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SETS OF WEAK EXPONENTS OF INDECOMPOSABILITY FOR IRREDUCIBLE BOOLEAN MATRICES

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Abstract. Let IB_n be the set of all irreducible matrices in B_n and let SIB_n be the set of all symmetric matrices in IB_n. Finding an upper bound for the set of indices of matrices in IB_n and SIB_n and determining gaps in the set of indices of matrices in IB_n and SIB_n has been studied by many researchers. In this paper, we establish a best upper bound for the set of weak exponents of indecomposability of matrices in SIB_n and IB_n, and show that there does not exist a gap in the set of weak exponents of indecomposability for any of class SIB_n and class IB_n.

1. Introduction

Let B_n be the set of all $n \times n$ Boolean matrices; that is, all $(0,1)$-matrices with the usual arithmetic except that $1+1=1$. Let r be an integer with $-n < r < n$. A matrix $A \in B_n$ is *r-indecomposable* if it contains no $k \times l$ zero submatrix with $1 \le k,\ l \le n$ and $k+l = n-r+1$. In particular, A is $(1-n)$-indecomposable if and only if $A \neq 0$, while A is $(n-1)$-indecomposable if and only if $A = J_n$, the all-1's matrix. A 1-indecomposable matrix is also said to be fully indecomposable, and a 0-indecomposable matrix is also called a Hall matrix.

By the definition of r-indecomposability, a matrix $A \in B_n$ is r-indecomposable if and only if, for each k such that $\max\{1, 1-r\} \le k \le \min\{n, n-r\}$, every $k \times n$ submatrix of A has at least $k+r$ columns with nonzero entries.

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A matrix $A \in B_n$ is reducible if there is a permutation matrix P such that

$$PAP^{-1} = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix},$$

where A_1 and A_2 are nonvacuous square matrices; otherwise A is irreducible. Note that for any $A \in IB_n$ with $n > 1$

$$A + A^2 + \cdots + A^n = J_n$$

and J_n is r-indecomposable for any r with $-n < r < n$. Hence for any $A \in IB_n$ and any integer r with $-n < r < n$, there exists a minimum positive integer p such that $A + A^2 + \cdots + A^p$ is r-indecomposable. Such an integer p is called the weak exponent of r-indecomposability of A, and is denoted by $w_r(A)$.

Brualdi and Liu[3] used $f_w(A)$, $h_w(A)$ for $w_1(A)$ and $w_0(A)$ and called them the weak fully indecomposable exponent and weak Hall exponent of A respectively. They suggested that further study on these weak exponents be done.

Liu[9] proved that $f_w(A) \leq \lfloor \frac{n}{2} \rfloor + 1$ and $h_w(A) \leq \lceil \frac{n}{2} \rceil$ for any $A \in IB_n$, where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greatest integer $\leq x$ and the smallest integer $\geq x$ respectively.

In [1], it is proven that $w_r(A) \leq \lfloor (n + r + 1)/2 \rfloor$, for any matrix $A \in IB_n$ and integer r with $-n < r < n$, and this upper bound is best possible.

Let SIB_n be the set of all symmetric matrices in IB_n. It can easily be seen that for any $A \in SIB_n$ with $n > 2$, $A + A^2 + \cdots + A^{n-1} = J_n$, which is equivalent to $w_{n-1}(A) \leq n - 1$ (See [5]). Let $A \in B_n$. The sequence of powers $A^0 = I$, A^1, A^2, ... forms a finite subsemigroup of B_n. Thus there is a least non-negative integer $k = k(A)$ and a least positive integer $p = p(A)$ such that $A^k = A^{k+p}$. The integers $k = k(A)$ and $p = p(A)$ are called the *index of convergence* or in short the *index* of A and the *period* of A, respectively. A matrix in IB_n with period 1 is called a primitive matrix. It is well known that $k(A) \leq n^2 - 2n + 2$ for any primitive matrix and the equality holds for the Wielandt matrix (see [4, p.82]).

The problem of finding an upper bound for the set of indices of a certain class of matrices in IB_n and SIB_n and determining gaps in the set of indices of a certain class of matrices in IB_n and SIB_n has been studied by many researchers. (See [6, 11, 13]. They especially studied the class of primitive matrices.) In this paper, we establish a best upper bound for the set of weak exponents of r-indecomposability of matrices in SIB_n and IB_n, and show that there is no gap in the set of weak exponents of r-indecomposability for any of class SIB_n and class IB_n.

2. Results

For a matrix $A = (a_{ij}) \in B_n$, the directed graph of A, $D(A)$, is the graph with vertex set $V(D(A)) = \{1, 2, \cdots, n\}$ and arc set $E(D(A)) = \{(i, j) : a_{ij} \neq 0\}$. It is well known that the (i, j) entry of A^k is nonzero if and only if there is a walk of length k from vertex i to vertex j in $D(A)$. If $A \in B_n$ is symmetric, then $D(A)$ corresponds naturally a graph $D_G(A)$ by replacing arcs (u, v) and (v, u) by an edge uv.

