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SOME DUALITY OF WEIGHTED BERGMAN SPACES OF THE HALF-PLANE

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Abstract. In the setting of the half-plane of the complex plane, we introduce a modified reproducing kernel and we show that for $r > -1/2$, $B^{1,r}$-cancellation property holds and the Bloch space is the dual space of $B^{1,r}$.

1. Introduction

It is well-known that the duality plays an important role in mathematics. We will discuss the Bloch space and weighted Bergman spaces.

Let $H = \{x + iy : y > 0\}$ denote the half-plane of the complex plane $\mathbb{C}$ and dA the area measure on H. Put $K(z,w) = -1/\pi(z-\overline{w})^2$. In fact, $K(z,\cdot)$ is the reproducing kernel for B^2, that is, for any $f \in B^2$, $f(z) = \int_H f(w)K(z,w)\,dA(w)$ for all $z \in H$. For $r > -1/2$, the weighted Bergman Spaces $B^{1,r}$ of the half-plane is the space of analytic functions in $L^1(H, dA_r)$, where $dA_r(z) = (2r+1)K(z,z)^{-r}\,dA$.

The main result of this paper is to show that for $r > -1/2$, the dual space of $B^{1,r}$ is the Bloch space.

In Section 2, using a modified reproducing kernel

$$M(\cdot,w)^{1+r} = K(\cdot,w)^{1+r} - K(\cdot,i)^{1+r},$$

we get bounded linear operators Q_r and we introduce a dense subspace of $B^{1,r}$. Moreover, we prove that $B^{1,r}$-cancellation property holds. Section 3 is devoted to the duality of $B^{1,r}$. To do so, for each element g

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of $\mathcal{B}^0(i)$, Λ_g is in $(B^{1,r})^*$, where for $r > -1/2$,

$$\mathcal{B}^r(i) = \{f \in A(H) : f(i) = 0 \text{ and } \|f\|_{\mathcal{B}^r} = \sup_{z=x+iy\in H} y^{1+2r}|f'(z)| < \infty\}$$

and $\Lambda_g : B^{1,r} \to \mathbb{C}$ is defined by $\Lambda_g(f) = \int_H f(z)\overline{g(z)}\, dA_r(z)$ for all $f \in B^{1,r}$. Using the bounded linear operator Q_r of L^∞ into $\mathcal{B}^0(i)$, we show that $\mathcal{B}^0(i)$ can be identified with the dual space of $B^{1,r}$. Throughout this paper, we use the symbol $A \lesssim B(A \approx B$, respectively) for nonnegative constants A and$B$ to indicate that A is dominated by B times some positive constant ($A \lesssim B$ and $B \lesssim A$, respectively).

2. Weighted Bergman spaces

In this section, we deal with bounded linear operators Q_r and dense subspaces of $B^{1,r}$.

PROPOSITION 2.1. *Suppose $r > -1/2$ and $f \in B^{1,r}$. Then for each $z = x + iy \in H$,*

$$|f(z)| \leq \frac{4}{(2r+1)(\pi y^2)^{1+r}}\|f\|_{1,r}.$$

Proof. Since $z = x+iy$ and f is analytic on H, the mean-value property implies that $|f(z)| \leq (1/|B(z,y/2)|)\int_{B(z,y/2)} |f(w)|\, dA(w)$. Since for any $w \in B(z,y/2)$, $\operatorname{Im} w > y/2$, $K(w,w) = 1/4\pi(\operatorname{Im} w)^2 \leq 1/\pi y^2$ and hence $|f(z)| \leq (1/(2r+1)(\pi y^2)^{1+r})\|f\|_{1,r}$. □

PROPOSITION 2.2. *For $r > -1/2$, $B^{1,r}$ is a closed subspace of $L^1(H, dA_r)$.*

Proof. It can be easily derived from Proposition 2.1. □

Let

$$\mathcal{B}^r(i) = \{f \in A(H) : \|f\|_{\mathcal{B}^r} = \sup_{z=x+iy\in H} y^{1+2r}|f'(z)| < \infty \text{ and } f(i) = 0\}.$$

