}^{**}, w^*)$, then $\pi^*(a_\alpha)\Box G) \longrightarrow \pi^*(F)\pi^*(G)$, and $a_\alpha \longrightarrow \pi^*(F)$ in $(\mathcal{A}, w^*)$, so $a_\alpha\pi^*(G) \longrightarrow \pi^*(F)\pi^*(G)$.
Hence $\pi^*(F\Box G) = \pi^*(F)\pi^*(G)$. In a similar way we can show that $\pi^*(F\triangle G) = \pi^*(F)\pi^*(G)$. □

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