FUZZY CONVERGENCE THEORY - II

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ABSTRACT. In this paper convergence of fuzzy filters and graded fuzzy filters have been studied in graded $L$-fuzzy topological spaces.

0. INTRODUCTION

This paper is the continuation of our earlier paper (Mondal & Samanta [10]) where convergence of fuzzy nets has been studied. In this paper we deal with the convergence of fuzzy filters. In 1979 a theory of convergence of fuzzy filters was developed by Lowen [9] for laminated spaces and afterwards it was extended to arbitrary fuzzy (Chang) spaces by Warren [13]. In 1995 Gahler [6, 7] introduced an idea of graded fuzzy filter in lattice valued setting (which he called $L$-fuzzy filter) and studied its convergence in Chang fuzzy topological spaces. Later on in the year of 1999 Burton, Muraleetharan & Garcia [1, 2] considered another type of graded fuzzy filter named as generalized filter ($g$-filter) by relaxing a condition imposed by Gahler [6, 7] but restricted themselves in $I$-fuzzy setting where $I = [0, 1]$ and studied relations among prime prefilters, prime $g$-filters and ultrafilters.

In this paper we study the convergence of both crisp fuzzy filters and graded fuzzy filters in $L$-fuzzy setting, where the underlying fuzzy topological space is a graded $L$-fuzzy topological space of the type as considered in Chattopadhyay, Hazra & Samanta [4], Höhle [8], and Šostak [12].

In Section 2 we study the graded convergence of Warren type fuzzy filters (cf. Warren [13]) and investigate its relation with the graded convergence of associated fuzzy nets.

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In Section 3 we deal with the convergence of $g$-filters. In doing so we have established decomposition theorem involving the convergence of a $g$-filter with the convergence of a family of Warren type fuzzy filters. Relationship between the convergence of $g$-filters and $gp$-mappings has been studied.

1. Notation and Preliminaries

In this paper $X$ denotes a nonempty set; unless otherwise mentioned, $L$ denotes a completely distributive order dense complete lattice with an order reversing involution $\iota$ whereas $L_0 = L \setminus \{0\}$. Let 0 and 1 denote respectively the least and the greatest elements of $L$. Let $L^X$ be the collection of all $L$-fuzzy subsets of $X$ and $\text{Pt}(L^X)$ the set of all $L$-fuzzy points of $X$. $M(L)$ denotes the set of all molecules of $L$ whereas $M(L^X)$ denotes the set of all molecule points of $L^X$. By $\tilde{0}$ and $\tilde{1}$ we denote the constant $L$-fuzzy subsets of $X$ taking values 0 and 1 respectively. For $p_x \in \text{Pt}(L^X)$ and $A, B \in L^X$ we say $p_x \notq A$ if $p_x \notq A'$ and $A \notq B$ if $A \notq B'$. For other notations we follow Liu [14].

**Definition 1.1** (Šostak [12]). A function $\tau : L^X \to L$ is called an $L$-fuzzy topology on $X$ if it satisfies the following conditions:

- (O1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$,
- (O2) $\tau(A_1 \land A_2) \geq \tau(A_1) \land \tau(A_2)$, for $A_1, A_2 \in L^X$, and
- (O3) $\tau(\bigvee_{i \in \Delta} A_i) \geq \bigwedge_{i \in \Delta} \tau(A_i)$ for any $\{A_i\}_{i \in \Delta} \subset L^X$.

The pair $(X, \tau)$ is called an $L$-fuzzy topological space and $\tau$ is also called a gradation of openness on $X$.

**Definition 1.2** (Šostak [12]). A function $\mathcal{F} : L^X \to L$ is called an $L$-fuzzy co-topology of $X$ if it satisfies the following conditions:

- (C1) $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$,
- (C2) $\mathcal{F}(A_1 \lor A_2) \geq \mathcal{F}(A_1) \land \mathcal{F}(A_2)$, for $A_1, A_2 \in L^X$, and
- (C3) $\mathcal{F}\left(\bigwedge_{i \in \Delta} A_i\right) \geq \bigvee_{i \in \Delta} \mathcal{F}(A_i)$ for any $\{A_i\}_{i \in \Delta} \subset L^X$.

The pair $(X, \mathcal{F})$ is called an $L$-fuzzy co-topological space and $\mathcal{F}$ is also called a gradation of closedness on $X$.