Since each element of $\mathcal{B}^r(i)$ vanishes at i, $\|\cdot\|_{\mathcal{B}^r}$ is the norm on $\mathcal{B}^r(i)$, in fact, for $r > -1/2$, is also a Banach space (see [4]). We note that the weighted Bergman kernel $K(\cdot,w)^{1+r} \notin L^1(H, dA_r)$ (see [5]) and hence we need a modified reproducing kernel $M(\cdot,w)^{1+r} = K(\cdot,w)^{1+r} - K(\cdot,i)^{1+r}$. For $r > -1/2$, we define $Q_r : L^\infty \to \mathcal{B}^0(i)$ by

$$Q_r(b)(w) = (2r+1)\int_H b(z)\overline{M(z,w)^{1+r}}K(z,z)^{-r}\, dA(z).$$

Then we have the following:

THEOREM 2.3. (1) *$M(\cdot,w)^{1+r}$ is in $L^1(H,dA_r)$ and hence in $B^{1,r}$.* (2) *For $r>-1/2$, Q_r is a bounded linear operator.*

Proof. (1) Suppose $2+2r=q/p$ for some $p,q\in\mathbb{Z}$. Since

$$\begin{aligned}
&|M(z,w)^{1+r}|\\
&=\frac{1}{\pi^{1+r}}\left|\frac{(z+i)^{2+2r}-(z-\overline{w})^{2+2r}}{(z-\overline{w})^{2+2r}\times(z+i)^{2+2r}}\right|\\
&=\frac{1}{\pi^{1+r}}\left|\frac{(z+i)^{\frac{q}{p}}-(z-\overline{w})^{\frac{q}{p}}}{(z-\overline{w})^{\frac{q}{p}}\times(z+i)^{\frac{q}{p}}}\right|\\
&=\frac{1}{\pi^{1+r}}\frac{|(z+i)^{\frac{q-1}{p}}+(z+i)^{\frac{q-2}{p}}(z-\overline{w})^{\frac{1}{p}}+\cdots+(z-\overline{w})^{\frac{q-1}{p}}|}{|z-\overline{w}|^{\frac{q}{p}}|z+i|^{\frac{q}{p}}}\\
&\qquad\times|(z+i)^{\frac{1}{p}}-(z-\overline{w})^{\frac{1}{p}}|\\
&=\frac{1}{\pi^{1+r}}\frac{|(z+i)^{\frac{q-1}{p}}+(z+i)^{\frac{q-2}{p}}(z-\overline{w})^{\frac{1}{p}}+\cdots+(z-\overline{w})^{\frac{q-1}{p}}|}{|(z+i)^{\frac{p-1}{p}}+(z+i)^{\frac{p-2}{p}}(z-\overline{w})^{\frac{1}{p}}+\cdots+(z-\overline{w})^{\frac{p-1}{p}}|}\\
&\qquad\times\frac{|z+i-z+\overline{w}|}{|z-\overline{w}|^{\frac{q}{p}}|z+i|^{\frac{q}{p}}},
\end{aligned}$$

$$\begin{aligned}
&\int_H|M(z,w)^{1+r}|K(z,z)^{-r}\,dA(z)\\
&\le\frac{4^r}{\pi}(1+|w|)\int_H\frac{|(z+i)^{\frac{q-1}{p}}+\cdots+(z-\overline{w})^{\frac{q-1}{p}}|\operatorname{Im}z^{\frac{q}{p}-2}\,dA(z)}{|z-\overline{w}|^{\frac{q}{p}}|z+i|^{\frac{q}{r}}|(z+i)^{\frac{p-1}{p}}+\cdots+(z-\overline{w})^{\frac{p-1}{p}}|}.
\end{aligned}$$

Since $\frac{q}{p}+\frac{q}{p}+\frac{p-1}{p}-\frac{q-1}{p}-\frac{q}{p}+2=3$, $M(\cdot,w)^{1+r}\in L^{1,r}$.

