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UNIFORM TOPOLOGY ON DIFFERENCE ALGEBRAS

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ABSTRACT. In this paper, we consider a collection of ideals of a difference algebra X. We use the concept of congruence relation with respect to ideals to construct a uniformity that induces a topology on X which makes this to a topological difference algebras. We study the properties of this topology regarding different ideals.

1. Introduction

In [2], Hausdorff introduced the order group, which is a general algebraic system combining a partial ordered set and a group. In [7], Meng introduced the concept of difference algebra as result of combining a partial order and set difference operation. In [8], E. Roh et. al. study some algebraic property of this algebraic structure. In this note we consider a collection of ideals and use congruence relation with respect to ideals to define a uniformity and make the difference algebra into a uniform topological space. Then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

DEFINITION 2.1. [7] A difference algebra is an algebra $(X, *, \leq, 0)$ with binary operation $*$ and a binary relation $\leq$ on X and constant $0 \in X$ such that:

(D1) $(X, \leq)$ is a poset,
(D2) $x \leq y$ implies $x * z \leq y * z$,
(D3) $(x * y) * z \leq (x * z) * y$,
(D4) $0 \leq x * x$,
(D5) $x \leq y$ if and only if $x * y \leq 0$.

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We shall write the binary relation "$\leq$" by putting $x \leq y$ if and only if $(x, y) \in \leq$, for convenience.

LEMMA 2.2. [7] *In each difference algebra X, the following relations hold for all $x, y, z \in X$:*

(1) $(x * y) * z = (x * z) * y$,
(2) $x * x = 0$,
(3) $x * y \leq z$ implies that $x * z \leq y$,
(4) $(x * (x * y)) * y = 0$,
(5) $x \leq y$ implies that $z * y \leq z * x$,
(6) $x * (x * (x * y)) = x * y$,
(7) $x * 0 = x$,
(8) $0 * (x * y) = (0 * x) * (0 * y)$,
(9) $(x * y) * (x * z) \leq z * y$.

DEFINITION 2.3. [8] A weak ideal of a difference algebra X is a nonempty subset I of X such that for all $x, y \in X$, we have

(I1) $0 \in I$,
(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

DEFINITION 2.4. [8] A weak ideal I of a difference algebra X is called ideal if it satisfies

(I3) $x \leq y$ and $y \in I$ imply $x \in I$.

DEFINITION 2.5. A congruence relation on a difference algebra X is an equivalence relation R on X moreover if xRy and uRv, then we have

(Cg1) $(x * u)R(y * v)$,
(Cg2) $(x * u)R(y * v)$ and $(u * x)R(v * y)$.
(Cg3) $(x \wedge u)R(y \wedge v)$ and $(x \vee u)R(y \vee v)$.

THEOREM 2.6. [8] *Let I be an ideal of a difference algebra X. Define:*

$$x \equiv_I y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

Then $\equiv_I$ is a congruence relation on X.

3. Uniformity in difference algebra

From now on $(X, *, \leq, 0)$ (briefly, X) is a difference algebra, unless otherwise is stated.

Let X be a nonempty set and U, V be any subset of $X \times X$. Define:

$$U \circ V = \{(x,y) \in X \times X \mid (z,y) \in U \text{ and } (x,z) \in V, \text{ for some } z \in X\},$$
$$U^{-1} = \{(x,y) \in X \times X \mid (y,x) \in U\},$$
$$\Delta = \{(x,x) \in X \times X \mid x \in X\}.$$

DEFINITION 3.1. [5] By a uniformity on X we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:
(U_1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$,
(U_2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,
(U_3) if $U \in \mathcal{K}$, then there exist a $V \in \mathcal{K}$, such that $V \circ V \subseteq U$,
(U_4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,
(U_5) if $U \in \mathcal{K}$, and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a *uniform structure* (uniform space).

THEOREM 3.2. *Let Λ be an arbitrary family of ideals of X which is closed under intersection. If*

$$U_I = \{(x,y) \in X \times X \mid x \equiv_I y\}$$

and

$$\mathcal{K}^* = \{U_I \mid I \in \Lambda\},$$

then $\mathcal{K}^*$ satisfies the conditions (U_1)-(U_4).

