Bull. Korean Math. Soc. **42** (2005), No. 2, pp. 421–432

POSITIVE p-HARMONIC FUNCTIONS ON GRAPHS

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ABSTRACT. Suppose that an infinite graph G of bounded degree has finite number of ends, each of which is p-regular, where $1 < p < \infty$. Then we can identify all the positive (bounded, respectively) p-harmonic functions on G.

1. Introduction

In this paper, we study the Liouville type property on a graph. By the Liouville property on a graph G, we mean that every bounded harmonic function on G is constant. It immediately follows that the set of all bounded harmonic functions on G having Liouville property is in one to one correspondence with the real line $\mathbf{R}$. It seems natural to regard the case that the set of all bounded harmonic functions on G is in one to one correspondence with $\mathbf{R}^l$ for some positive integer l as a generalized version of the Liouville property. In line with this view point, we consider the case of the p-Laplacian operator ($1 < p < \infty$) and the positive (bounded, respectively) p-harmonic functions on graphs of bounded degree. In the case that a graph G has a finite number of ends, each of which is p-regular, we identify all the positive (bounded, respectively) p-harmonic functions on G in section 4.

On the other hand, Kanai[5] proved that if G and H are roughly isometric graphs of bounded degree, then H is parabolic whenever G is. In [9], Soardi proved that if G and H are roughly isometric graphs of bounded degree, and if G has no nonconstant harmonic functions with finite energy, then neither has H. Later, the second author [8] proved that the dimension of the space of harmonic functions with finite energy is preserved under rough isometries between graphs of bounded degree. In this paper, we also discuss its rough isometric invariance in section 3

Received November 11, 2004.
2000 Mathematics Subject Classification: 31C20, 31C45.
Key words and phrases: p-harmonic function, Liouville property.

and extend our result to graphs being rough isometric to those satisfying some analytic properties in section 4.

2. Preliminaries

In this section we introduce briefly some notions and results related to the discrete potential theory. We refer to [3] for a more detailed results. Let $G = (V, E)$ be an infinite graph with no self-loops, where V and E denote the set of vertices and the set of edges of G, respectively. If vertices x and y are the endpoints of the same edge, we say that x and y are neighbors to each other and write $y \in N_x$ and $x \in N_y$. A sequence $\mathbf{x} = (x_0, x_1, \cdots, x_l)$ in G is called a path from x_0 to x_l with the length l if $x_k \in N_{x_{k-1}}$ for $k = 1, 2, \cdots, l$. Then for any $x, y \in V$, we define a distance function $d(x, y)$ to be the minimum of the lengths of paths from x to y. It is easy to check that d defines a metric on G. For this metric d, define an l-neighborhood $N_l(x) = \{y \in V : d(x, y) \leq l\}$ for each $x \in V$ and each $l \in \mathbf{N}$. In particular, we say that a subset H of V is connected if for any $x, y \in H$, there is a path $\mathbf{x} = (x_0, x_1, \cdots, x_l)$ in H such that $x_0 = x$ and $x_l = y$. Through this paper, we assume that each graph G is connected and has bounded degree, that is, $\sup_{x \in V} |N_x| < \infty$, where V and $|A|$ denote the set of vertices of G and the cardinality of the set A, respectively. Through this section, V stands for the vertex set of an infinite connected graph G of bounded degree.

Given any $U \subset V$, the outer boundary ∂U and the inner boundary δU of U are defined by $\partial U = \{x \in V : d(x, U) = 1\}$ and $\delta U = \{x \in V : d(x, V \setminus U) = 1\}$. For each real valued function u on $U \cup \partial U$, define the norm of the p-gradient $(1 < p < \infty)$ of u at $x \in U$ by $|Du|(x) = \left(\sum_{y \in N_x} |u(y) - u(x)|^p\right)^{1/p}$. Through this paper, we also assume that $1 < p < \infty$.

Definition 2.1. Let U be a finite subset of V with $\partial U \neq \emptyset$. Let u be a real valued function on $U \cup \partial U$. We say that u is p-harmonic on U, if u is a minimizer of the p-energy, that is, $\sum_{x \in U} |Du|^p(x) \leq \sum_{x \in U} |Dv|^p(x)$ for every real valued function v on $U \cup \partial U$ such that $v = u$ on ∂U.

