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PRELIMINARY DETECTION FOR ARCH-TYPE HETEROSCEDASTICITY IN A NONPARAMETRIC TIME SERIES REGRESSION MODEL

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ABSTRACT

In this paper a nonparametric method is proposed for detecting conditionally heteroscedastic errors in a nonparametric time series regression model where the observation points are equally spaced on $[0, 1]$. It turns out that the first-order sample autocorrelation of the squared residuals from the kernel regression estimates provides essential information. Illustrative simulation study is presented for diverse errors such as ARCH(1), GARCH(1,1) and threshold-ARCH(1) models.

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1. INTRODUCTION

In a seminal paper, Engle(1982, section 5) introduced a linear regression with conditionally heteroscedastic errors in the context of time series regression models, *viz.*,

$$Y_i = x_i'\beta + \epsilon_i \tag{1.1}$$

$$\epsilon_i = \sqrt{v_i} \cdot u_i \tag{1.2}$$

where u_i's are iid N(0,1) variates, x_i (in the mean function $x_i'\beta$) is a vector of explanatory variables and v_i denotes conditional heteroscedastic variance, *i.e.*,

$$v_i = \mathrm{Var}(\epsilon_i | \Psi_{i-1}) \tag{1.3}$$

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where Ψ_{i-1} is used for the information set consisting of $(Y_{i-1}, x_{i-1}), (Y_{i-2}, x_{i-2}), \cdots$ Typically v_i is taken as a linear combination of squared errors;

$$v_i = \alpha_0 + \alpha_1 \epsilon_{i-1}^2 \tag{1.4}$$

which is referred to as first order (Engle's) ARCH (autoregressive conditional heteroscedastic) model. Notice that $\alpha_1 = 0$ gives usual *iid* errors for $\{\epsilon_i\}$.

Engle(1982) compared OLS-estimate $\hat{\beta}_{OLS}$ and WLS-estimate $\hat{\beta}_{WLS}$ for the regression parameter β. The OLS-estimate $\hat{\beta}_{OLS}$ is obtained by minimizing, ignoring ARCH effect, $\sum_{i=1}^n (Y_i - x_i'\beta)^2$. Instead, WLS-estimate $\hat{\beta}_{WLS}$ is computed, taking v_i into account, by minimizing $\sum_{i=1}^n (Y_i - x_i'\beta)^2/v_i$. He demonstrated that the relative efficiency eff$(\hat{\beta}_{WLS}, \hat{\beta}_{OLS})$ of $\hat{\beta}_{WLS}$ over $\hat{\beta}_{OLS}$ goes to infinity as ARCH effect becomes prominent. It is noted that eff$(\hat{\beta}_{WLS}, \hat{\beta}_{OLS}) \geq 1$ with equality holding if and only if v_i is constant(almost surely). Thus, statistical inference ignoring conditional heteroscedasticity may be distorted. Confidence interval (of approximate 95% level) for Y_i given Ψ_{i-1} is then given by

$$x_i'\hat{\beta}_{WLS} \pm 2\sqrt{v_i}$$

which must be contrasted with constant v_i (with respect to $i = 1, \cdots, n$) for the standard regression. In addition, since ARCH modeling requires more complicated statistical procedures than otherwise, it may be desirable to detect whether ARCH is present before going into the effort to handle it. Consequently test for presence of conditional heteroscedasticity in time series regression model deserves much investigation.

For testing ARCH, Li and Mak(1994) proposed a chi-square test based on squared residuals ($\hat{\epsilon}_i^2$) autocorrelations. Also, Hwang *et al.*(1994) suggested a chi-square test, for model diagnostics, using residual ($\hat{\epsilon}_i$) autocorrelations obtained after fitting a random coefficient autoregressive process exhibiting conditional heteroscedasticity. However traditional methods including Engle(1982) and those mentioned above presumed that the functional form of the mean function (such as $x_i'\beta$) is *known*, and hence essentially dealing with parametric case. Thus it would be useful if one could develop a nonparametric ARCH detection procedure where the functional form of the mean function is *unknown*. To this end this article provides ARCH detection procedures in a simple nonparametric time series regression model where the observation points are equally spaced. Our result may give much useful insights to more general nonparametric setup. Error inference in the nonparametric regression model has mainly focused on correlated errors because it is known that they cause fundamental problems such as bandwidth selection problem (see, *e.g.*, Chiu(1989), Hart(1991, 1994) and Opsomer *et al.*(2001)). Recently Kim *et al.* (2004) and Park *et al.* (2004) provided nonparametric methods for handling correlated errors, and our detection technique in this paper extends their ideas towards ARCH context via analyzing squared errors (ϵ_i^2).