Proof. (U_1): Since I is an ideal of X then we have $x \equiv_I x$ for any $x \in X$, hence $\Delta \subseteq U_I$, for all $U_I \in \mathcal{K}^*$.
(U_2): For any $U_I \in \mathcal{K}^*$, we have

$$(x,y) \in (U_I)^{-1} \Leftrightarrow (y,x) \in U_I \Leftrightarrow y \equiv_I x \Leftrightarrow x \equiv_I y \Leftrightarrow (x,y) \in U_I.$$

(U_3): For any $U_I \in \mathcal{K}^*$, the transitivity of $\equiv_I$ implies that $U_I \circ U_I \subseteq U_I$.
(U_4): For any $U_I, U_J \in \mathcal{K}^*$, we claim that $U_I \cap U_J = U_{I \cap J}$. Let $(x,y) \in U_I \cap U_J$. Then $x \equiv_I y$ and $x \equiv_J y$. Hence $x * y \in I$, $y * x \in I$ and $x * y \in J$, $y * x \in J$. Then $x \equiv_{I \cap J} y$ and hence $(x,y) \in U_{I \cap J}$.

Conversely, let $(x,y) \in U_{I \cap J}$. Then $x \equiv_{I \cap J} y$, hence $x * y \in I \cap J$ and $y * x \in I \cap J$. Then $x * y \in I$, $y * x \in I$, $x * y \in J$ and $y * x \in J$. Therefore $x \equiv_I y$ and $x \equiv_J y$. Then $(x,y) \in U_I \cap U_J$. So $U_I \cap U_J = U_{I \cap J}$. Since $I, J \in \Lambda$, then $I \cap J \in \Lambda$, $U_I \cap U_J \in \mathcal{K}^*$. □

THEOREM 3.3. *Let $\mathcal{K} = \{U \subseteq X \times X \mid U_I \subseteq U$ for some $U_I \in \mathcal{K}^*\}$. Then $\mathcal{K}$ satisfies a uniformity on X and the pair $(X, \mathcal{K})$ is a uniform structure.*

Proof. By Theorem 3.2, the collection $\mathcal{K}$ satisfies the conditions (U_1)–(U_4). It suffices to show that $\mathcal{K}$ satisfies (U_5). Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_I \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem. □

Let $x \in X$ and $U \in \mathcal{K}$. Define

$$U[x] := \{y \in X \mid (x, y) \in U\}.$$

THEOREM 3.4. *Given a difference algebra X, then*

$$T = \{G \subseteq X \mid \forall\, x \in G, \exists\, U \in \mathcal{K}, U[x] \subseteq G\}$$

is a topology on X.

Proof. It is clear that $\emptyset$ and the set X belong to T. Also from the definition, it is clear that T is closed under arbitrary union. Finally to show that T is closed under finite intersection, let $G, H \in T$ and suppose $x \in G \cap H$. Then there exist U and V $\in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W = U \cap V$, then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$ therefore $G \cap H \in T$. Thus T is topology on X. □

Note that for any x in X, $U[x]$ is an open neighborhood of x.

DEFINITION 3.5. Let $(X, \mathcal{K})$ be a uniform structure. Then the topology T is called the uniform topology on X induced by $\mathcal{K}$.

PROPOSITION 3.6. *Topological space (X, T) is completely regular.*

Proof. See Theorem 14.2.9, [5]. □

4. Topological property of space (X, T)

Let X be a difference algebra and C, D subsets of X. Then we define $C * D$ as follows:

$$C * D = \{x * y \mid x \in C, y \in D\}$$

Let X be a difference algebra and T a topology defined on the set X. Then we say that the pair (X, T) is a topological difference algebra if the operation $*$ is continuous with respect to T. The continuity of the operations $*$ is equivalent to having the following property satisfied:

(C): Let O be an open set and $a, b \in X$ such that $a * b \in O$. Then there are open sets O_1 and O_2 such that $a \in O_1$, $b \in O_2$ and $O_1 * O_2 \subseteq O$.