Proposition 2.2. *Let U be a finite subset of V with $\partial U \neq \emptyset$. Then a real valued function u is p-harmonic on U if and only if*

$$\sum_{x \in U} \sum_{y \in N_x} |u(y) - u(x)|^{p-2}(u(y) - u(x))(w(y) - w(x)) = 0 \tag{2.1}$$

for any real valued function w on $U \cup \partial U$ such that $w = 0$ on ∂U.

Proof. Suppose that u is p-harmonic on U. Then by the p-harmonicity, $\frac{d}{dt}\Big|_{t=0} \sum_{x\in U} |D(u+tw)|^p(x) = 0$, where w is a real valued function with $w = 0$ on ∂U and $t \in \mathbf{R}$. The equation (2.1) immediately follows from this.

Conversely, assume that the equation (2.1) holds. Let v be a real valued function on $U \cup \partial U$ such that $v = u$ on ∂U. Putting $w = u - v$,

$$\sum_{x\in U}\sum_{y\in N_x} |u(y)-u(x)|^{p-2}(u(y)-u(x))(w(y)-w(x)) = 0.$$

This implies that, by the Hölder inequality,

$$\begin{aligned}
&\sum_{x\in U} |Du|^p(x) \\
&= \sum_{x\in U}\sum_{y\in N_x} |u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x)) \\
&\le \left(\sum_{x\in U}\sum_{y\in N_x} |u(y)-u(x)|^p\right)^{(p-1)/p} \left(\sum_{y\in N_x} |v(y)-v(x)|^p\right)^{1/p} \\
&\le \left(\sum_{x\in U} |Du|^p(x)\right)^{(p-1)/p} \left(\sum_{x\in U} |Dv|^p(x)\right)^{1/p}.
\end{aligned}$$

Hence we have the conclusion. □

From Proposition 2.2, we have an equivalent definition of the p-harmonicity:

DEFINITION 2.3. Let U be a subset of V with $\partial U \neq \emptyset$. We say that a real valued function u is p-harmonic (p-superharmonic, p-subharmonic, respectively) on U if

$$\sum_{x\in U}\sum_{y\in N_x} |u(y)-u(x)|^{p-2}(u(y)-u(x))(w(y)-w(x)) = 0 \tag{2.2}$$

(≥ 0, ≤ 0, respectively) for any finitely supported (nonnegative, respectively) real valued function w on $U \cup \partial U$ such that $w = 0$ on ∂U.

PROPOSITION 2.4 (Local Harnack Inequality). *Let U be a subset of V and u be a nonnegative p-superharmonic function on U. Then for each $x \in U$, we have*

$$\max_{y\in N_x} u(y) \le \left(|N_x|^{1/(p-1)} + 1\right) u(x).$$

Proof. By adding some constant, we may assume that $u > 0$ on U. Further replacing u with $u/u(x)$ allows us to assume that $u(x) = 1$. Let $u(y_0) = \max_{y\in N_x} u(y)$ for some $y_0 \in N_x$. Choose a real valued function w on G such that $w(x) = 1$ and $w(y) = 0$ for all $y \in G \setminus \{x\}$. Then we have

$$\begin{aligned} 0 &\le \frac{1}{2}\sum_{z\in U}\sum_{y\in N_z} |u(y)-u(z)|^{p-2}(u(y)-u(z))(w(y)-w(z)) \\ &= \frac{1}{2}\sum_{y\in N_x} |u(x)-u(y)|^{p-2}(u(x)-u(y)) \\ &\quad -\frac{1}{2}\sum_{y\in N_x} |u(y)-u(x)|^{p-2}(u(y)-u(x)) \\ &= -\sum_{y\in N_x} |u(y)-u(x)|^{p-2}(u(y)-u(x)) \\ &= -\sum_{y\in N_x^+} (u(y)-1)^{p-1} + \sum_{y\in N_x^-} (1-u(y))^{p-1}, \end{aligned}$$

where $N_x^+ = \{y \in N_x : u(y) > 1\}$ and $N_x^- = \{y \in N_x : u(y) < 1\}$. Hence we get $(u(y_0) - 1)^{p-1} \le |N_x|$. □