(2) Clearly, each Q_r is linear and $Q_r(b)(i)=0$. Take any closed contour C in H. By Fubini's Theorem,

$$\begin{aligned}
&\int_C Q_r(b)(w)\,dA(w)\\
&=(2r+1)\int_H b(z)\overline{\int_C M(z,w)\,dA(w)}K(z,z)^{-r}\,dA(z)=0
\end{aligned}$$

and hence $Q_r(b)$ is analytic on H. Let $z=x+iy$ and $w=s+it$ be in H. Since

$$\begin{aligned}
&\frac{d}{dz}\overline{M(w,z)^{1+r}}\\
&= \left(-\frac{1}{\pi}\right)^{1+r} \frac{2+2r}{(\overline{w}-z)^{3+2r}}\, y \left|\frac{d}{dz}Q_r(b)(z)\right|\\
&= (2r+1)y\left|\int_H b(w)\left(-\frac{1}{\pi}\right)^{1+r}\frac{2+2r}{(\overline{w}-z)^{3+2r}}K(w,w)^{-r}\,dA(w)\right|\\
&\leq \frac{(2r+1)(2r+2)\|b\|_\infty}{\pi^{1+r}}\, y \int_0^\infty\int_{-\infty}^\infty \frac{(4\pi)^r t^{2r}(y+t)}{\{(s-x)^2+(y+t)^2\}(y+t)^{2+2r}}\,ds\,dt\\
&\leq 4^r(2r+1)(2r+2)\|b\|_\infty\, y\int_0^\infty \frac{1}{(y+t)^2}\,dt\\
&\approx \|b\|_\infty.
\end{aligned}$$

This implies that $\|Q_r(b)\|_{\mathcal{B}^0} \lesssim \|b\|_\infty$, that is, Q_r is bounded. □

LEMMA 2.4. *$B^{1,r}\cap H^\infty$ is dense in $B^{1,r}$.*

Proof. Take any h in $B^{1,r}$ and any w in H. For any $\delta > 0$, let $h_\delta(z) = h(z+i\delta)$ for all $z \in H$. Since

$$\begin{aligned}
|h_\delta(w)| &= |h(w+i\delta)|\\
&= \left|\frac{1}{|B(w+i\delta,\delta/2)|}\int_{B(w+i\delta,\delta/2)} h\,dA(z)\right|\\
&\lesssim \int_{B(w+i\delta,\delta/2)} |h(z)|K(z,z)^{-r}\,dA(z) \lesssim \|h\|_{1,r},
\end{aligned}$$

h_δ is in H^∞. On the other hand,

$$\begin{aligned}
\|h_\delta\|_{1,r} &= (2+1)\int_H |h_\delta(z)|K(z,z)^{-r}\,dA(z)\\
&\lesssim \int_H |h(z)|K(z,z)^{-r}\,dA(z) \approx \|h\|_{1,r}
\end{aligned}$$

and hence $h_\delta \in B^{1,r}\cap H^\infty$. Since $C_C(H)$ is dense in $L^{1,r}$, for any f in $B^{1,r}$ and any $\epsilon > 0$, there is $g \in C_C(H)$ such that $\|g-f\|_{1,r} < \epsilon$. Since $\|f_\delta - f\|_{1,r} \leq \|f_\delta - g_\delta\|_{1,r} + \|g_\delta - g\|_{1,r} + \|g-f\|_{1,r}$, $B^{1,r}\cap H^\infty$ is dense in $B^{1,r}$. □

In order to prove that for any $f \in B^{1,r}$ and any $g \in \mathcal{B}^0(i)$, $f \cdot g \in B^{1,r}$, we define the set

$$\mathcal{D}_r = \{f \in B^{1,r} : \text{there is a constant } C_f \text{ such that } |f(z)| \le \frac{C_f}{(1+|z|)^{3+2r}} \text{ for all } z \in H\}.$$

The set $\mathcal{D}_r$ satisfies the following property.

PROPOSITION 2.5. *$\mathcal{D}_r$ is dense in $B^{1,r}$.*

Proof. For each $n \in \mathbb{N}$, let $\varphi_n(z) = (ni)^{3+2r}/(ni+z)^{3+2r}$ for all $z \in H$. Then for $z = x+iy$ in H,

$$|\varphi_n(z)| = \frac{n^{3+2r}}{(x^2+(y+n)^2)^{\frac{3+2r}{2}}} \le \left(\frac{\sqrt{2}n}{1+|z|}\right)^{3+2r}$$

and hence $\varphi_n \in \mathcal{D}_r$. Take any f in $B^{1,r}$ and any $\epsilon > 0$. By Lemma 2.4, there is $g \in B^{1,r} \cap H^\infty$ such that $\|g-f\|_{1,r} < \epsilon$. Since $g\varphi_n \in \mathcal{D}_r$ and $|g(z)\varphi_n(z) - g(z)| \le |g(z)|(1+|\varphi_n(z)|) \le 2|g(z)|$, Lebesque dominated convergence theorem implies that $\lim_{n\to\infty} \int_H |g(z)\varphi_n - g(z)|\, dA_r(z) = 0$ and hence $\lim_{n\to\infty} \|g\varphi_n - g\|_{1,r} = 0$. Since $\|g\varphi_n - f\|_{1,r} \le \|g\varphi_n - g\|_{1,r} + \|g-f\|_{1,r}$, $\mathcal{D}_r$ is dense in $B^{1,r}$. □

