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ESTIMATING THE CORRELATION COEFFICIENT IN A BIVARIATE NORMAL DISTRIBUTION USING MOVING EXTREME RANKED SET SAMPLING WITH A CONCOMITANT VARIABLE

MOHAMMAD FRAIWAN AL-SALEH[1] AND AHMAD MOHAMMAD AL-ANANBEH[1]

ABSTRACT

In this paper, we consider the estimation of the correlation coefficient in the bivariate normal distribution, based on a sample obtained using a modification of the moving extreme ranked set sampling technique (MERSS) that was introduced by Al-Saleh and Al-Hadhrami (2003a). The modification involves using a concomitant random variable. Nonparametric-type methods as well as the maximum likelihood estimation are considered under different settings. The obtained estimators are compared to their counterparts that are obtained based simple random sampling (SRS). It appears that the suggested estimators are more efficient

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1. INTRODUCTION

Ranked set sampling technique (RSS) was suggested by McIntyre (1952) for estimating the means of pasture and forage yields. The procedure consists of choosing randomly m sets of size m each. The elements in each set are judgment ranked with respect to the variable of interest. Then, for $i = 1, \dots, m$, the i^{th} judgment ranked unit from the i^{th} set is chosen for actual quantification. The set of these carefully chosen m units is called a RSS. The advantage of RSS over SRS in estimating the population mean is well established. In practice, to avoid

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[1]Department of Statistics, Yarmouk University, Irbid, Jordan

ranking errors the set size m should be small; a sample of larger size can be obtained via the iterating of the procedure.

Many applications of RSS and its variations have been pointed out by authors. Estimating the weights of browse and herbage in a pine -hardwood forest was carried out, using RSS, by Halls and Dell (1966). Its application in seedling counts was carried out by Evans (1967). The application of RSS for the estimation of average length of bacterial cells was pointed out by Takahasi and Wakimoto (1968). Al-Saleh and Al-Shrafat (2001) used RSS for the estimation of average milk yields of sheep. Al-Saleh and Al-Omari (2002) used RSS for the estimation of average yield of olives per tree. Stokes and Sager (1988) suggested the application of RSS for consumer expenditure survey and consumer price index. The use of RSS for Monte Carlo approximation of integrals was investigated by Al-Saleh and Samawi (2000). For other application of RSS and some of its modification, see Zheng and Al-Saleh (2002).

For one population mean μ, McIntyre (1952) proposed the estimator

$$\hat{\mu}_{RSS} = \frac{1}{mk}\sum_{j=1}^{k}\sum_{i=1}^{m} X_{(i:m)j} ,$$

where, $X_{(i:m)j}$ is the i^{th} order statistics of the i^{th} set in the j^{th} cycle and k is the number of cycles. Takahasi and Wakimoto (1968) established a rigorous statistical foundation for the theory of ranked set sampling. They showed that the efficiency of $\hat{\mu}_{RSS}$ with respect to the sample mean of a SRS of size m, $\bar{X}$, satisfies the following inequalities:

$$1 \leq eff(\hat{\mu}_{RSS}; \bar{X}) = \frac{var(\bar{X})}{var(\hat{\mu}_{RSS})} \leq \frac{m+1}{2}.$$

Stokes (1977) studied RSS with concomitant variables. She assumed that the variable of interest X has a linear relation with another variable Y. Stokes (1980) introduced a modified ranked set sampling procedure in which only the largest judgment (or the smallest) ranked unit is chosen for quantification. The procedure was used to estimate the correlation coefficient of the bivariate normal distribution using a concomitant variable. The method, denoted by ERSS, was investigated further by Samawi et al (1996). It is noted that the procedure may be appropriate to use for some distributions especially symmetric ones. One problem with the procedure is that it lacks the balancing property inherited in RSS.

Moving extreme ranked set sampling (MERSS) as introduced by Al-Odat and Al-Saleh (2001) and Al-Saleh and Al-Hadrami (2003a,b), is a modification of RSS, that does not need a complete ranking, as for RSS, but only the lowest or largest unit of sets of **varied** sizes are to be measured, assuming that they can be detected visually. Its aim is to reduce the error of ranking, while keeping some of the balancing behavior. This is achieved by varying the set size. Thus, MERSS can be thought of as a compromise between the balanced RSS and the unbalanced one in which only the two extremes of samples of fixed set sizes are measured (ERSS). Stokes (1980) used ERSS for estimating the correlation of the bivariate normal distribution. She noticed that the procedure may be highly non robust to departure from bivariate normality.