PROPOSITION 2.5 (Maximum Principle). *Let U be a connected subset of V and u be a p-subharmonic function on U. If u attains an interior maximum in U, then u must be constant on $U \cup \partial U$.*

Proof. Suppose that u attains its maximum at a point $x_0 \in U$. Choose a real valued function w on G such that $w(x_0) = 1$ and $w(y) = 0$ for all $y \in G \setminus \{x_0\}$. Then from the p-subharmonicity of u, we have

$$\sum_{y\in N_{x_0}} |u(y)-u(x_0)|^{p-2}(u(y)-u(x_0)) \ge 0.$$

By the maximality of u at x_0, $u(y) - u(x_0) \le 0$ for all $y \in N_{x_0}$. This implies that $u(y) = u(x_0)$ for all $y \in N_{x_0}$. By the connectedness of U, $u(y) = u(x_0)$ for all $y \in U \cup \partial U$. □

3. p-regular ends and rough isometric invariants

Let o be a fixed vertex in a graph G. For each $l \in \mathbf{N}$, we denote by $\sharp(l)$ the number of unbounded components of $G \setminus N_l(o)$. If $\lim_{l\to\infty} \sharp(l) = k$, then we say that the number of ends of G is k. If k is finite, then we can choose an integer l_0 such that $\sharp(l) = k$ for all $l \ge l_0$. In this case, there exist mutually disjoint unbounded components $E_1, E_2, \cdots, E_k$ of $G \setminus N_{l_0}(o)$ and we call each E_i an end of G for $i = 1, 2, \cdots, k$.

By using the standard Moser iteration, Holopainen and Soardi[4] pointed out that the Harnack inequality for nonnegative p-harmonic functions holds on each ball satisfying the volume doubling condition and the Poincaré inequality. Following their program, one can prove that the Harnack inequality on each ball at infinity of ends for nonnegative p-harmonic functions holds if each end satisfies the following conditions: Let E be an end of G and L be a sufficiently large integer.

(VD) For a given integer $n \geq 2$, there is a constant $C < \infty$ depending only on n such that for any point $x \in \partial N_L(o) \cap E$ and any integer $0 < l < L/2n$,

$$|N_{nl}(x)| \leq C|N_l(x)|.$$

(P) There exist a constant $C < \infty$ and an integer n such that for any point $x \in \partial N_L(o) \cap E$, any integer $0 < l < L/2n$ and any real valued functions f on $N_{nl}(x)$,

$$\frac{1}{|N_l(x)|} \sum_{y \in N_l(x)} |f(y) - \overline{f}| \leq Cl \left(\frac{1}{|N_{nl}(x)|} \sum_{y \in N_{nl}(x)} |Df|^p(y) \right)^{1/p},$$

where $\overline{f} = |N_l(x)|^{-1} \sum_{y \in N_l(x)} f(y)$.

In order to obtain the Harnack inequality at infinity for nonnegative p-harmonic functions, we need the uniform connectedness at infinity of each end as follows:

(FC) For a given integer n, there exist an integer $m = m(n)$ and points $x_1, x_2, \cdots, x_m$ in $\partial C_{E,L}$ such that for sufficiently large integer L

$$\partial C_{E,L} \subset \bigcup_{i=1}^{m} N_l(x_i)$$

and $\bigcup_{i=1}^{m} N_l(x_i)$ is connected, where $C_{E,L}$ denotes the unbounded component of $E \setminus N_L(o)$ and l is the largest integer such that $0 < l \leq L/n$.

From the finite covering condition (FC) together with (VD) and (P), we get the following:

THEOREM 3.1 (Harnack Inequality). *Let E be an end of a graph G satisfying the conditions (VD), (P), and (FC). Then there exists a* *constant $C < \infty$ such that for any nonnegative p-harmonic function f on E and for sufficiently large integer L,*

$$\sup_{\partial C_{E,L}} f \leq C \inf_{\partial C_{E,L}} f.$$

EXAMPLE. Here we give some examples of graphs that satisfy conditions (VD), (P), and (FC). (a) $\mathbf{Z}^n \# \mathbf{Z}^n, n \geq 3$. (b) Nets of a connected sum of complete Riemannian manifolds with nonnegative Ricci curvature.