In Section 2, a nonparametric detection procedure for conditional heteroscedasticity is proposed and is justified. For illustration, a simulation study is conducted in Section 3, from which it is shown that our method is easy to implement but providing reasonably good powers for various ARCH-type models.

2. THE MODEL AND THE PROPOSED DETECTION PROCEDURE

In this article, we consider the following model:

$$Y_i = m(x_i) + \epsilon_i, \qquad i = 1, \cdots, n, \tag{2.1}$$

where m is an *unknown* smooth function and the error ϵ_i follows conditionally heteroscedastic model in (1.2). The regression points x_i's are assumed to be equidistant on the interval $[0,1]$, *i.e.*, $x_i = i/n, i = 1, \cdots, n$, as is one of the usual settings considered by several authors including Härdle *et al.*(1988). Here we do not assume any specific functional form of conditional heteroscedasticity, and thus v_i may have generalized ARCH, so called first order GARCH(*cf.* Bollerslev(1986)) defined by

$$v_i - \phi_1 v_{i-1} = \alpha_0 + \alpha_1 \epsilon_{t-1}^2. \tag{2.2}$$

First order structure of v_i in (1.4) and (2.2) can be straightforwardly extended to higher order models and thus we retain first order structure in (1.4) and (2.2) for simplicity of presentation. Also, various (G)ARCH-type processes in the literature can be accommodated, See, for instance, Hwang and Kim(2004). Consider the following condition.

(C1). The innovation $\{u_i\}$ is (not necessarily normal) iid sequence of random variables with mean zero and variance unity. Further, the distribution of u_i is symmetric about zero, *i.e.*, $u_i \stackrel{d}{=} -u_i$.

It will be assumed that $\{\epsilon_i\}$ is stationary, and define autocovariances γ_p and γ_p^* for $\{\epsilon_i\}$ and squared process $\{\epsilon_i^2\}$ respectively, *viz.*,

$$\gamma_p = E(\epsilon_{i+p}\epsilon_i) \quad \text{and} \quad \gamma_p^* = E(\epsilon_{i+p}^2 \epsilon_i^2) - E(\epsilon_{i+p}^2)E(\epsilon_i^2), \ p \geq 0.$$

Here and in what follows * is used in order to indicate squared errors. Consider, in particular, autocorrelations of lag 1

$$\rho_1 = \gamma_1/\gamma_0 \quad \text{and} \quad \tau_1 = \rho_1^* = \gamma_1^*/\gamma_0^*. \tag{2.3}$$

It is worth noting for our model defined by (2.1) and (1.2) that (i) when there is no conditional heteroscedasticity, that is, when ϵ_i's are iid, it holds that $\tau_1 = 0$ and $\rho_1 = 0$; (ii) If conditional heteroscedasticity does exist in $\{\epsilon_i\}$, then $\tau_1 > 0$ and $\rho_1 = 0$.

For detecting conditionally heteroscedastic errors, we propose a nonparametric procedure consisting of the following two steps:

[Step1] Obtain nonparametric estimate $\hat{\tau}_1$ for τ_1. $\hat{\tau}_1$ will be specified in (2.6);

[Step2] When $\hat{\tau}_1$ is significantly greater than zero, conditional heteroscedasticity in $\{\epsilon_i\}$ is declared.

Note that the null hypothesis is given as the constancy of ν_i with respect to i and the alternative is restricted to a class of ARCH-type errors specified in (1.2) and (1.3). To be more specific with [Step2], analogous to standard one sided test, we will use as critical region

$$\hat{\tau}_1 > 2/\sqrt{n} \quad [\text{ or,} \quad \hat{\tau}_1 > 1.7/\sqrt{n}\ ] \tag{2.4}$$

In fact, the proposed detection rule (2.4) is conservative in comparison with parametric case. When $m(\cdot)$ is specified (i.e., parametric case), it is known in time series literature that the SACF's (sample auto-correlation function) based on residuals with lower lags (in particular, for lag 1) have asymptotic variance smaller than unity whereas for higher lags, they converges to unity. See, for instance, Hwang et al.(1994, Table 6.1). Thus, in view of the fact that for ϵ_i following GARCH, the squared process $\{\epsilon_i^2\}$ obeys ARMA(autoregressive moving average) models, the asymptotic variance of $\hat{\tau}_1$ is expected to be smaller than 1 provided $m(\cdot)$ is known. Consequently $2/\sqrt{n}(1.7/\sqrt{n})$ provides asymptotically conservative value, compared to parametric case.