**Definition 1.3** (Mondal & Samanta [10]). Let $(X, \tau)$ be an $L$-fuzzy topological space and let $Q : \text{Pt}(L^X) \times L^X \to L$ be a mapping defined by

$$Q(p_x, A) = \bigvee \{\tau(U); p_x \notq U \subseteq A\}.$$
Then $Q$ is said to be a *gradation* of $\mathcal{q}$-neighborhoodness in $(X, \tau)$.

**Definition 1.4** (Mondal & Samanta [10]). Let $(X, \tau)$ be an $L$-fuzzy topological space and let $Q : \text{Pt}(L^X) \times L^X \rightarrow L$ be a mapping defined by

$$Q(p_x, A) = \bigvee \{\tau(U) ; \ p_x \mathcal{q} U \subseteq A\}.$$  

Then $Q$ is said to be a *gradation* of $\mathcal{q}$-neighborhoodness.

**Proposition 1.5** (Mondal & Samanta [10]). Let $Q$ be a gradation of $\mathcal{q}$-neighborhoodness in an $L$-fuzzy topological space $(X, \tau)$. Then

(QN1): $\forall \ p_x \in \text{Pt}(L^X), \ Q(p_x, \bar{1}) = 1, \ Q(p_x, \bar{0}) = 0$.

(QN2): $Q(p_x, A) \leq Q(p_x, B)$ if $A, B \in L^X, \ A \subseteq B$.

(QN3): $\forall \ p_x \in \text{Pt}(L^X)$ and $\forall \ A, B \in L^X, \ Q(p_x, A \wedge B) = Q(p_x, A) \wedge Q(p_x, B)$.

(QN4): $Q(p_x, A) \not\leq k$ implies that there exists a $B_p \in L^X$ such that $p_x \mathcal{q} B_p \subseteq A$ and

$$\wedge_{r_y \mathcal{q} B_p} Q(r_y, B_p) \not\leq k.$$  

**Proposition 1.6** (Mondal & Samanta [10]). Let $Q : \text{Pt}(L^X) \times L^X \rightarrow L$ be a mapping satisfying (QN1)–(QN3) of Proposition 1.5. Let $\bar{\tau} : L^X \rightarrow L$ be defined by $\bar{\tau}(A) = \wedge_{p_x \mathcal{q} A} Q(p_x, A)$. Then $(X, \bar{\tau})$ forms an $L$-fuzzy topological space. If further the condition (QN4) of Proposition 2.4 is satisfied by $Q$ then the mapping $\bar{Q} : \text{Pt}(L^X) \times L^X \rightarrow L$ defined by

$$\bar{Q}(p_x, A) = \bigvee \{\bar{\tau}(U) ; \ p_x \mathcal{q} U \subseteq A\}$$  

is identical with $Q$.

**Proposition 1.7** (Mondal & Samanta [10]). Let $Q$ be a gradation of $\mathcal{q}$-neighborhoodness in an $L$-fuzzy topological space $(X, \tau)$ and $\bar{\tau} : L^X \rightarrow L$ be defined by $\bar{\tau}(A) = \bigvee_{p_x \mathcal{q} A} Q(p_x, A)$ then $\bar{\tau}$ is an $L$-fuzzy topology on $X$ and $\bar{\tau} = \tau$.

**Definition 1.8** (Mondal & Samanta [10]). Let $(X, \tau)$ be an $L$-fuzzy topological space and $e \in \text{Pt}(L^X)$. The $\mathcal{q}$-neighborhood system of the fuzzy point $e$ with respect to the Chang fuzzy topology $\tau_\mathcal{q}$, denoted by $\bar{Q}_\mathcal{q}(e)$, is defined by $\bar{Q}_\mathcal{q}(e) = \{U \in L^X ; \ \exists V \in \tau_\mathcal{q} \text{satisfying } e \mathcal{q} V \subseteq U\}$.

**Definition 1.9** (Mondal & Samanta [10]). Let $(X, \tau)$ be an $L$-fuzzy topological space and $N : \text{Pt}(L^X) \times L^X \rightarrow L$ be a mapping defined by

$$N(p_x, A) = \bigvee \{\tau(U) ; \ p_x \in U \subseteq A\}.$$  

Then $N$ is said to be a *gradation of neighborhoodness* in $(X, \tau)$.  