Formally, MERSS can be described by the following steps:

1. Select m SRS of size $1, 2, \ldots, m$, respectively.
2. Identify by **judgment** the maximum of each set with respect to the characteristic of interest.
3. Measure accurately the selected judgment identified maxima.
4. Repeat steps 1 to 3, but for minima.

The above four steps produce an MERSS of size $2m$.

5. Repeat the above steps r times, if necessary, until the desired sample size, $n = 2rm$ is achieved.

Note that to obtain a MERSS of size $2m$, we need to identify $m^2 + m$ units, while we need to identify $2m^2$ in the case of ERSS or usual RSS.
For more on recent work on RSS and its variations, see Al-Saleh and Al-Omari (2002), Al-Saleh and Zheng (2002, 2003), Samawi and Al-Saleh (2004) and Al-Saleh (2004), Patil *et al.*, (1999) (a bibliography); Kour *et al.* (1995), Zheng and Al-Saleh (2002, 2003) and Barabesi & Pisani (2004).

In this paper, we propose to use the MERSS with concomitant variable for the estimation of the correlation coefficient of the bivariate normal distribution.

2. MERSS WITH CONCOMITANT VARIABLE

Assume that (X, Y) has a bivariate normal distribution and suppose that the variable Y is difficult to measure or to order by judgment, but the concomitant variable X, which is correlated with Y is easier to measure or to order by judgment. The variable X may be used to acquire the rank of Y as follows:

a. Select m bivariate units using m SRS of sizes $1, 2, \ldots, m$ respectively.

b. Identify by judgment the maximum of each set with respect to the variable X.

c. Measure accurately the selected judgment identified units for both variables.

d. Repeat steps 1, 2, 3 but for the minimum.

e. Repeat the above steps r times, if necessary, until the desired sample size, $n = 2rm$ is obtained.

The n pairs are called a Moving Extremes Ranked Set Sampling (MERSS) with concomitant variable.

Let $X_{(i:k)}$ and $Y_{[i:k]}$ be the i^{th} smallest value of X and the corresponding value of Y obtained from the k^{th} sample, where $i = 1, k$ and $k = 1, 2, \ldots, m$. Note that $Y_{[i:k]}$ represent the i^{th} induced order statistic of a sample of size ; the stronger the relation between X and Y, the closer $Y_{[i:k]}$ to $Y_{(i:k)}$.

Let (X, Y) be a bivariate normal random vector with density denoted by $BN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$. It is well known that, $f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|X = x)$, where $f_X(x)$ is the marginal pdf of X and $f_{Y|X}(y|x)$ is the conditional pdf of $(Y|X = x)$; also $(Y|X = x) \sim N(\mu_y + \rho(\sigma_y/\sigma_x)(x - \mu_x), \sigma_y^2(1 - \rho^2))$.

Let $\{(X_{(l:k)}, Y_{[l:k]}), (X_{(k:k)}, Y_{[k:k]}), k = 1, 2, \ldots, m\}$ be a MERSS from $f_{X,Y}(x, y)$ If judgment ranking is as good as actual ranking, then for $k = 1, 2, \ldots, m$, $(X_{(k:k)}, Y_{[k:k]})$ and $(X_{(l:k)}, Y_{[l:k]})$ have the densities $f_{k:k}(x, y)$ and $f_{l:k}(x, y)$, respectively, given by:

$$f_{k:k}(x, y) = f_{X_{(k:k)}}(x) f_{Y|X}(y|x)$$

and

$$f_{l:k}(x, y) = f_{X_{(l:k)}}(x) f_{Y|X}(y|x)$$

where $f_{X_{(i:k)}}(x)$ is the density of the i^{th} order statistic of a SRS of size k from the univariate normal distribution, $i = l, k$. $f_{Y_{[k:k]}}(x)$ and $f_{Y_{[l:k]}}(x)$ are the densities of the corresponding induced rank of Y, see Yang 1977 and Stokes 1980.

Thus,

$$f_{k:k}(x, y) = k\Phi^{k-1}(\frac{x - \mu_x}{\sigma_x}) \frac{1}{\sigma_x} \phi(\frac{x - \mu_x}{\sigma_x}) \frac{1}{\sigma_y\sqrt{1 - \rho^2}} \phi(\frac{y - b}{\sigma_y\sqrt{1 - \rho^2}}) \quad (2.1)$$

$$f_{l:k}(x, y) = k(1 - \Phi(\frac{x - \mu_x}{\sigma_x}))^{k-l} \frac{1}{\sigma_x} \phi(\frac{x - \mu_x}{\sigma_x}) \frac{1}{\sigma_y\sqrt{1 - \rho^2}} \phi(\frac{y - b}{\sigma_y\sqrt{1 - \rho^2}})$$

where $b = \mu_y + \rho(\sigma_y/\sigma_x)(x - \mu_x)$, Φ & ϕ are, respectively, the cdf and pdf of the standard normal density, $N(0,1)$.