A map, not necessarily continuous, $\varphi : X \to Y$ is called a rough isometry, introduced by Kanai in [5] and [6], between two metric spaces X and Y if φ satisfies the following condition:

(R) For some $\tau > 0$, the τ-neighborhood of the image $\varphi(X)$ covers Y; there exist constants $a \geq 1$ and $b \geq 0$ such that

$$\frac{1}{a} d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq a\, d(x_1, x_2) + b$$

for all $x_1, x_2 \in X$, where d denotes the distances of X and Y induced from their metrics, respectively.

From now on, τ, a and b always mean those which appear in (R). Especially, being roughly isometric is an equivalence relation (See [5]). The conditions (VD), (P), and (FC) slightly modified are rough isometric invariants. It has to be emphasized that these modified conditions still validate the Harnack inequality at infinity for nonnegative p-harmonic functions.

Following the arguments of the authors [7], we have:

LEMMA 3.2. *Let $\varphi : G \to H$ be a rough isometry between graphs G and H. Suppose that G has k ends, say $D_1, D_2, \cdots, D_k$. Then H also has k ends, say $E_1, E_2, \cdots, E_k$. Moreover, φ induces a rough isometry between ends in such a way that for each $i = 1, 2, \cdots, k$ the restriction map $\varphi|_{D_i} : D_i \to E_i$ is also a rough isometry.*

We say that an end E of G is p-nonparabolic if for some $l_1 \geq l_0$, there exists a real valued function u_E, called a p-harmonic measure, on $E \setminus N_{l_1}(o)$ such that u_E is p-harmonic in $E \setminus N_{l_1}(o)$, $u_E = 0$ on $\delta N_{l_1}(o) \cap E$ and $\sup_{E \setminus N_{l_1}(o)} u_E = 1$. Otherwise, E is called p-parabolic.

Following the programs in [1], [5], and [7], one can prove the rough isometric invariance of each of the p-parabolicity, (VD), (P), and (FC) as follows:

LEMMA 3.3. *Let D and E be ends of graphs G and H, respectively. Let $\varphi : D \to E$ be a rough isometry. Then we have the followings: For a fixed vertex o in H,*

(i) *D is p-parabolic if and only if E is p-parabolic.*

(ii) *If D satisfies the condition (VD), then for each integer $\tilde{n} \geq 2$, there exists a constant $C < \infty$ depending only on $\tilde{n}$ such that for any $x \in \partial N_L(o) \cap E$ and any integer $0 < l < L/8a^2\tilde{n}$,*

$$|N_{\tilde{n}l}(x)| \leq C|N_l(x)|.$$

(iii) *If D satisfies the condition (P), then there exist a constant $C < \infty$ and an integer $\tilde{n}$ such that for any point $x \in \partial N_L(o) \cap E$, any integer $0 < l < L/2\tilde{n}$ and any real valued function f on $N_{\tilde{n}l}(x)$,*

$$\frac{1}{|N_l(x)|} \sum_{y \in N_l(x)} |f(y) - \overline{f}| \leq Cl \left(\frac{1}{|N_{\tilde{n}l}(x)|} \sum_{y \in N_{\tilde{n}l}(x)} |Df|^p(y) \right)^{1/p},$$

where $\overline{f} = |N_l(x)|^{-1} \sum_{y \in N_l(x)} f(y)$.

(iv) *If D satisfies the condition (FC), then there exist a sequence $\{H_L\}$ of finite subsets of E and positive integers $\tilde{m}, \tilde{n}$ such that each H_L divides E into a bounded subset and the unbounded component of $E \setminus H_L$, $d(o, H_L) \to \infty$ as $L \to \infty$ and*

$$H_L \subset \bigcup_{i=1}^{\tilde{m}} N_l(\varphi(x_i)),$$

where $\bigcup_{i=1}^{\tilde{m}} N_l(\varphi(x_i))$ is connected and l is the largest integer such that $l \leq L/\tilde{n}$.