Definition 1.10. Let \((X, \tau)\) be an \(L\)-fuzzy topological space and \(e \in \text{Pt}(L^X)\). The neighborhood system of the fuzzy point \(e\) with respect to the Chang fuzzy topology \(\tau_r\), denoted by \(\tilde{N}_r(e)\), is defined by
\[
\tilde{N}_r(e) = \{U \in L^X; \ \exists V \in \tau_r \text{ satisfying } e \in V \subseteq U\}.
\]

Definition 1.11 (Liu [14]). Let \(L\) be a complete lattice. Define a relation \(\ll\) in \(L\) as follows: \(\forall a, b \in L, \ a \ll b\) if and only if \(\forall S \subset L, \ \sqrt{S} \geq b \Rightarrow \exists s \in S\) such that \(s \geq a, \ \forall a \in L\), denote \(\beta(a) = \{b \in L; b \ll a\}\), \(\beta^0(a) = M(\beta(a))\).

Definition 1.12 (Chattopadhyay, Hazra & Samanta [4]). Let \((X, \tau)\) and \((Y, \delta)\) be two \(L\)-fuzzy topologies and \(f : X \to Y\) be a mapping. Then \(f\) is called a gradation preserving map (gp-map) if for each \(B \in L^Y, \ \delta(B) \leq \tau(f^{-1}(B))\).

2. Fuzzy Filter and its Convergence

Definition 2.1. Let \(X\) be a nonempty crisp set. A fuzzy filter on \(L^X\) is a non-empty family \(\mathcal{G}\) of \(L\)-fuzzy subsets of \(X\) such that
(i) \(\tilde{\varnothing} \notin \mathcal{G}\),
(ii) \(\mathcal{G}\) is closed under finite intersection, and
(iii) \(\forall A, B \in L^X, \ \text{if } B \in \mathcal{G}\) and \(B \subseteq A\) then \(A \in \mathcal{G}\).

Example 2.2. Let \((X, \tau)\) be an \(L\)-fuzzy topological space with \(\tau\) as a gradation of openness on \(X, \ e \in M(L^X)\). Then, for every \(r \in L_0, \ \tilde{Q}_r(e)\) and \(\tilde{N}_r(e)\) are fuzzy filters on \(L^X\).

Example 2.3. Let \(X\) be an infinite crisp set then for each \(r \in L_0\) the collection \(\{A \in L^X; \ A'_r\text{'cut of } A\}\) is a fuzzy filter on \(L^X\) where \(A'_r\) is the \(r\)-cut of \(A\).

Definition 2.4. Let \((X, \tau)\) be an \(L\)-fuzzy topological space \(\mathcal{G} \subset L^X\) be a fuzzy filter on \(L^X, \ e \in \text{Pt}(L^X)\). Then \(e\) is called a cluster point of \(\mathcal{G}\) of upper grade \(l\) (respectively, lower grade \(k\)), denoted by \(\mathcal{G} \uparrow l e\) (respectively, \(\mathcal{G} \downarrow k e\)), if
\[
l' = \bigwedge \{r \in L_0; \ U \cap A \neq \tilde{\varnothing}, \ \forall U \in \tilde{Q}_r(e) \text{ and } A \in \mathcal{G}\}
\]
(respectively, if
\[
k' = \bigvee \{r \in L_0; \ \exists U \in \tilde{Q}_r(e) \text{ and } \exists A \in \mathcal{G}\text{ such that } A \cap U = \tilde{\varnothing}\}).
And \( e \) is called a **limit point** of \( G \) of upper grade \( l \) (respectively, lower grade \( k \)), denoted by \( G \rightarrow^l e \) (respectively, \( G \rightarrow_k e \)), if \( l' = \bigwedge \{ r \in L_0 ; \ \tilde{Q}_r(e) \subseteq G \} \) (respectively, \( k' = \bigvee \{ r \in L_0 ; \ \tilde{Q}_r(e) \not\subseteq G \} \)).

**Proposition 2.5.** For any fuzzy filter \( G \) in an \( L \)-fuzzy topological space \((X, \tau)\), we have the following properties.

(i) \( G \preceq^l e \) and \( G \preceq_k e \Rightarrow k \neq l \).

(ii) \( G \rightarrow^l e \) and \( G \rightarrow_k e \Rightarrow k \neq l \).