We now propose and study nonparametric estimate $\hat{\tau}_1$. Employing Priestly-Chao estimator of m given by

$$\hat{m}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right) Y_i$$

where K is kernel and h is the bandwidth, define sample auto-covariances(SACV) of order p for both the residuals e_i and their squares e_i^2:

$$SACV_p(h) = (n-p)^{-1} \sum_{i=1}^{n-p} e_i e_{i+p} - \bar{e}^{(1)} \bar{e}^{(2)}$$

$$SACV_p^*(h) = (n-p)^{-1} \sum_{i=1}^{n-p} e_i^2 e_{i+p}^2 - \bar{e}^{*(1)} \bar{e}^{*(2)}$$

where $e_i = Y_i - \hat{m}(x_i)$, $\bar{e}^{(1)} = (n-p)^{-1} \sum_{i=1}^{n-p} e_i$, $\bar{e}^{(2)} = (n-p)^{-1} \sum_{i=1}^{n-p} e_{i+p}$, $\bar{e}^{*(1)} = (n-p)^{-1} \sum_{i=1}^{n-p} e_i^2$, and $\bar{e}^{*(2)} = (n-p)^{-1} \sum_{i=1}^{n-p} e_{i+p}^2$. Then we have the following estimators

$$\hat{\rho}_1 = SACV_1(\hat{h}_{S1})/SACV_0(\hat{h}_{S0}) \tag{2.5}$$

and

$$\hat{\tau}_1 = SACV_1^*(\hat{h}_{S1}^*)/SACV_0^*(\hat{h}_{S0}^*), \tag{2.6}$$

where for $p = 0, 1$

$$\hat{h}_{Sp} = \text{argmin}_{h>0} SACV_p(h), \tag{2.7}$$

and

$$\hat{h}_{Sp}^* = \text{argmin}_{h>0} SACV_p^*(h). \tag{2.8}$$

For $\hat{\rho}_1$, Park *et al.* (2004) argued some asymptotic optimality properties for $\hat{\rho}_1$. With modifications due to ARCH component, adapting the lines in Kim *et al.* (2004) and Park *et al.* (2004) to the case of squared errors (for $\hat{\tau}_1 = \hat{\rho}_1^*$) we provide justifications of $\hat{\tau}_1$. First we state regularity conditions on the kernel and mean function.

(C2-1). The kernel function K is a square-integrable symmetric probability density with finite second moment. Also, K satisfies Lipschitz condition of order one, and has a local minimum zero at zero, *i.e.* $K(0) = 0$.

(C2-2). The bandwidth h satisfies $h \to 0$ and $nh^2 \to \infty$ as $n \to \infty$.

(C2-3). The mean function m is twice continuously differentiable on $[0, 1]$.

For the statement of the theorem, we introduce

$$
\begin{aligned}
C_1 &= \gamma_0^* + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k^*,\\
C_2 &= \frac{1}{16}\left(\int u^2K\right)^4\int\left((m'')^2-\int(m'')^2\right)^2,\\
C_2' &= \gamma_0\left(\int u^2K\right)^2\int(m'')^2,\\
C_3 &= 2\gamma_0^2\left(\int K^2\right)^2+4\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)K^2\left(\frac{k}{nh}\right)\gamma_k^*,\\
C_3' &= 4\gamma_0^2\int K^2,\\
C_4 &= \gamma_0\left(\int u^2K(u)\right)^2\int K^2\int(m'')^2.
\end{aligned}
$$

We have the following results which give the approximations of the expected values of $SACV_1^*(h)$ and $SACV_0^*(h)$.