The following theorem provides a best possible upper bound for the set of weak exponents of r-indecomposability of matrices in SIB_n.

THEOREM 1. *For any matrix $A \in SIB_n$ with $n > 2$, and any integer r with $-n < r < n$, we have*

$$w_{1-n}(A) = w_{2-n}(A) = 1;$$

$$w_r(A) \leq \begin{cases} r & \text{if } 2 \leq r \leq n-1, \\ 2 & \text{if } 3-n \leq r \leq 1, \end{cases}$$

and this bound is best possible.

Proof. We consider the following three cases: $r = 1 - n$ or $2 - n$; $2 \leq r \leq n - 1$; $3 - n \leq r \leq 1$.

Suppose that $r = 1 - n$ or $2 - n$. For any $A \in SIB_n$, A has neither zero rows nor zero columns, implying that A is r-indecomposable. So $w_r(A) = 1$.

Suppose that $2 \leq r \leq n - 1$. Assume that $A + A^2 + \cdots + A^r$ is not r-indecomposable. Then it contains a $k \times l$ zero submatrix with $1 \leq k,\ l \leq n$ and $k + l = n - r + 1$. Let $D = D(A)$. Then there are subsets $V_1, V_2 \subseteq V(D)$ with $|V_1| = k$, $|V_2| = l$ such that for any integer m with $1 \leq m \leq r$, there is no walk of length m from any vertex in V_1 to any vertex in V_2. Since A is symmetric, $V_1 \cap V_2 = \emptyset$. On the other hand, by the strong connectivity of D, there is a vertex $u \in V_1$ and a vertex $v \in V_2$ such that the distance from u to v is at most $n - |V_1| - |V_2| + 1 = n - (n - r + 1) + 1 = r$, which is a contradiction. So $A + A^2 + \cdots + A^r$ is r-indecomposable and $w_r(A) \leq r$.

In order to show the sharpness of the bound, take $A_0 \in SIB_n$ where $D_G(A_0)$ is the path on n vertices $1, 2, \ldots, n$ with edges $i(i + 1)$, $i = 1, 2, \ldots, n-1$. It is easy to see that the $1 \times (n-r+1)$ submatrix indexed by the first row and the last $n-r$ columns in $A_0 + A_0^2 + \cdots + A_0^{r-1}$ is zero. This implies that $w_r(A_0) \geq r$. Hence $w_r(A_0) = r$.

Finally suppose that $3-n \leq r \leq 1$. Note that an r-indecomposable matrix is also $(r-1)$-indecomposable. In this case, $w_r(A) \leq w_1(A) \leq w_2(A) \leq 2$.

To show the bound is best possible, take $A_0 \in SIB_n$, where $D_G(A_0)$ is the star $K_{1,n-1}$. Clearly $w_r(A) = 2$. □

The following theorem completely determines the set of weak exponents of r-decomposability of matrices in SIB_n.

THEOREM 2. *Let* $w_r(SIB_n) = \{w_r(A) : A \in SIB_n\}$ *with* $n > 2$. *Then*

$$w_r(SIB_n) = \begin{cases} \{1\} & \text{if } r = 1-n, 2-n, \\ \{1,2\} & \text{if } 3-n \leq r \leq 1, \\ \{1,2,\ldots,r\} & \text{if } 2 \leq r \leq n-1. \end{cases}$$

Proof. Note that $J_n \in SIB_n$, $w_r(J_n) = 1$ for all $1-n \leq r \leq n-1$. The case $1-n \leq r \leq 2$ follows from Theorem 1. Note also that $w_r(A_0) = 2$ for all $3 \leq r \leq n-1$, where $D_G(A_0)$ is the star $K_{1,n-1}$. Suppose $3 \leq r \leq n-1$. By Theorem 1 we only need to show that $\{3,\ldots,r-1\} \subseteq w_r(SIB_n)$ for $3 \leq r \leq n-1$.

For any integer $3 \leq k \leq r-1$, take $A_1 \in SIB_n$, where $D_G(A_1) = G$ is a graph on vertices $1,2,\ldots,n$ with edges $i(n-k+1)$, $i = 1,2,\ldots,n-k$ and $i(i+1)$, $i = n-k+1,\ldots,n$. It is easy to see that $A_1 + A_1^2 + \cdots + A_1^{k-1}$ contains an $(n-k) \times 1$ zero submatrix, so A_1 is not k-indecomposable and hence not r-indecomposable. But $A_1 + A_1^2 + \cdots + A_1^k = J_n$. We have $w_r(A_1) = k$, and hence $\{3,\ldots,r-1\} \subseteq w_r(SIB_n)$ for $3 \leq r \leq n-1$. □

Let $A \in B_n$ and let $X \subseteq V(D(A))$. By $R_t(A,X)$, we denote the set of all vertices reachable from a vertex in X via a walk of length t. Clearly, $R_1(A^i,X) = R_i(A,X)$. Then $A \in B_n$ is r-indecomposable if and only if, for each $X \subseteq V(D(A))$ with $\max\{1,1-r\} \leq |X| \leq \min\{n,n-r\}$, $|R_1(A,X)| \geq |X| + r$.