LEMMA 2.6. *For any $g \in \mathcal{B}^0(i)$ and any $z = x+iy \in H$,*

$$|g(z)| \le \|g\|_{\mathcal{B}^0}(1+|\log y| + 2\log(1+|x|)).$$

Proof. Suppose $z = x+iy \in H$. Since $g(i) = 0$,

$$\begin{aligned}
|g(z)| &\le |g(x+iy) - g(x+i(1+|x|))| \\
&\quad + |g(x+i(1+|x|)) - g((1+|x|)i)| + |g((1+|x|)i) - g(i)| \\
&= \left| \int_0^1 \frac{g'(x+((y-|x|-1)t+1+|x|)i)((y-1-|x|)t+1+|x|)}{(y-1-|x|)t+1+|x|} \right. \\
&\quad \left. \times (y-1-|x|)\, dt \right| + \left| \int_0^1 \frac{g'(tx+(1+|x|)i)(1+|x|)}{1+|x|} x\, dt \right| \\
&\quad + \left| \int_0^1 \frac{g'((t|x|+1)i)(t|x|+1)}{t|x|+1} |x| i\, dt \right| \\
&\le \|g\|_{\mathcal{B}^0}(|\log y| + 1 + 2\log(1+|x|)).
\end{aligned}$$

□

PROPOSITION 2.7. *Suppose $f \in \mathcal{D}_r$ and $g \in \mathcal{B}^0(i)$. Then $f \cdot g \in B^{1,r}$.*

Proof. Since $f \in \mathcal{D}_r$, there is a constant C_f such that $|f(z)| \le C_f/(1+|z|)^{3+2r}$ for all $z \in H$. Put $z = x + iy$. By Lemma 2.6,

$$\begin{aligned}
&\int_H |f(z)g(z)|\, dA_r(z) \\
&= (2r+1)\int_H |f(z)g(z)|K(z,z)^{-r}\, dA(z) \\
&\le C_f\|g\|_{\mathcal{B}^0}\int_H \frac{1+|\log y|+2\log(1+|x|)}{(1+|z|)^{3+2r}}(4\pi)^r(\operatorname{Im} z)^{2r}\, dA(z) \\
&\lesssim \|g\|_{\mathcal{B}^0}\int_H \frac{1+|\log y|+2\log(1+|x|)}{(1+|z|)^3}\, dA(z).
\end{aligned}$$

Since there is a compact subset K of H such that for $z \notin K$,

$$\log(1+|z|) < (1+|z|)^{\frac{1}{2}}, f\cdot g \in B^{1,r}. \qquad \square$$

3. The dual space of $B^{1,r}$

The main goal of this section is to prove that the dual space of weighted Bergman spaces $B^{1,r}$ can be identified with $\mathcal{B}^0(i)$. To do so, we define the following linear functional Λ_g on $\mathcal{D}_r$ as for g in $\mathcal{B}^0(i)$, $\Lambda_g(f) = (2r+1)\int_H f(z)\overline{g(z)}K(z,z)^{-r}\, dA(z)$. By Proposition 2.7, Λ_g is well-defined and linear. Let $h(z) = \frac{1+z}{1-z}i$. The h is a Riemann map from $\mathbb{D}$ onto H. Let $A^{1,r}(\mathbb{D})$ denote the space of analytic functions in $L^p(\mathbb{D}, dA_r)$, where $dA_r(z) = (2r+1)K_{\mathbb{D}}(z,z)^{-r}\, dA(z)$. Then $B^{1,r}$ can be identified with $A^{1,r}(\mathbb{D})$ via $\Psi(f)(w) = 2^{2+4r}f(h(w))/(1-w)^{4+4r}$ (see [3]). Let $\mathcal{B}(\mathbb{D})$ be the Bloch space of $\mathbb{D}$. Then we have the following:

LEMMA 3.1. (1) *For any* $f \in B^{1,r}$, $f(h(z))/(1-z)^{4+4r}$ *is in* $A^{1,r}(\mathbb{D})$. (2) *For any* $g \in \mathcal{B}^0(i)$, $\|g \circ h\|_{\mathcal{B}(\mathbb{D})} = 2\|g\|_{\mathcal{B}^0}$.