Combining Theorem 3.1 and Lemma 3.3, we get the following theorem:

THEOREM 3.4. *Let D be an end of a graph G satisfying the conditions (VD), (P), and (FC). Let E be an end of a graph H being roughly isometric to D. Then there exist a constant $C < \infty$ such that for any nonnegative p-harmonic function f on E and any sufficiently large integer L,*

$$\sup_{H_L} f \leq C \inf_{H_L} f,$$

where the set H_L is given in (iv) of Lemma 3.3.

Definition 3.5. We say that an end E of a graph G is p-regular if there exist a sequence $\{H_L\}$ of finite subsets of E and a constant $C < \infty$ such that for any nonnegative p-harmonic function f on E,

$$\sup_{H_L} f \leq C \inf_{H_L} f,$$

where $d(o, H_L) \to \infty$ as $L \to \infty$, and each H_L divides E into a bounded subset and the unbounded component of $E \setminus H_L$.

4. Proof of main results

A p-parabolic end can be characterized by the following property:

Lemma 4.1. *Let e be a p-parabolic end of a graph G and f be a nontrivial p-harmonic function bounded above on $e \setminus N_{l_0}(o)$ for some $l_0 \in \mathbf{N}$. Then*

$$\sup_e f > \limsup_{x \to \infty,\, x \in e} f(x).$$

Proof. Suppose that $\limsup_{x \to \infty, x \in e} f(x) = \sup_e f = M$. Since f is nontrivial, there exists a proper subset Ω of $e \setminus N_{l_0}(o)$ such that

$$\Omega = \{x \in e \setminus N_{l_0}(o) : f(x) > M - \epsilon\}$$

for sufficiently small $\epsilon > 0$. Put $v = \max\{(f - M + \epsilon)/\epsilon, 0\}$. Then v is a nonnegative p-subharmonic in $e \setminus N_{l_0}(o)$, $v \equiv 0$ on $e \setminus \Omega$, and $\sup_{e \setminus N_{l_0}(o)} v = 1$. By the maximum principle, we can construct a p-harmonic function u_e on $e \setminus N_{l_0}(o)$ such that $u_e = 0$ on $\delta N_{l_0}(o) \cap e$, and $\sup_{e \setminus N_{l_0}(o)} u_e = 1$. This is a contradiction to the p-parabolicity of the end e. □

Combining Lemma 4.1 and the maximum principle, we get the Liouville theorem for positive p-harmonic functions for the graphs with only p-parabolic ends.

Theorem 4.2 (Liouville Theorem). *Let G be a graph with only p-parabolic (not necessarily p-regular) ends. Then every positive p-harmonic function on G must be constant.*

Let E be a p-regular end of G and f be a nonnegative p-harmonic function on E. Then there exists a constant $0 \leq C \leq \infty$ such that

$$\lim_{x \to \infty,\, x \in E} f(x) = C. \tag{4.1}$$

Moreover, if E is p-nonparabolic, it is easy to see that the constant C in (4.1) is finite and one can construct a bounded p-harmonic function f_E on G in such a way that

$$\lim_{x\to\infty,\, x\in E} f_E(x) = 1 \quad \text{and} \quad \lim_{x\to\infty,\, x\in E'} f_E(x) = 0$$

for any other p-nonparabolic end E'. If e is a p-regular p-parabolic end, then, by local Harnack inequality, for some $l_0 \in \mathbf{N}$, we can construct a p-harmonic function v_e on $e \setminus N_{l_0}(o)$ in such a way that $v_e \geq 0$ on $e \setminus N_{l_0}(o)$, $v_e = 0$ on $\delta N_{l_0}(o) \cap e$, and $\lim_{x\to\infty,\, x\in e} v_e(x) = \infty$. If we further assume that G has at least one p-nonparabolic end, by the strong maximum principle, one can construct a nonnegative p-harmonic function h_e on G in such a way that $\lim_{x\to\infty,\, x\in e} h_e(x) = \infty$, $\lim_{x\to\infty,\, x\in E} h_e(x) = 0$ for any p-nonparabolic end E, and $0 \leq h_e \leq c < \infty$ on $G \setminus e$.

From now on, each end of G is assumed to be p-regular unless specified otherwise.