The following theorem completely determines the set of weak exponents of r-decomposability of matrices in IB_n. We need the following Lemma to prove it.

LEMMA 3 ([1, Lemma 1], [9]). *Suppose that* $A \in IB_n$, $X \subseteq V(D(A))$, *and* $1 \leq t \leq n$. *If* $R_1(\sum_{i=1}^{t} A^i, X) \neq V(D(A))$, *then*

$$\left| R_1(\sum_{i=1}^{t} A^i, X) \right| \geq |R_1(A,X)| + t - 1.$$

THEOREM 4. *Let $w_r(IB_n) = \{w_r(A) : A \in IB_n\}$ with $-n < r < n$, $n > 1$. Then*

$$w_r(IB_n) = \left\{1, 2, \ldots, \left\lfloor \frac{n+r+1}{2} \right\rfloor \right\}.$$

Proof. Note that [1] $w_r(A) \leq \lfloor (n+r+1)/2 \rfloor$ for any $A \in IB_n$. The case $r = 1-n, 2-n$ is trivial. Suppose in the following $3-n \leq r \leq n-1$. We need only to show that

$$\{1, 2, \ldots, \lfloor (n+r+1)/2 \rfloor\} \subseteq w_r(IB_n).$$

For integer a with $\max\{1-r, 1\} \leq a \leq \lfloor (n-r+1)/2 \rfloor$, take $A_0 \in IB_n$ with $D(A) = D$, where $V(D) = \{1, 2, \cdots, n\}$ and $E(D) = \{(i, a+1) : 1 \leq i \leq a\} \cup \{(i, i+1) : a+1 \leq i \leq n-1\} \cup \{(n, i) : 1 \leq i \leq a\}$. It can be easily seen that all columns except columns $a+1, \ldots, 2a+r-1$ are zero in rows $1, 2, \ldots, a$ of $A_0 + A_0^2 \cdots + A_0^{a+r-1}$; hence $A_0 + A_0^2 + \cdots + A_0^{a+r-1}$ contains a $a \times (n-a-r+1)$ zero submatrix with $a + (n-a-r+1) = n-r+1$, which implies that $w_r(A_0) \geq a+r$. It can be checked that for each $X \subseteq V(D)$ with $\max\{1, 1-r\} \leq |X| \leq \min\{n, n-r\}$,

$$|R_1(A_0, X)| \geq |X| - a + 1,$$

and hence, by Lemma 3, $|R_1(A_0 + A_0^2 + \cdots + A_0^{a+r}, X)| \geq |R_1(A_0, X)| + a + r - 1 \geq |X| + r$. This implies that $A_0 + A_0^2 + \cdots + A_0^{a+r}$ is r-indecomposable. We have $w_r(A_0) = a + r$.

Suppose that $3 - n \leq r \leq -1$. We take $a = 1-r, 2-r, \ldots, \lfloor (n-r+1)/2 \rfloor$ to obtain $\{1, 2, \ldots, \lfloor (n+r+1)/2 \rfloor\} \subseteq w_r(IB_n)$.

If $1 \leq r \leq n-1$, then we first take $a = 1, 2, \lfloor (n-r+1)/2 \rfloor$ to obtain $\{r+1, r+2, \ldots, \lfloor (n+r+1)/2 \rfloor\} \subseteq w_r(IB_n)$. Then by Theorem 2, we have $\{1, 2, \ldots, r\} \subseteq w_r(IB_n)$.

In either case, $\{1, 2, \ldots, \lfloor (n+r+1)/2 \rfloor\} \subseteq w_r(IB_n)$. It completes the proof. □

3. Closing remark

Let $A \in B_n$. If there exists a positive integer k such that A^k is r-indecomposable, then the smallest positive integer k is called the *exponent of r-indecomposability* of A. If there exists a positive integer k such that A^i is r-indecomposable for all $i \geq k$, then the smallest positive integer k is called the *strict exponent of r-indecomposability* of A. These (strict) exponents of r-indecomposability of primitive matrices have been investigated in [7, 12]. The cases when $r = 1$ (fully indecomposable exponent) and $r = 0$ (Hall exponent) have already been studied extensively (see [2, 3, 8, 10]). In this paper, we studied weak exponent of r-indecomposability of irreducible matrices some of whose special cases are the weak fully indecomposable exponent and weak Hall exponent initiated by Brualdi and Liu [3]. Theorems 2 and 4 tell us that there is no gap in the set of weak exponents of r-indecomposability for any of class SIB_n and class IB_n.

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