3. ESTIMATION

3.1. *Method of Moments Estimator of ρ: All Other Parameters are Known*

Let $\{(X_{(l:k)}, Y_{[l:k]}), (X_{(k:k)}, Y_{[k:k]})\}, k = 1, 2, \ldots, m$ be a MERSS from bivariate normal distribution. We may assume WLOG that $\mu_x = \mu_y = 0$ and $\sigma_x^2 = \sigma_y^2 = 1$. The method of moments estimator of ρ can be obtained by equating the population mixed moments to the sample mixed moments. Alternatively, we may consider the estimator:

$$\hat{\rho}^*_{MERSS} = c\sum_{k=1}^{m}[X_{(l:k)} \times Y_{[l:k]} + X_{(k:k)} \times Y_{[k:k]}].$$

THEOREM 3.1. *$\hat{\rho}^*_{MERSS}$ is an unbiased estimator of*

$$\rho \Leftrightarrow c = \frac{1}{2\sum_{k=1}^{m} E(X^2_{(k:k)})}.$$

PROOF. $E[X_{(l:k)} \times Y_{[l:k]}] = \rho E(X^2_{(l:k)})$ and $E[X_{(k:k)} \times Y_{[k:k]}] = \rho E(X^2_{(k:k)})$. Also, since $X_{(k:k)}$ has the same distribution as $-X_{(l:k)}$, $E(X^2_{(k:k)}) = E(X^2_{(l:k)})$. Therefore,

$$E(\hat{\rho}^*_{MERSS}) = 2c\rho\sum_{k=1}^{m} E(X^2_{(k:k)}).$$

For to be unbiased we should have

$$c = \frac{1}{2\sum_{k=1}^{m} E(X^2_{(k:k)})}.$$

□

Thus,

$$\hat{\rho}^*_{MERSS} = \frac{\sum_{k=1}^{m}[X_{(l:k)} \times Y_{[l:k]} + X_{(k:k)} \times Y_{[k:k]}]}{2\sum_{k=1}^{m} E(X^2_{(k:k)})}$$

is an unbiased estimator of ρ.

The corresponding unbiased estimator of ρ based on a SRS of size $2m$ is

$$\hat{\rho}^*_{SRS} = \frac{\sum_{i=1}^{2m} X_i Y_i}{2m}.$$

TABLE 3.1 # *of values of* $\hat{\rho}^*_{MERSS}(\hat{\rho}^*_{SRS})$ *that are out side the interval [-1,1]*

$\rho \rightarrow$ $m \downarrow$	0	0.2	0.4	0.6	0.8
1	664(1358)	1189(1468)	1640(1809)	2250(2363)	2909(3000)
2	283(536)	640(750)	1258(1283)	2055(2066)	3033(3060)
3	88(221)	332(432)	856(971)	1786(1781)	2913(2941)
4	21(98)	137(244)	551(727)	1425(1597)	2815(2827)
5	5(51)	57(118)	353(539)	1062(1398)	2593(2644)
6	1(24)	19(107)	188(410)	761(1176)	2340(2598)
7	0(6)	9(58)	101(304)	631(1079)	2126(2416)
8	0(5)	5(47)	47(231)	447(935)	1999(2390)
9	0(1)	0(23)	32(179)	325(826)	1737(2253)
10	0(2)	0(19)	17(153)	217(741)	1538(2222)

Note that the estimator $\hat{\rho}^*_{MERSS}$ and $\hat{\rho}^*_{SRS}$ may take values (with positive probability) out side $[-1,1]$ and hence are inadmissible estimators. To overcome this problem we may consider the following modified estimators:

$$\hat{\rho}^{**}_{MERSS} = \begin{cases} -1 & if\ \hat{\rho}^*_{MERSS} \leq -1 \\ \hat{\rho}^*_{MERSS} & if\ -1 < \hat{\rho}^*_{MERSS} < 1 \\ 1 & if\ \hat{\rho}^*_{MERSS} \geq 1 \end{cases}$$

and

$$\hat{\rho}^{**}_{SRS} = \begin{cases} -1 & if\ \hat{\rho}^*_{SRS} \leq -1 \\ \hat{\rho}^*_{SRS} & if\ -1 < \hat{\rho}^*_{SRS} < 1 \\ 1 & if\ \hat{\rho}^*_{SRS} \geq 1 \end{cases}$$

The efficiency of these estimators can't be put in closed form. So, we used simulation to compare $\hat{\rho}^{**}_{MERSS}$ and $\hat{\rho}^{**}_{SRS}$. All computations are done using Mathematica 4. The number of times that $\hat{\rho}^{**}_{MERSS}$ and $\hat{\rho}^{**}_{SRS}$ are outside the interval $[-1,1]$ with 10,000 iterations are given in Table 3.1.