THEOREM 4.1. *The pair (X, T) is a topological difference algebra.*

Proof. Let us first prove (C). Indeed assume that $x * y \in G$, with $x, y \in X$ and G an open subset of X. Then there exist $U \in \mathcal{K}$, $U[x*y] \subseteq G$ and an ideal I such that $U_I \subseteq U$. We claim that the following relation holds:

$$U_I[x] * U_I[y] \subseteq U[x * y]$$

Indeed for $h \in U_I[x]$ and $k \in U_I[y]$ we get that $x \equiv_I h$ and $y \equiv_I k$. Hence $x * y \equiv_I h * k$. From that $(x * y, h * k) \in U_I \subseteq U$. Hence $h * k \in U_I[x * y] \subseteq U[x * y]$. Then $h * k \in G$. Thus the condition (C) is verified. □

THEOREM 4.2. [5] *Let X be a set and $\mathcal{S} \subset \mathcal{P}\ (X \times X)$ be a family such that for every $U \in \mathcal{S}$ the following conditions hold:*

(a) $\Delta \subseteq U$,
(b) U^{-1} *contains a member of* $\mathcal{S}$, *and*
(c) *there exists a* $V \in \mathcal{S}$, *such that* $V \circ V \subseteq U$.

Then there exists a unique uniformity $\mathcal{U}$, for which $\mathcal{S}$ is a subbase.

THEOREM 4.3. *If we let $\mathcal{B} = \{U_I \mid I \text{ is an ideal of } X\}$, then $\mathcal{B}$ is a subbase for a uniformity of X. We denote this topology by S.*

Proof. Since $\equiv_I$ is an equivalence relation, then it is clear that $\mathcal{B}$ satisfies the axioms of Theorem 4.2. □

We say that topology σ is finer than τ if $\tau \subseteq \sigma$ as subsets of the power set. Then we have:

COROLLARY 4.4. *Topology S is finer than T.*

THEOREM 4.5. *Any ideal in the collection Λ is a clopen subset of X.*

Proof. Let I be an ideal of X in Λ and $y \in I^c$. Then $y \in U_I[y]$ and we get that $I^c \subseteq \bigcup\{U_I[y] \mid y \in I^c\}$. We claim that, $U_I[y] \subseteq I^c$, for all $y \in I^c$. Let $z \in U_I[y]$, then $z \equiv_I y$. Hence $y * z \in I$. If $z \in I$ then $y \in I$, that is a contradiction. So $z \in I^c$ and we get $\bigcup\{U_I[y] \mid y \in I^c\} \subseteq I^c$. Hence $I^c = \bigcup\{U_I[y] \mid y \in I^c\}$ and since $U_I[y]$ is open for all $y \in X$, I is a closed subset. We show that $I = \bigcup\{U_I[y] \mid y \in I\}$. If $y \in I$ then $y \in U_I[y]$ and we get $I \subseteq \bigcup\{U_I[y] \mid y \in I\}$. Let $y \in I$, if $z \in U_I[y]$ then $z \equiv_I y$ and so $z * y \in I$. Since $y \in I$ hence $z \in I$ and we get that $\bigcup\{U_I[y] \mid y \in I\} \subseteq I$. So I is also an open subset of X. □

THEOREM 4.6. *For any $x \in X$ and $I \in \Lambda$, $U_I[x]$ is a clopen subset of X.*

Proof. We show that $(U_I[x])^c$ is open. Let $y \in (U_I[x])^c$, then $x * y \in I^c$ or $y * x \in I^c$. Let $y * x \in I^c$. Hence by Theorems 4.1 and 4.2, $(U_I[y] * U_I[x]) \subseteq U_I[y * x] \subseteq I^c$. We claim that: $U_I[y] \subseteq (U_I[x])^c$. Let

$z \in U_I[y]$, then $z * x \in (U_I[z] * U_I[x])$. So $z * x \in I^c$ then we get $z \in (U_I[x])^c$. Hence $U_I[x]$ is closed. It is clear that $U_I[x]$ is open. So $U_I[x]$ is clopen subset of X. □

A topological space X is connected if and only if has only X and $\emptyset$ as clopen subsets. Therefore we have

COROLLARY 4.7. *The space (X, T) is not a connected space.*

We denote the uniform topology obtained by an arbitrary family Λ, by T_Λ and if $\Lambda = \{I\}$, we denote it by T_I.

THEOREM 4.8. *$T_\Lambda = T_J$, where $J = \bigcap\{I \mid I \in \Lambda\}$.*

Proof. Let $\mathcal{K}$ and $\mathcal{K}^*$ be as in Theorems 3.2 and 3.3. Now consider $\Lambda_0 = \{J\}$, define:

$$(\mathcal{K}_0)^* = \{U_J\}$$

and

$$\mathcal{K}_0 = \{U \mid U_J \subseteq U\}.$$

Let $G \in T_\Lambda$. So for all $x \in G$, there exist $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq I$ we get that $U_J \subseteq U_I$, for all ideals I of X. Since $U \in \mathcal{K}$, there exist $I \in \Lambda$ such that $U_I \subseteq U$. Hence $U_J[x] \subseteq U_I[x] \subseteq G$. Since $U_J \in \mathcal{K}_0$, $G \in T_J$. So $T_\Lambda \subseteq T_J$.