THEOREM 4.3. *Let G be an infinite graph with p-regular p-nonparabolic ends $E_1, E_2, \cdots, E_l$. Then for any $a_1, a_2, \cdots, a_l \in \mathbf{R}$, there exists a bounded p-harmonic function f on G such that for each $i = 1, 2, \cdots, l$*

$$\lim_{x\to\infty,\, x\in E_i} f(x) = a_i. \tag{4.2}$$

Furthermore, such a function f has finite p-energy, that is,

$$\sum_{x\in G} |Df|^p(x) < \infty. \tag{4.3}$$

In particular, if a bounded p-harmonic function f on G is asymptotically zero on each p-nonparabolic end E_i, that is, every value a_i in equation (4.2) equals zero, then f is identically zero.

Proof. Without loss of generality, we may assume that $0 < a_1 \leq a_2 \leq \cdots \leq a_l \leq 2a_1$. Construct a sequence of real valued functions $\{f_l\}_{l>l_0}$ such that $f_l = 0$ is p-harmonic on $N_l(o)$, $f_l = a_i$ on $\delta N_l(o) \cap E_i$ and $f_l = 0$ on $\delta N_l(o) \setminus (\bigcup_{i=1}^{l} E_i)$, where $i = 1, 2, \cdots, l$. Then $a_i u_{E_i} \leq f_l \leq a_i(2 - u_{E_i})$ on $(\partial N_{l_0}(o) \cup \partial N_l(o)) \cap E_i$, where u_{E_i} is a p-harmonic measure of E_i. Hence by the comparison principle, we have $a_i u_{E_i} \leq f_l \leq a_i(2 - u_{E_i})$ on $N_l(o) \cap E_i$ for $i = 1, 2, \cdots, l$.

By Ascoli's theorem, there exists a subsequence, denoted again by $\{f_l\}$, converging uniformly on any finite subset of G. The limit function $f = \lim_{l\to\infty} f_l$ is also p-harmonic on G such that

$$a_i u_{E_i} \leq f \leq a_i(2 - u_{E_i}) \quad \text{on} \quad E_i.$$

Therefore, we get (4.2).

Since the subsequence $\{f_l\}$ converges uniformly on any finite subset, by the minimizing property of p-harmonic functions, we have

$$\sum_{x\in E_i} |Df|^p(x) < \infty.$$

Applying this argument to other ends, we get (4.3).

We next prove the last statement. Assume that $\sup_G f = m > 0$. Then by the maximum principle, there exists a p-parabolic end e such that $\lim_{x\to\infty,\, x\in e} f(x) = m$. By Lemma 4.1, this is a contradiction. □

Combining Theorem 4.3 and the comparison principle, we get a generalized Liouville property for bounded p-harmonic functions.

THEOREM 4.4. *Let G be an infinite graph with only p-regular p-nonparabolic ends $E_1, E_2, \cdots, E_l$. Then for each bounded p-harmonic function f on G, there exist real numbers*

$$a_i = \lim_{x\to\infty,\, x\in E_i} f(x) \tag{4.4}$$

for each $i = 1, 2, \cdots, l$. Conversely, given any $a_1, a_2, \cdots, a_l \in \mathbf{R}$, there exists a unique bounded p-harmonic function f with finite p-energy on G satisfying (4.4) for each $i = 1, 2, \cdots, l$. Moreover, every nonnegative p-harmonic function on G is bounded and has finite p-energy.

We are ready to prove the existence theorem for positive p-harmonic functions. This result directly generalizes that of Holopainen[2].

THEOREM 4.5. *Let G be an infinite graph with p-regular p-nonparabolic ends $E_1, E_2, \cdots, E_l$ and p-regular p-parabolic ends $e_1, e_2, \cdots, e_s$, respectively. Then for any nonnegative $a_1, a_2, \cdots, a_l \in \mathbf{R}$ and any integer $0 \le t \le s$, there exists a nonnegative p-harmonic function f on G such that for each $i = 1, 2, \cdots, l$ and $j = 1, 2, \cdots, t$,*

(i) $\lim_{x\to\infty,\, x\in E_i} f(x) = a_i$;
(ii) $\lim_{x\to\infty,\, x\in e_j} f(x) = \infty$;
(iii) *f is bounded on $G \setminus \bigcup_{j=1}^t e_j$.*