It can be seen from Table 3.1 that the number of inadmissible values tends to decrease in m for fixed $|\rho|$ and to increase in $|\rho|$ for fixed m. Thus, the two estimators are not valid for large $|\rho|$ or small m.

The efficiency $\hat{\rho}^{**}_{MERSS}$ w.r.t. $\hat{\rho}^{**}_{SRS}$ is

$$eff(\hat{\rho}^{**}_{MERSS}; \hat{\rho}^{**}_{SRS}) = \frac{MSE(\hat{\rho}^{**}_{SRS})}{MSE(\hat{\rho}^{**}_{MERSS})}.$$

Table 3.2 gives the numerical values of $eff(\hat{\rho}^{**}_{MERSS}; \hat{\rho}^{**}_{SRS})$. Based on this Table we conclude that the efficiency tends to decrease in $|\rho|$ and to increase in

TABLE 3.2 $eff(\hat{\rho}^{*}_{MERSS}; \hat{\rho}^{**}_{SRS})$

$\rho \rightarrow$ $m \downarrow$	0	0.2	0.4	0.6	0.8
1	1.00	0.97	1.01	1.02	0.99
2	1.01	1.01	1.02	1.00	1.01
3	1.04	1.09	1.07	1.03	1.06
4	1.16	1.15	1.15	1.14	1.13
5	1.31	1.35	1.25	1.19	1.26
6	1.43	1.40	1.35	1.34	1.34
7	1.50	1.50	1.44	1.36	1.41
8	1.70	1.61	1.64	1.52	1.57
9	1.71	1.68	1.67	1.64	1.54
10	1.88	1.85	1.85	1.79	1.69

m. $\hat{\rho}^{**}_{MERSS}$ is significantly more efficient than $\hat{\rho}^{**}_{SRS}$.

3.2. Maximum Likelihood Estimator of ρ: Other Parameters are Known

The MLE of ρ can be found by solving the following equation for ρ:

$$\frac{\partial}{\partial\rho}L^{*}(\omega) = \frac{\partial}{\partial\rho}\{c - mlog(\sigma_x^{@}\sigma_y^{2}) + \sum_{k=1}^{m} log\phi_{x,y}\left(\frac{x_{(k:k)}-\mu_x}{\sigma_x}, \frac{y_{[k:k]}-\mu_y}{\sigma_y}\right) +$$

$$\sum_{k=1}^{m}(k-1)\left\{log\ \Phi\left(\frac{x_{(k:k)}-\mu_x}{\sigma_x}\right) + log\left[1-\Phi\left(\frac{x_{(l:k)}-\mu_x}{\sigma_x}\right)\right]\right\} +$$

$$\sum_{i=1}^{m} log\ \phi_{x,y}\left(\frac{x_{(l:k)}-\mu_x}{\sigma_x}, \frac{y_{[l:k]}-\mu_y}{\sigma_y}\right)\} = 0,$$

which is simplified to,

$$\frac{2m\rho}{1-\rho^2} + \left(\frac{1+\rho^2}{(1-\rho^2)^2}\right)\sum_{k=1}^{m}\left\{\begin{array}{c}\left(\frac{Y_{[l:k]}-\mu_y}{\sigma_y}\right)\left(\frac{X_{(l:k)}-\mu_x}{\sigma_x}\right)\\ +\left(\frac{Y_{[l:k]}-\mu_y}{\sigma_y}\right)\left(\frac{X_{(k:k)}-\mu_x}{\sigma_x}\right)\end{array}\right\}$$

$$-\left(\frac{\rho}{(1-\rho^2)^2}\right)\sum_{k=1}^{m}\left\{\begin{array}{c}\left(\frac{Y_{[l:k]}-\mu_y}{\sigma_y}\right)^2 + \left(\frac{X_{(l:k)}-\mu_x}{\sigma_x}\right)^2\\ +\left(\frac{X_{(k:k)}-\mu_x}{\sigma_x}\right)^2 + \left(\frac{Y_{[k:k]}-\mu_y}{\sigma_y}\right)^2\end{array}\right\} = 0$$