THEOREM 2.1. *Suppose that (C1) and (C2) hold. Then*

$$
\begin{aligned}
E\{SACV_1^*(h)\} &= \gamma_1^* - C_1n^{-1} + C_2h^8 + C_3n^{-2}h^{-2} + C_4n^{-1}h^3 \qquad (2.9)\\
&\quad + o\left(h^8+n^{-1}h^3+n^{-2}h^{-2}\right)\\
E\{SACV_0^*(h)\} &= \gamma_0^* + C_2'h^4 + C_3'n^{-1}h^{-1} + o\left(h^4+n^{-1}h^{-1}\right). \qquad (2.10)
\end{aligned}
$$

From the above Theorem one may easily notice that $SACV_1^*(h)$ and $SACV_0^*(h)$ may serve as good estimators of γ_1^* and γ_0^* respectively because their biases can be controlled by appropriate choices of h. Since C_1, C_2', C_3, C_3' and $C_4 > 0$, optimal h for $SACV_0^*(h)$ and $SACV_1^*(h)$ can be defined as the minimizer of $E(SACV_0^*(h))$ and $E(SACV_1^*(h))$ respectively. *i.e.*,

$$h_{S0}^* = (C_3'/(4C_2'))^{1/5}n^{-1/5}$$

and

$$h_{S1}^* = (16C_2)^{-1/5}\{-3C_4+[9C_4^2+64C_2C_3]^{1/2}\}^{1/5}n^{-1/5}.$$

Thus their estimators given by $\hat{h}_{S0}^*$ and $\hat{h}_{S1}^*$ could be justified. Note that minimizing $E(SACV_0^*(h))$ and $E(SACV_1^*(h))$ over h is independent of γ_0^* and γ_1^*. Now the the choice of the kernel K which satisfies $K(0) = 0$ is worthy to be mentioned. Indeed in the proof of the theorem such kernel makes the $n^{-1}h^{-1}$ terms in the expansion of $E\{SACV_1^*(h)\}$ vanish, and thus reducing the bias of $SACV_1^*(h)$ as an estimator of γ_1^*. In fact Park (*et al.*) (2004) find usefulness of such kernel in correlated error inference. From our argument one may naturally expect underestimation of τ_1 due to positive bias from denominator and negative bias from numerator of $\hat{\tau}_1$, *i.e.* $E(SACV_1^*(h_{S0}^*)) - \gamma_1^* = -C_1 n^{-1}$ and $E(SACV_0^*(h_{S0}^*)) - \gamma_0^* = O(n^{-4/5})$. It may be noted that underestimation of τ_1 makes the detection rule (2.4) conservatively declared. Our simulation study reveals such underestimation tendency clearly.

PROOF. Define the following quantities:

$$b_i = m(x_i) - (nh)^{-1} \sum_j K\left(\frac{x_i - x_j}{h}\right) m(x_j),$$

$$s_i = \epsilon_i - (nh)^{-1} \sum_j K\left(\frac{x_i - x_j}{h}\right) \epsilon_j.$$

Then, we may write $e_i = b_i + s_i$ and thus

$$\begin{aligned} &\frac{1}{n-1} E\left(\sum_i e_i^2 e_{i+1}^2\right) - \frac{1}{(n-1)^2} E\left(\sum_i e_i^2 \sum_j e_{j+1}^2\right) \\ &= \frac{1}{n-1} \sum_i E\left(b_i^2 b_{i+1}^2 + b_i^2 s_{i+1}^2 + 4 b_i b_{i+1} s_i s_{i+1} + s_i^2 b_{i+1}^2 + s_i^2 s_{i+1}^2\right) \\ &\quad - \frac{1}{(n-1)^2} \sum_i \sum_j E\left(b_i^2 b_{j+1}^2 + b_i^2 s_{j+1}^2 + 4 b_i b_{j+1} s_i s_{j+1} + s_i^2 b_{j+1}^2 + s_i^2 s_{j+1}^2\right). \end{aligned}$$

The above equation follows from the facts that $E\epsilon_i = E\epsilon_i^3 = 0$ and that $E\epsilon_i\epsilon_j = 0$ for $i \neq j$ and $E\epsilon_i^2\epsilon_j = 0$ for $i \leq j$ since these entail for all i and j

$$E(b_i^2 b_{i+1} s_{i+1}) = E(b_i s_i b_{i+1}^2) = E(b_i^2 b_{j+1} s_{j+1}) = E(b_i s_i b_{j+1}^2) = 0,$$

$$E(b_i s_i s_{i+1}^2) = E(b_{i+1} s_{i+1} s_i^2) = E(b_i s_i s_{j+1}^2) = E(b_{j+1} s_{j+1} s_i^2) = 0.$$