Proof. (1) Since

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|f(h(z))|}{|1-z|^{4+4r}}K_{\mathbb{D}}(z,z)^{-r}\, dA(z) \\
&= \int_H \frac{|f(h(h^{-1}(z)))|}{|1-h^{-1}(z)|^{4+4r}}K_{\mathbb{D}}(h^{-1}(z),h^{-1}(z))^{-r}|(h^{-1})'(z)|^2\, dA(z) \\
&= \pi^r\int_H \frac{|f(z)|}{\left|1-\frac{z-i}{z+i}\right|^{4+4r}}\left(1-\left|\frac{z-i}{z+i}\right|^2\right)^{2r}\left|\frac{2i}{(z+i)^2}\right|^2\, dA(z)
\end{aligned}$$

$$= \frac{\pi^r}{4^{1+r}} \int_H |f(z)|(\operatorname{Im} z)^{2r}\, dA(z) \approx \|f\|_{1,r} < \infty,$$

$f(h(z))/|1-z|^{4+4r}$ is in $A^{1,r}(\mathbb{D})$.

(2) Suppose $g \in \mathcal{B}^0(i)$. Then

$$\begin{aligned} \|g \circ h\|_{\mathcal{B}(\mathbb{D})} &= \sup_{z=x+iy\in\mathbb{D}} (1-|z|^2)|(g\circ h)'(z)| + |g\circ h(0)| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)|g'(h(z))||h'(z)| \\ &= \sup_{w=s+it\in H} \frac{|w+i|^2 - |w-i|^2}{|w+i|^2}|g'(w)|\frac{|w+i|^2}{2} \\ &= \sup_{w=s+it\in H} 2t|g'(w)| = 2\|g\|_{\mathcal{B}^0}. \end{aligned}$$ □

PROPOSITION 3.2. *For $g \in \mathcal{B}^0(i)$, the linear functional $\Lambda_g : \mathcal{D}_r \to \mathbb{C}$ defined by $\Lambda_g(f) = (2r+1)\int_H f(z)\overline{g(z)}K(z,z)^{-r}\, dA(z)$ is bounded.*

Proof. Since

$$\begin{aligned} \Lambda_g(f) &= (2r+1)\int_H f(z)\overline{g(z)}K(z,z)^{-r}\, dA(z) \\ &= (2r+1)\int_{\mathbb{D}} f(h(z))\overline{g(h(z))}K(h(z),h(z))^{-r}|h'(z)|^2\, dA(z) \end{aligned}$$

and $(A^{1,r})^* = \mathcal{B}(\mathbb{D})$, $|\Lambda_g(f)| \lesssim \|g\circ h\|_{\mathcal{B}(\mathbb{D})}\|f\|_{1,r} \lesssim \|g\|_{\mathcal{B}^0}\|f\|_{1,r}$ and hence $\|\Lambda_g\| \lesssim \|g\|_{\mathcal{B}^0}$, that is, Λ_g is bounded. □

PROPOSITION 3.3. *For $g \in \mathcal{B}^0(i)$, we define $\Lambda_g : B^{1,r} \to \mathbb{C}$ by*

$$\Lambda_g(f) = (2r+1)\int_H f(z)\overline{g(z)}K(z,z)^{-r}\, dA(z).$$

Then Λ_g is bounded.