WLOG, take $\mu_x = \mu_y = 0$ and $\sigma_x^2 = \sigma_y^2 = 1$. Then the above ML equation is simplified to

TABLE 3.3 $eff(\hat{\rho}_{MERSS}; \hat{\rho}_{SRS})$

$\rho \rightarrow$ $m \downarrow$	0	0.2	0.4	0.6	0.8
1	0.99	1.01	1.00	1.00	1.01
2	1.01	0.98	1.01	0.99	0.97
3	1.05	1.05	1.08	1.12	1.05
4	1.14	1.16	1.19	1.22	1.10
5	1.25	1.24	1.35	1.35	1.08
6	1.35	1.40	1.44	1.42	1.12
7	1.57	1.55	1.58	1.45	1.16
8	1.64	1.70	1.62	1.40	1.17
9	1.79	1.75	1.70	1.41	1.19
10	1.89	1.87	1.77	1.47	1.20

$$\begin{aligned} h(\rho) = \; & 2m\rho(1-\rho^2) + (1+\rho^2)\sum_{k=1}^{m}[(X_{(l:k)})(Y_{[l:k]}) + (X_{(k:k)})(Y_{[k:k]})] - \\ & \rho\sum_{k=1}^{m}[X^2_{(l:k)} + X^2_{(k:k)} + Y^2_{[l:k]} + Y^2_{[k:k]}] = 0 \end{aligned} \tag{3.1}$$

Since the left hand side of the above equation is positive for $\rho = -1$ and negative for $\rho = 1$; there must be always a real root between -1 and $+1$. Kendall and Stuart (1963) showed that, the probability that the above equation has more than one real root, tends to zero. In any specific case, however, it is possible to have three roots between -1 and 1; The MLE should be the root that maximizes the likelihood equation. Lets this root be denoted by $\hat{\rho}_{MERSS}$.

For SRS, the MLE can be found(Johnson and Kotz 1972), by solving for ρ, the cubic equation

$$h^*(\rho) = 2m\rho(1-\rho^2) + (1+\rho^2)\left(\sum_{k=1}^{2m} X_i Y_i\right) - \rho\left[\sum_{k=1}^{2m}(X_i^2 + Y_i^2)\right] = 0 \tag{3.2}$$

In order to study the properties of $\hat{\rho}_{MERSS}$ numerically, Equation 1 and 2 have been solved numerically with 30000 iterations. At each iteration, $\hat{\rho}_{MERSS}$ and $\hat{\rho}_{SRS}$ are obtained and the MSE of each of them is approximated using these 30000 values. The results of the simulation are shown in Table 3.3. It can be seen from this Table that The efficiency of $\hat{\rho}_{MERSS}$ w.r.t. $\hat{\rho}_{SRS}$ is equal 1 at $m = 1, 2$

for all $|\rho|$. The slight discrepancy from 1 is due to sampling error; it tends to increase in m for fixed $|\rho|$ and to decrease in $|\rho|$ for fixed m.

THEOREM 3.2. *The Fisher information number about ρ in a MERSS of size $2m$ is given by*

$$I_{\rho MERSS} = \frac{\sum_{k=1}^{m} 2\left(E(X^2_{(l:k)}) + \frac{2\rho^2}{1-\rho^2}\right)}{1-\rho^2}.$$

PROOF.

$$\frac{\partial^2}{\partial\rho^2}(L^*(\rho)) = \frac{2m(1+\rho^2)}{(1-\rho^2)^2} + \left\{\frac{(2\rho)}{(1-\rho^2)^2} + \frac{4\rho(1+\rho^2)}{(1-\rho^2)^3}\right\} Z - \left(\frac{1}{(1-\rho^2)^2} + \frac{4\rho^2}{(1-\rho^2)^3}\right) V,$$

where,

$$V = \sum_{k=1}^{m}\left(Y^2_{[l:k]} + X^2_{(l:k)} + Y^2_{[k:k]} + X^2_{(k:k)}\right)$$

$$Z = \sum_{k=1}^{m}\left(Y_{[l:k]}X_{(l:k)} + Y_{[k:k]}X_{(k:k)}\right)$$

Thus,

$$I_{\rho MERSS} = -\frac{2m(1+\rho^2)}{(1-\rho^2)^2} - \left\{\frac{2\rho}{(1-\rho^2)^2} + \frac{4\rho(1+\rho^2)}{(1-\rho^2)^3}\right\} E(Z) + \left(\frac{1}{(1-\rho^2)^2} + \frac{4\rho^2}{(1-\rho^2)^3}\right) E(V).$$