Proof. Assume that $t \ge 1$. Then as mentioned before, we can choose a nonnegative p-harmonic function h_j on G for each $j = 1, 2, \cdots, t$ in such a way that $\lim_{x\to\infty,\, x\in e_j} h_j(x) = \infty$, $\lim_{x\to\infty,\, x\in E_i} h_j(x) = 0$ for each $i = 1, 2, \cdots, l$, and $0 \le h_j \le C_j < \infty$ on $G \setminus e_j$. Put $C = \max\{\sup_{M\setminus e_j} h_j : j = 1, 2, \cdots, t\}$. We may assume that $0 < a_1 \le a_2 \le \cdots \le a_l \le 2a_1$ and $C \le 2a_1 - a_l$. Then by the comparison principle, we have $h_j + a_i \le a_i(2 - u_{E_i})$ on each E_i, where u_{E_i} is a p-harmonic measure of E_i. Now construct a sequence of real valued functions $\{f_l\}_{l>l_0}$ such that f_l is p-harmonic in $N_l(o)$, $f_l = a_i$ on $\partial N_l(o) \cap E_i$, $f_l = h_j$ on $\partial N_l(o) \cap e_j$ and $f_l = a_1$ on $\partial N_l(o) \setminus \bigcup_{j=1}^t e_j$, where $i = 1, 2, \cdots, l$ and $j = 1, 2, \cdots, t$. By the comparison principle, $f_l \leq h_j + a_i$ on $N_l(o) \cap E_i$, hence $a_i u_{E_i} \leq f_l \leq a_i(2 - u_{E_i})$ on $N_l(o) \cap E_i$. On the other hand, $h_j \leq f_l \leq h_j + a_l$ on $N_l(o) \cap e_j$.

By Ascoli's theorem, there exists a subsequence, denoted again by $\{f_l\}$, converging uniformly on any finite subset of G. The limit function $f = \lim_{l\to\infty} f_l$ is a nonnegative p-harmonic on G such that $a_i u_{E_i} \leq f \leq a_i(2 - u_{E_i})$ on E_i for each $i = 1, 2, \cdots, l$, $h_j \leq f \leq h_j + a_l$ on e_j for each $j = 1, 2, \cdots, t$. Hence we get

$$\lim_{x\to\infty,\, x\in E_i} f(x) = a_i \quad \text{and} \quad \lim_{x\to\infty,\, x\in e_j} f(x) = \infty,$$

where $i = 1, 2, \cdots, l$ and $j = 1, 2, \cdots, t$. On the other hand, since $f_l \leq \max\{h_j : j = 1, 2, \cdots, t\} + a_l$ on $N_l(o) \setminus \bigcup_{j=1}^t e_j$, f is bounded on $G \setminus \bigcup_{j=1}^t e_k$.

If $t = 0$, then theorem follows from Theorem 4.3. □

Combining Lemma 3.2, Theorem 3.4, Theorem 4.2, Theorem 4.3, Theorem 4.4 and Theorem 4.5, we have the following result:

THEOREM 4.6. *Let G be an infinite graph with l p-nonparabolic ends and s p-parabolic ends, each of which satisfies the conditions (VD), (P), and (FC). Let H be an infinite graph being roughly isometric to G. Then H has l p-nonparabolic ends, say $E_1, E_2, \cdots, E_l$ and s p-parabolic ends, say $e_1, e_2, \cdots, e_s$, respectively, each of which is p-regular. Hence, for each nonnegative p-harmonic function f on H, there exist nonnegative real numbers $a_1, a_2, \cdots, a_l$ and a subset P of $\{1, 2, \cdots, s\}$ such that*

$$\lim_{x\to\infty,\, x\in E_i} f(x) = a_i \quad \text{and} \quad \lim_{x\to\infty,\, x\in e_j} f(x) = \infty, \tag{4.5}$$

where $i = 1, 2, \cdots, l$ and $j \in P$. Conversely, given any nonnegative real numbers $a_1, a_2, \cdots, a_l$ and a subset P of $\{1, 2, \cdots, s\}$, there exists a nonnegative p-harmonic function f on H satisfying (4.5).

In particular, if G has only p-nonparabolic ends, then every nonnegative p-harmonic function on H is bounded, and each bounded p-harmonic function on H is uniquely determined by the values at infinity of each end and has finite p-energy.

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