Proof. Take any f in $B^{1,r}$. Since $\mathcal{D}_r$ is dense in $B^{1,r}$, there is a sequence $\{f_n\}$ in $\mathcal{D}_r$ such that $\lim_{n\to\infty}\|f_n - f\|_{1,r} = 0$. Since $\{\Lambda_g(f_n)\}$ is a Cauchy sequence in $\mathbb{C}$, $\lim_{n\to\infty}\Lambda_g(f_n)$ exists. Suppose that $\{h_n\}$ is a sequence in $\mathcal{D}_r$ such that $\lim_{n\to\infty}\|h_n - f\|_{1,r} = 0$. Since $|\Lambda_g(f_n - h_n)| \leq \|\Lambda_g\|\|f_n - h_n\|_{1,r} = \|\Lambda_g\|(\|f_n - f\|_{1,r} + \|f - h_n\|_{1,r})$, we define $\Lambda_g(f) = \lim_{n\to\infty}\Lambda_g(f_n)$. Since

$$|\Lambda_g(f)| = \lim_{n\to\infty}|\Lambda_g(f_n)| \leq \lim_{n\to\infty}\|\Lambda_g\|\|f_n\|_{1,r} = \|\Lambda_g\|\|f\|_{1,r},$$

Λ_g is a bounded linear functional on $B^{1,r}$. □

THEOREM 3.4. *We define $\Phi : \mathcal{B}^0(i) \to (B^{1,r})^*$ by $\Phi(g) = \Lambda_g$ for all $g \in \mathcal{B}^0(i)$. Then Φ is a bounded linear functional.*

Proof. It is immediate from Proposition 3.2 and Proposition 3.3. □

Main result of this paper is to show that $(B^{1,r})^*$ can be identified with $\mathcal{B}^0(i)$. To do so, we need the $B^{1,r}$-cancellation property.

LEMMA 3.5. *For any* $f \in B^{1,r}$, $\int_H f(w)K(w,w)^{-r}\,dA(w) = 0$.

Proof. Take any f in $B^{1,r}$. Then

$$\begin{aligned}
&\int_H f(w)K(w,w)^{-r}\,dA(w)\\
&= \int_{\mathbb{D}} f(h(z))K(h(z),h(z))^{-r}|h'(z)|^2\,dA(z)\\
&= 4^{1+r}\pi^r \int_{\mathbb{D}} \frac{f(g(z))}{|1-z|^{4+4r}}(1-|z|^2)^{2r}\,dA(z)\\
&= 4^{1+r}\pi^r \int_{\mathbb{D}} \frac{f(g(z))}{(1-z)^{4+4r}}\frac{(1-z)^{2+2r}}{(1-\overline{z})^{2+2r}}(1-|z|^2)^{2r}\,dA(z)\\
&= 4^{1+r}\pi^r \lim_{t\to 1^-}\int_{\mathbb{D}} \frac{f(g(z))}{(1-z)^{4+4r}}\frac{(1-tz)^{2+2r}}{(1-t\overline{z})^{2+2r}}(1-|z|^2)^{2r}\,dA(z) \text{ (Lemma 3.1)}\\
&= \frac{4^{1+r}\pi^{1+r}}{2r+1}\lim_{t\to 1^-}\int_{\mathbb{D}} \frac{f(g(z))(1-tz)^{2+2r}}{(1-z)^{4+4r}}(2r+1)\left\{\frac{1}{\pi(1-t\overline{z})^2}\right\}^{1+r}\\
&\qquad\qquad \times \pi^r(1-|z|^2)^{2r}\,dA(z)\\
&= \frac{4^{1+r}\pi^{1+r}}{2r+1}\lim_{t\to 1^-}\frac{f(g(t))}{(1-t)^{4+4r}}(1-t^2)^{2+2r}.
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{t\to 1^-}\frac{|f(g(t))|}{(1-t)^{4+4r}}\\
&= \lim_{t\to 1^-}\left|\frac{1}{|B(t,(\frac{1-t^2}{2})^{1+r})|}\int_{B(t,(\frac{1-t^2}{2})^{1+r})}\frac{f(g(z))}{(1-z)^{4+4r}}\,dA(z)\right|\\
&\le \lim_{t\to 1^-}\frac{4^{1+r}}{\pi(1-t^2)^{2+2r}}\int_{B(t,(\frac{1-t^2}{2})^{1+r})}\left|\frac{f(g(z))}{(1-z)^{4+4r}}\right|\,dA(z),\\
&\lim_{t\to 1^-}\frac{|f(g(t))|}{(1-t)^{4+4r}}(1-t^2)^{2+2r}\\
&\le \lim_{t\to 1^-}\frac{4^{1+r}}{\pi}\int_{B(t,(\frac{1-t^2}{2})^{1+r})}\left|\frac{f(g(z))}{(1-z)^{4+4r}}\right|\,dA = 0
\end{aligned}$$

and hence $\int_H f(w,w)K(w,w)^{-r}\,dA(w) = 0$. □

PROPOSITION 3.6. *Let Φ be a map defined in Theorem 3.4. Then Φ is 1-1 and onto.*