But,

$$E(Z) = 2\rho\sum_{k=1}^{m} E(X^2_{(l:k)})$$

and

$$E(V) = \sum_{k=1}^{m}\left(2E(X^2_{(l:k)}) + 2 - 2\rho^2[1 - E(X^2_{(l:k)})]\right),$$

therefore,

$$I_{\rho MERSS} = \frac{2\sum_{k=1}^{m}\left(E(X^2_{(l:k)}) + \frac{2\rho^2}{1-\rho^2}\right)}{1-\rho^2}.$$

□

TABLE 3.4 *Asymptotic Efficiency of* $\hat{\rho}_{MERSS}$ *w.r.t.* $\hat{\rho}_{SRS}$

$\rho \rightarrow$ $m \downarrow$	0	0.2	0.4	0.6	0.8
3	1.09	1.08	1.07	1.04	1.02
4	1.21	1.19	1.15	1.10	1.05
5	1.33	1.30	1.24	1.15	1.07
6	1.44	1.41	1.32	1.21	1.10
7	1.55	1.51	1.40	1.26	1.12
8	1.66	1.61	1.48	1.31	1.14
9	1.76	1.70	1.55	1.36	1.17
10	1.85	1.79	1.62	1.40	1.19

On the other hand, the Fisher information number for ρ in a SRS of size $2m$ is

$$I_{\rho SRS} = \frac{2m(1+\rho^2)}{(1-\rho^2)^2}.$$

Note that $I_{\rho MERSS} \geq I_{\rho SRS}$ because $\sum_{k=1}^{m} E(X_{(l:k)}^2) \geq 1$ for $m > 1$. If $n = 2rm$, then the Fisher information in MERSS is $rI_{\rho MERSS}$. Therefore, the Asymptotic Efficiency (AE) of $\hat{\rho}_{MERSS}$ w.r.t. $\hat{\rho}_{SRS}$ is

$$AE = \frac{I_{\rho MERSS}}{I_{\rho SRS}} = \frac{2(1-\rho^2)\sum_{k=1}^{m}\{E(X_{(l:k)}^2) + \frac{2\rho^2}{1-\rho^2}\}}{2m(1+\rho^2)}.$$

The Asymptotic Efficiency of w.r.t. is given in Table 3.4.
Based on Table 3.4, we conclude that the asymptotic efficiency of $\hat{\rho}_{MERSS}$ w.r.t. $\hat{\rho}_{SRS}$ is always greater than 1, decreasing in $|\rho|$ for fixed m and is increasing in m for fixed $|\rho|$. Thus, MERSS provides the most benefit over SRS when ρ is close to zero.

3.3. Estimation of ρ: μ_x, σ_x^2 are Known

When the concomitant variable X, can be ranked easily and accurately, it is more reasonable to assume that μ_x and σ_x^2 are known than to assume that μ_y and σ_y^2. Therefore, in this section we consider the MLE of ρ when μ_x and σ_x^2 are known. The MLE's of μ_y, σ_y^2, and ρ based on a SRS are given by Johnson and

TABLE 3.5 *The efficiency of* $\hat{\rho}_{MERSS}$ *w.r.t.* $\hat{\rho}_{SRS}$

$\rho \rightarrow$ $m \downarrow$	0	0.2	0.4	0.6	0.8	0.95
3	1.09	1.11	1.14	1.19	1.29	1.40
4	1.19	1.19	1.24	1.34	1.42	1.21
5	1.28	1.29	1.35	1.43	1.44	1.12
6	1.39	1.41	1.46	1.50	1.46	1.12
7	1.51	1.51	1.54	1.57	1.42	1.08
8	1.65	1.67	1.67	1.67	1.47	1.10
9	1.77	1.76	1.76	1.69	1.44	1.11
10	1.86	1.83	1.82	1.71	1.47	1.17
20	2.61	2.53	2.34	1.99	1.53	1.08

Kotz (1972):

$$\begin{aligned} \hat{\mu}_{ySRS} &= \bar{Y} - rS_y\frac{(\bar{X} - \mu_x)}{S_x} \\ \hat{\sigma}^2_{ySRS} &= S^2_y(1 - r^2 + r^2\frac{\sigma^2_x}{S^2_x}) \qquad (3.3) \\ \hat{\rho}_{SRS} &= (r\frac{\sigma_x}{S_x})(1 - r^2 + r^2\frac{\sigma^2_x}{S^2_x})^{-\frac{1}{2}} \end{aligned}$$

where

$$S^2_x = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}, S^2_y = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n}$$

and

$$r = \frac{n^{-1}\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{S_xS_y}.$$