Recall from (C2) that $K(0) = 0$. Using this it is not too hard to verify that

$$\frac{1}{n-1} \sum_i b_i^2 b_{i+1}^2 - \frac{1}{(n-1)^2} \sum_i \sum_j b_i^2 b_{j+1}^2 = C_2 h^8 + o(h^8),$$

$$\frac{1}{n-1}\sum_i b_i^2 E s_{i+1}^2 - \frac{1}{(n-1)^2}\sum_i\sum_j b_i^2 E s_{j+1}^2 = O\left(\frac{h}{n^2}\right),$$

$$\frac{1}{n-1}\sum_i b_{i+1}^2 E s_i^2 - \frac{1}{(n-1)^2}\sum_i\sum_j b_{j+1}^2 E s_i^2 = O\left(\frac{h}{n^2}\right),$$

$$\frac{1}{n-1}\sum_i E(s_i^2 s_{i+1}^2) - \frac{1}{(n-1)^2}\sum_i\sum_j E(s_i^2 s_{j+1}^2) = \gamma_1^* - \frac{C_1}{n} + \frac{C_3}{n^2h^2} + o\left(\frac{1}{n^2h^2}\right)$$

Here, we would like to point out that if $K(0) \neq 0$ then the right hand side of the last equation would be $\gamma_1^* - 4\gamma_1^* K(0)n^{-1}h^{-1} + o(n^{-1}h^{-1})$. Finally, one can show

$$\frac{1}{n-1}\sum_i b_i b_{i+1} E(s_i s_{i+1}) - \frac{1}{(n-1)^2}\sum_i\sum_j b_i b_{j+1} E(s_i s_{j+1}) = \frac{C_4 h^3}{4n}.$$

This completes the verification of (2.9).

Similar but simpler arguments as above readily yield (2.10). Details are omitted. Main differences lie in the calculations of

$$\frac{1}{n}\sum_i E s_i^4 - \frac{1}{n^2}\sum_i\sum_j E s_i^2 s_j^2 = \gamma_0^* + \frac{C_3'}{nh} + o\left(\frac{1}{nh}\right),$$

$$\frac{1}{n}\sum_i b_i^2 E(s_i^2) - \frac{1}{n^2}\sum_i\sum_j b_i b_j E(s_i s_j) = \frac{C_2' h^4}{4} + o(h^4).$$

□

3. SIMULATION STUDY

In our simulation, the mean function m is taken as $m(x) = 300x^3(1-x)^3$, and the kernel K is chosen as a bimodal

$$K(x) = 630(4x^2-1)^2 x^4 I_{[-1/2,1/2]}(x).$$

The points x_i's are taken as equidistant on the interval $[0,1]$, *i.e.*, $x_i = i/n, i = 1, \cdots, n$. We consider three cases, *viz.*, (i) ARCH(1) (ii) GARCH(1, 1) and (iii) threshold-ARCH(1). For each cases, simulated sample size $n = 200$ and 100 replicates are made. The innovations u_i's are zero mean *iid* normal variates. The variance of error ϵ_i is set to be unity for each cases.

TABLE 3.1 *Sample means and standard errors of various statistics:* $v_i = 0.5 + \alpha_1 \epsilon_{i-1}^2$

α_1	statistic	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\rho}_1$	$\hat{\gamma}_0^*$	$\hat{\gamma}_1^*$	$\hat{\tau}_1 = \hat{\rho}_1^*$
0	mean	1.042	.050	.047	2.070	-.024	-0.013
	s.e.	.107	.077	.072	.572	.121	.057
1/6	mean	1.030	.046	.041	2.216	.303	.127
	s.e.	.142	.087	.082	.721	.242	.083
2/6	mean	1.063	.045	.041	3.481	1.076	.250
	s.e.	.190	.110	.100	3.273	1.485	.138
3/6	mean	1.024	.051	.050	5.479	2.324	.328
	s.e.	.298	.154	.124	12.326	5.965	.139
4/6	mean	.960	.017	.027	7.072	3.257	.383
	s.e.	.304	.155	.144	16.677	8.615	.140
5/6	mean	.746	.035	.056	6.508	2.920	.431
	s.e.	.359	.203	.165	14.886	6.381	.144