Conversely, let $H \in T_J$ then for all $x \in H$, there exist $U \in \mathcal{K}_0$ such that $U[x] \subseteq H$. So $U_J[x] \subseteq H$ and sine Λ is closed under intersection, $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $H \in T_\Lambda$. Thus $T_J \subseteq T_\Lambda$. □

COROLLARY 4.9. *Let I and J be ideals of X and $I \subseteq J$. Then J is clopen in topological space (X, T_I).*

Proof. Consider $\Lambda = \{I, J\}$. Then by Theorem 4.8, $T_\Lambda = T_I$ and therefore J is clopen in topological space (X, T_I). □

THEOREM 4.10. *Let I and J be ideals of X. Then $T_I \subseteq T_J$ if $J \subseteq I$.*

Proof. Let $J \subseteq I$. Consider:
$\Lambda_1 = \{I\}$, ${\mathcal{K}_1}^* = \{U_I\}$, $\mathcal{K}_1 = \{U \mid U_I \subseteq U\}$ and
$\Lambda_2 = \{J\}$, ${\mathcal{K}_2}^* = \{U_J\}$, $\mathcal{K}_2 = \{U \mid U_J \subseteq U\}$.

Let $G \in T_I$. Then for all $x \in G$, there exist $U \in \mathcal{K}_1$ such that $U[x] \subseteq G$. Since $J \subseteq I$, then $U_J \subseteq U_I$ and since $U_I[x] \subseteq G$, we get $U_J[x] \subseteq G$. $U_J \in \mathcal{K}_2$ and so $G \in T_J$. □

Recall that a uniform space $(X, \mathcal{K})$ is totally bounded if for each $U \in \mathcal{K}$, there exists $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} U[x_i]$ and X is compact if any open cover of X has a finite subcover.

THEOREM 4.11. *Let I be an ideal of X. Then the following conditions are equivalent:*

(1) *Topological space (X, T_I) is compact,*
(2) *Topological space (X, T_I) is totally bounded,*
(3) *There exists $P = \{x_1, x_2, \ldots, x_n\} \subseteq X$ such that for all $a \in X$ there exists $x_i \in P$ where $a * x_i \in I$ and $x_i * a \in I$.*

Proof. (1) $\to$ (2): It is clear by Theorem 14.3.8 of [5].

(2) $\to$ (3): Let $U_I \in \mathcal{K}$ since (X, T_I) is totally bounded, then there exists $x_1, x_2, \ldots, x_n \in I$ such that $X = \bigcup_{i=1}^{n} U_I[x_i]$. Now let $a \in X$ then there exist x_i such that $a \in U_I[x_i]$, therefore $a * x_i \in I$ and $x_i * a \in I$.

(3) $\to$ (1): For all $a \in X$ by hypothesis there exists $x_i \in P$ where $a * x_i \in I$ and $x_i * a \in I$. Hence $a \in U_I[x_i]$, thus $X = \bigcup_{i=1}^{n} U_I[x_i]$. Now let $X = \bigcup_{\alpha \in \Omega} O_\alpha$, where each O_α is an open set of X, then for any $x_i \in X$ there exists $\alpha_i \in \Omega$ such that $x_i \in O_{\alpha_i}$, since O_{α_i} is an open set then $U_I[x_i] \subseteq O_{\alpha_i}$, so $X = \bigcup_{i=1}^{n} U_I[x_i] \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$, therefore $X = \bigcup_{i=1}^{n} O_{\alpha_i}$ which means that (X, T_I) is compact. □

THEOREM 4.12. *If I is an ideal of X, then $U_I[x]$ is a compact set in topological space (X, T_I), for all $x \in X$.*

Proof. Let $U_I[x] \subseteq \bigcup_{\alpha \in \Omega} O_\alpha$, where each O_α is an open set of X. Since $x \in U_I[x]$, then there exists $\alpha \in \Omega$ such that $x \in O_\alpha$. Then $U_I[x] \subseteq O_\alpha$. Hence $U_I[x]$ is compact. □

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