If the sample is a MERSS, then the MLE's of μ_y, σ^2_y, and ρ can be easily shown to be

$$\hat{\mu}_{yMERSS} = \bar{Y} - rS_y\left(\frac{\bar{X} - \mu_x}{S_x}\right) \qquad (3.4)$$

$$\hat{\sigma}^2_{yMERSS} = S^2_y(1 - r^2 + r^2\frac{\sigma^2_x}{S^2_x}) \qquad (3.5)$$

$$\hat{\rho}_{MERSS} = (r\frac{\sigma_x}{S_x})(1 - r^2 + r^2\frac{\sigma^2_x}{S^2_x})^{-\frac{1}{2}}$$

where

$$\bar{Y} = \frac{\sum_{k=1}^m (Y_{[l:k]} + Y_{[k:k]})}{2m}, \ \bar{X} = \frac{\sum_{k=1}^m (X_{(l:k)} + X_{(k:k)})}{2m},$$

$$S_x^2 = \frac{\sum_{k=1}^{m}((X_{(l:k)} - \bar{X})^2 + (X_{(k:k)} - \bar{X})^2)}{2m},$$
$$S_y^2 = \frac{\sum_{k=1}^{m}((Y_{[l:k]} - \bar{Y})^2 + (Y_{[k:k]} - \bar{Y})^2)}{2m}$$

and

$$r = \frac{(2m)^{-1}\sum_{k=1}^{m}((Y_{[l:k]} - \bar{Y})(X_{(l:k)} - \bar{X}) + (Y_{[k:k]} - \bar{Y})(X_{(k:k)} - \bar{X}))}{S_x S_y}.$$

Table 3.5 gives the efficiency $\hat{\rho}_{MERSS}$ w.r.t. $\hat{\rho}_{SRS}$. It can be seen from this Table that $\hat{\rho}_{MERSS}$ is more efficient than $\hat{\rho}_{SRS}$; the efficiency tends to decrease in $|\rho|$ for fixed m to increase in m for fixed $|\rho|$.

3.4. Estimation of ρ: All Parameters are Unknown

Suppose that $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and ρ are all unknown. The MLE's of the parameters based on a SRS are (Johnson and Kotz (1972)) $\hat{\mu}_x = \bar{X}$, $\hat{\mu}_y = \bar{Y}, \hat{\sigma}_x^2 = S_x^2, \hat{\sigma}_y^2 = S_y^2$ and $\hat{\rho} = r$, where

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n},\ \bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n},\ S_x^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n},$$
$$S_y^2 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{n} \quad \textit{and} \quad r = \frac{n^{-1}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{S_x S_y}.$$

In the case of MERSS, the maximum likelihood equations are not easy to deal with and don't appear to be easily solvable even by numerical methods. As suggested by one referee, an EM algorithm can be used to obtain the MLEs. For more details about this approach, see Balakrishnan and Kim (2004).We provide two options to deal with this problem. The first is similar to that given by Stokes (1980). In this approach, we solve the ML equation pretending that μ_x and σ_x^2 are known; then replacing them at the end with some suitable unbiased estimators. The second option is based on the modified likelihood function. Now,

$$E(X_{(l:k)}) = \mu_x + \sigma_x E(Z_{(l:k)})$$

and

$$E(X_{(k:k)}) = \mu_x + \sigma_x E(Z_{(k:k)});$$

where $k = 1, 2, \ldots, m$ and $\{Z_{(l:k)}, Z_{(k:k)}\}$ are respectively, the first and the k^{th} order statistics based on $N(0,1)$. $E(Z_{(l:k)}) = -E(Z_{(k:k)})$ implies,

$$\hat{\mu}_{xMERSS} = \frac{\sum_{k=1}^{m}(X_{(l:k)} + X_{(k:k)})}{2m},$$

TABLE 3.6 *The efficiency of $\hat{\rho}_{MERSS}$ w.r.t. $\hat{\rho}_{SRS}$*

$\rho \rightarrow$ $m \downarrow$	0	0.2	0.4	0.6	0.8
1	1.00	1.00	1.01	1.00	0.97
2	0.96	0.98	0.98	1.01	1.01
3	1.01	1.03	1.05	1.11	1.18
4	1.09	1.10	1.14	1.19	1.31
5	1.17	1.15	1.22	1.32	1.46
6	1.26	1.29	1.37	1.45	1.55
7	1.38	1.38	1.42	1.50	1.63
8	1.52	1.52	1.53	1.59	1.71
9	1.63	1.63	1.63	1.67	1.80
10	1.70	1.70	1.73	1.74	1.78