TABLE 3.2 *Detection rates for ARCH(1):* $v_i = 0.5 + \alpha_1 \epsilon_{i-1}^2$

α_1		0	1/6	2/6	3/6	4/6	5/6
cut	$2/\sqrt{n}$	0	43	75	92	99	99
value	$1.7/\sqrt{n}$	0	54	84	94	99	99

ARCH(1)

The errors $\{\epsilon_i\}$ are generated from ARCH(1) given by

$$\epsilon_i = \sqrt{v_i} \cdot u_i$$

$$v_i = \alpha_0 + \alpha_1 \epsilon_{i-1}^2 \tag{3.1}$$

where the variance of u_i is $(\alpha_0 + \alpha_1)^{-1}$ so that $\mathrm{Var}(\epsilon_i) = 1$. (3.1) is stationary when $0 \leq \alpha_1 < 1$. Notice that $\rho_1 = 0$ and $\tau_1 = \rho_1^* = \alpha_1$ in our simulation setup. Simulated sample (of $n = 200$) are obtained with $\alpha_0 = 1/2$ and for each values of α_1 =0, 1/6, 2/6, 3/6, 4/6 and 5/6. Table 1 contains some results for $\hat{\rho}_1$ and $\hat{\tau}_1$. Observe that $\hat{\rho}_1$ divided by corresponding standard error($s.e.$) is very close to zero across all α_1 values. This supports the validity of $\rho_1 = 0$ for (3.1). Note, by comparing $\hat{\tau}_1$ and $\tau_1 = \alpha_1$, that $\hat{\tau}_1$ has negative bias and therefore one can argue that the rule (2.4) is somewhat conservative in detecting ARCH(1) errors. False detection rate (corresponding to $\alpha_1 = 0$) turns out to be zero in Table 3.2, which is partly due to underestimation tendency of $\hat{\tau}_1$.

TABLE 3.3 *Sample means and standard errors of various statistics:* $v_i - 0.2v_{i-1} = 0.5 + \alpha_1\epsilon_{i-1}^2$

α_1	statistic	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\rho}_1$	$\hat{\gamma}_0^*$	$\hat{\gamma}_1^*$	$\hat{\tau}_1 = \hat{\rho}_1^*$
0	mean	1.030	.037	.033	2.092	-.034	-0.017
($\phi = 0$)	s.e.	.104	.082	.078	.543	.145	.072
.1	mean	1.066	.058	.054	2.295	.186	.072
	s.e.	.135	.095	.085	.649	.227	.085
.2	mean	1.040	.032	.028	2.604	.553	.157
	s.e.	.161	.102	.095	1.844	1.235	.124
.4	mean	1.066	.044	.040	4.945	1.621	.287
	s.e.	.291	.147	.107	14.625	4.576	.130
.6	mean	.930	.034	.026	6.857	3.669	.354
	s.e.	.386	.193	.140	15.117	10.976	.152
.7	mean	.739	.001	.017	8.120	4.098	.396
	s.e.	.404	.179	.168	15.993	9.839	.167

TABLE 3.4 *Detection rates for GARCH(1,1):* $v_i - 0.2v_{i-1} = 0.5 + \alpha_1\epsilon_{i-1}^2$

α_1		$0(\phi = 0)$	0.1	0.2	0.4	0.6	0.7
cut	$2/\sqrt{n}$	1	22	51	90	93	95
value	$1.7/\sqrt{n}$	3	31	66	91	96	98

The percent (out of 100 replications) satisfying $\hat{\tau}_1 > 2/\sqrt{n}$ increases significantly as α_1 is getting larger; 43, 75, 92, 99 and 99 % for α_1 =1/6, 2/6, 3/6, 4/6 and 5/6 respectively. Accordingly it seems that our methods provide reasonably good power for detecting ARCH(1).