Proof. Take any g in $ker\Phi$. Let $\varphi_n(z) = (ni)^{3+2r}/(ni+z)^{3+2r}$ for all $z \in H$ and for each $n \in \mathbb{N}$. Then $g \in \mathcal{B}^0(i)$ and $\varphi_n(z) \in \mathcal{D}_r$ and hence $g \cdot \varphi_n \in B^{1,r}$. Since φ_n is bounded, for any $f \in B^{1,r}$,

$$\begin{aligned}&\lim_{n\to\infty} \int_H f(z)\overline{\varphi_n(z)g(z)}K(z,z)^{-r}\,dA(z)\\ &= \int_H f(z)\overline{g(z)}K(z,z)^{-r}\,dA(z) = 0.\end{aligned}$$

Since $M(\cdot,w)^{1+r} \in B^{1,r}$,

$$\begin{aligned}0 &= \lim_{n\to\infty} \int_H \varphi_n(z)g(z)\overline{M(z,w)^{1+r}}K(z,z)^{-r}\,dA(z)\\ &= \lim_{n\to\infty} \left(\int_H \varphi_n(z)g(z)\overline{K(z,w)^{1+r}}K(z,z)^{-r}\,dA(z)\right.\\ &\qquad \left. - \int_H \varphi_n(z)g(z)\overline{K(z,i)^{1+r}}K(z,z)^{-r}\,dA(z)\right)\\ &= \frac{1}{2r+1}\lim_{n\to\infty}(\varphi_n(w)g(w) - \varphi_n(i)g(i))\\ &= \frac{1}{2r+1}g(w)\end{aligned}$$

and hence $g=0$, that is, $ker\Phi = \{0\}$. This implies that Φ is one-to-one. Take any Ψ in $(B^{1,r})^*$. By Hahn-Banach Extension Theorem, there is a unique $\tilde{\Psi}$ in $(L^{1,r})^*$ such that $\tilde{\Psi} = \Psi$ on $B^{1,r}$ and $\|\tilde{\Psi}\| = \|\Psi\|$. The Riesz Representation Theorem implies that there is a unique b in L^∞ such that $\tilde{\Psi}(b) = \int_H f(w)\overline{b(z)}K(w,w)^{-r}\,dA(w)$ for all $f \in B^{1,r}$. Put $g = Q_r(b)$. Then $g \in \mathcal{B}^0(i)$ and for any $f \in B^{1,r}$,

$$\begin{aligned}\Phi(g)(f) &= \Lambda_g(f)\\ &= \int_H f(w)\overline{g(w)}K(w,w)^{-r}\,dA(w)\\ &= \int_H f(w)\overline{Q_r(b)(w)}K(w,w)^{-r}\,dA(w)\end{aligned}$$

$$= (2r+1)\int_H f(w)\overline{\int_H b(z)\overline{M(z,w)^{1+r}}K(z,z)^{-r}\,dA(z)}$$
$$\times K(w,w)^{-r}\,dA(w)$$
$$= (2r+1)\int_H f(w)\int_H \overline{b(z)}(K(z,w)^{1+r} - K(z,i)^{1+r})K(z,z)^{-r}\,dA(z)$$
$$\times K(w,w)^{-r}\,dA(w)$$
$$= \int_H \overline{b(z)}(2r+1)\int_H f(w)(\overline{K(w,z)^{1+r}} - \overline{K(i,z)^{1+r}})K(w,w)^{-r}\,dA(w)$$
$$\times K(z,z)^{-r}\,dA(z)$$
$$= \int_H \overline{b(z)}f(z)K(z,z)^{-r}\,dA(z) = \Psi(f),$$

that is, $\Phi(g) = \Psi$. This completes the proof. □

THEOREM 3.7. *For $r > -1/2$, $\mathcal{B}^0(i)$ is the dual space of $B^{1,r}$.*

Proof. The open mapping theorem implies that Φ is an isomorphism. □

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