is an unbiased estimator of μ_x. Similarly,

$$\hat{\sigma}^2_{xMERSS} = \frac{\sum_{k=1}^{m}(X_{(k:k)} - X_{(l:k)})^2}{2\Big(\sum_{i=1}^{m} E(Z^2_{(k:k)}) + \sum_{i=1}^{m}\big(E(Z_{(k:k)})\big)^2\Big)}$$

is an unbiased estimator of σ^2_x. Using equations (4), and (5), the estimators of μ_y, σ^2_y and ρ then become:

$$\hat{\mu}_{yMERSS} = \frac{\sum_{k=1}^{m}(Y_{[l:k]} + Y_{[k:k]})}{2m},$$

$$\hat{\sigma}^2_{yMERSS} = S^2_y(1 - r^2 + r^2\frac{\hat{\sigma}^2_{xMERSS}}{S^2_x})$$

and

$$\hat{\rho}_{MERSS} = (r\frac{\hat{\sigma}_{xMERSS}}{S_x})(1 - r^2 + r^2\frac{\hat{\sigma}^2_{xMERSS}}{S^2_x})^{-\frac{1}{2}}.$$

Note that $\hat{\sigma}^2_{xMERSS}$ is an option; of course there are other options. The main purpose is to replace the unknown value of σ^2_x with a suitable estimator for the purpose of estimating ρ. So, our main concern is in the parameter ρ, not the means or the variances. Table 3.6 gives the efficiency of $\hat{\rho}_{MERSS}$ w.r.t. $\hat{\rho}_{SRS}$.

Based on Table 3.6 we conclude that $\hat{\rho}_{MERSS}$ is more efficient than $\hat{\rho}_{SRS}$. The efficiency of $\hat{\rho}_{MERSS}$ w.r.t. $\hat{\rho}_{SRS}$ is increasing in m for fixed $|\rho|$ and increasing in $|\rho|$ for fixed m.

TABLE 3.7 *The efficiency of* $\hat{\rho}_{mMERSS}$ *w.r.t.* $\hat{\rho}_{SRS}$

$\rho \rightarrow$ $m \downarrow$	0	0.2	0.4	0.6	0.8
1	1.00	1.00	1.01	1.00	0.97
2	1.00	1.01	1.02	1.04	1.08
3	1.01	1.03	1.04	1.15	1.31
4	0.99	1.03	1.08	1.23	1.47
5	0.96	0.98	1.09	1.27	1.68
6	0.96	0.99	1.11	1.30	1.66
7	0.98	0.99	1.07	1.21	1.48
8	1.02	1.03	1.05	1.17	1.32
9	1.02	1.00	1.02	1.05	1.14
10	0.99	0.97	0.93	0.92	0.97

For the second approach, we use the same formulas of the MLE's estimators using SRS, with the SRS data replaced by MERSS:

$$\hat{\mu}_{xmMERSS} = \bar{X};\ \hat{\mu}_{ymMERSS} = \bar{Y}$$
$$\hat{\sigma}^2_{xmMERSS} = S^2_x;\ \hat{\sigma}^2_{ymMERSS} = S^2_y;\ \hat{\rho}_{mMERSS} = r$$

These are called the modified MLE's of $\mu_x, \mu_y, \sigma^2_x, \sigma^2_y$ and ρ. Table 3.7 gives the efficiency of $\hat{\rho}_{mMERSS}$ w.r.t. $\hat{\rho}_{SRS}$. Based on this Table we may conclude that the efficiency of $\hat{\rho}_{mMERSS}$ w.r.t $\hat{\rho}_{SRS}$. tends to increase in m for $m \leq 6$ and to decrease in m for $m \leq 6$ when $|\rho| \geq 0.4$. It also tends to increase in $|\rho|$ for fixed m.

4. Concluding Remarks

MERSS with concomitant variable is a useful modification of RSS, that can be used with bivariate data to estimate the correlation coefficient. The method is easier to use in practice than the usual RSS procedure, because only one of the two extremes is needed to be identified by judgment. It appears that the use of MERSS with concomitant variable is highly beneficial when compared to SRS for estimating the correlation coefficient. No systematic investigation of robustness to departure from bivariate normality has been undertaken, but we anticipate that that MERSS procedure will be more robust than using the unbalanced RSS with only measuring the two extremes, *i.e.*, ERSS.

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