GARCH(1, 1)

In order to allow long range dependence of v_i on errors, it would be useful to study GARCH(1, 1) model specified by

$$v_i - \phi v_{i-1} = \alpha_0 + \alpha_1\epsilon_{i-1}^2; \ \phi \geq 0, \ \alpha_0 > 0, \ \alpha_1 \geq 0. \tag{3.2}$$

This is stationary when $\phi + \alpha_1 < 1$. See Bollerslev(1986). We choose $\phi = 0.2$, $\alpha_0 = 1/2$ and α_1 =0.1, 0.2, 0.4, 0.6 and 0.7 in the stationarity region. It can be shown that

$$\tau_1 = \rho_1^* = \alpha_1(1 - \phi^2 - \phi\alpha_1)/(1 - \phi^2 - 2\phi\alpha_1).$$

Simulation results are summarized in Tables 3.3 and 3.4. Notice that corresponding to $\alpha_1 = 0$ and $\phi = 0$, false detection rate is 1(3)% when $\hat{\tau}_1 > 2/\sqrt{n}$

TABLE 3.5 *Sample means and standard errors of various statistics:* $v_i = 0.5 + \alpha_{11}(\epsilon_{i-1}^{+})^2 + \alpha_{12}(\epsilon_{i-1}^{-})^2$

$(\alpha_{11}, \alpha_{12})$	statistic	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\rho}_1$	$\hat{\gamma}_0^*$	$\hat{\gamma}_1^*$	$\hat{\tau}_1 = \hat{\rho}_1^*$
(0,0)	mean	1.041	.048	.045	2.125	-.011	-0.004
	s.e.	.106	.071	.066	.454	.142	.068
(0.1,0.5)	mean	.995	.026	.023	2.771	.668	.178
	s.e.	.166	.102	.095	3.046	1.596	.107
(0.5,0.1)	mean	1.029	.040	.037	3.133	0.780	.186
	s.e.	.171	.083	.081	2.799	1.583	.134
(0.2,0.8)	mean	.998	.044	.036	4.485	1.834	.293
	s.e.	.253	.135	.115	8.102	6.052	.136
(0.8,0.2)	mean	1.006	.040	.028	5.413	2.282	.289
	s.e.	.286	.158	.130	12.778	8.287	.128
(0.8,0.6)	mean	.991	.065	.050	17.440	10.594	.404
	s.e.	.603	.301	.152	81.276	61.433	.164
(0.9,1.0)	mean	.422	.017	.056	13.198	6.445	.431
	s.e.	.530	.167	.174	55.014	30.649	.150

($\hat{\tau}_1 > 1.7/\sqrt{n}$) is used. Similar conclusions to those in ARCH(1) case continue to hold.

Threshold-ARCH(1)

ARCH(1) and GARCH(1, 1) are typical examples of symmetric heteroscedastic models in the sense that v_i is symmetric with respect to ϵ_{i-1}. As an illustration of non-symmetric ARCH, consider threshold version of ARCH(1), so called, threshold-ARCH(1) given by

$$v_i = \alpha_0 + \alpha_{11}(\epsilon_{i-1}^{+})^2 + \alpha_{12}(\epsilon_{i-1}^{-})^2 \tag{3.3}$$

where $\alpha_0 > 0$, $\alpha_{11} \geq 0$, $\alpha_{12} \geq 0$; $\epsilon_{i-1}^{+} = \max(\epsilon_{i-1}, 0)$ and $\epsilon_{i-1}^{-} = \max(-\epsilon_{i-1}, 0)$. It is known that (3.3) is stationary when $\alpha_{11} + \alpha_{12} < 2$. Refer to Hwang and Kim (2004) for detailed account of threshold-ARCH(1) model. Note that $\alpha_{11} = \alpha_{12} = 0$ corresponds to the case that ϵ_i's are *iid.*

Tables 3.5 and 3.6 summarize simulation results for various pairs of $(\alpha_{11}, \alpha_{12})$ in the stationarity region. Here we fix $\alpha_0 = 1/2$.

It is seen that false detection is of 2(5)% rate for the rule $\hat{\tau}_1 > 2/\sqrt{n}$ ($\hat{\tau}_1 > 1.7/\sqrt{n}$) and detection power quickly converges to one as threshold-ARCH effect becomes prominent.

TABLE 3.6 *Detection rates of Threshold-ARCH(1) errors:* $v_i = 0.5 + \alpha_{11}(\epsilon_{i-1}^{+})^2 + \alpha_{12}(\epsilon_{i-1}^{-})^2$

(α_{11}, α_{12})		(0,0)	(0.1,0.5)	(0.5,0.1)	(0.2,0.8)	(0.8,0.2)	(0.8,0.6)	(0.9,1.0)
cut	$2/\sqrt{n}$	2	56	56	86	91	95	98
value	$1.7/\sqrt{n}$	5	67	68	97	92	98	99

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