**Proof.** (i) Let \( U = \{ r \in L_0 ; \ \forall \ U \in \tilde{Q}_r(e) \text{ and } V \in G, \ U \cap V \neq \emptyset \} \) and \( V = \{ r \in L_0 ; \ \exists \ U \in \tilde{Q}_r(e), \ V \in G, \ U \cap V = \emptyset \} \). Then obviously \( U \cap V = \emptyset \) and \( U \cup V = L_0 \).

Also from the definition of limit points of upper grade and lower grade of a fuzzy filter we have \( l' = \bigwedge U \) and \( k' = \bigvee V \). If \( \bigwedge U > \bigvee V \) then there exists \( m \in L_0 \) such that \( \bigwedge U > m > \bigvee V \Rightarrow m \not\in U \) and \( m \not\in V \), which is contradictory to the fact that \( U \cup V = L_0 \). So, \( l' = \bigwedge U > \bigvee V = k' \) is not possible. This implies \( k \neq l \).

(ii) Similar to (i). \( \square \)

**Proposition 2.6.** If \( L \) be an order dense chain then, in an \( L \)-fuzzy topological space \((X, \tau)\), we have the following properties.

(i) \( G \preceq^l e \) and \( G \preceq_k e \Rightarrow k = l \).

(ii) \( G \rightarrow^l e \) and \( G \rightarrow_k e \Rightarrow k = l \).

**Proof.** (i) As in Proposition 2.5, if we consider the partitions \( U \) and \( V \) of \( L_0 \) and \( l' = \bigwedge U, k' = \bigvee V \) then we have \( k \leq l \). If possible let \( k < l \) then \( k' > l' \Rightarrow \exists m \in L_0 \) such that \( k' > m > l' \Rightarrow V > m > \bigwedge U \Rightarrow m \in V \text{ and } m \in U \), which is contradictory to the fact that \( U \cap V = \emptyset \). Hence \( k \neq l \).

(ii) Similar to (i) \( \square \)

**Note 2.7.** If in addition \( L \) is a chain then in the \( L \)-fuzzy topological space \((X, \tau)\), as there is no difference between \( G \preceq^l e \) and \( G \preceq_l e \) so they will be commonly denoted by \( G \preceq (l) e \). Similarly, \( G \rightarrow^l e \) and \( G \rightarrow_l e \) will be commonly denoted by \( G \rightarrow (l) e \).

**Proposition 2.8.** Let \((X, \tau)\) be an \( L \)-fuzzy topological space with \( \tau \) as a gradation of openness on \( X \), \( G \subseteq L^X \) be a fuzzy filter on \( L^X \), \( e \in \text{Pt}(L^X) \). Then, for \( k \in L \), we have

(i) \( G \rightarrow^k e \Rightarrow G \preceq^l e \text{ for some } l \geq k \),

(ii) \( G \preceq^k e \geq f \Rightarrow G \preceq^l f \text{ for some } l \geq k \),

(iii) \( G \rightarrow^k e \geq f \Rightarrow G \rightarrow^l f \text{ for some } l \geq k \),
(iv) \( G \otimes_k e \Rightarrow G \rightarrow^l e \) for some \( l \leq k \),

(v) \( G \otimes_k e \leq f \Rightarrow G \otimes_l f \) for some \( l \leq k \), and

(vi) \( G \rightarrow_k e \leq f \Rightarrow G \rightarrow^l f \) for some \( l \leq k \).

The proof is straightforward.

**Definition 2.9.** Let \((X, \tau)\) be an \(L\)-fuzzy topological space and \(G, H\) be any two fuzzy filters on \(L^X\). Say \(H\) is finer than \(G\) or subfilter of \(G\), or say \(G\) is coarser than \(H\) if \(G \subseteq H\).

**Proposition 2.10.** Let \((X, \tau)\) be an \(L\)-fuzzy topological space and \(G, H\) be fuzzy filters on \(L^X\), \(H\) be coarser than \(G\), \(e \in \text{Pt}(L^X)\). Then, for \(k \in L\), we have

(i) \( H \rightarrow^k e \Rightarrow G \rightarrow^l e \) for some \( l \geq k \),

(ii) \( G \otimes_k e \Rightarrow H \otimes^l e \) for some \( l \geq k \),

(iii) \( H \rightarrow^k e \Rightarrow G \rightarrow^l e \) for some \( l \leq k \), and

(iv) \( G \otimes_k e \Rightarrow H \otimes^l e \) for some \( l \leq k \).

**Proposition 2.11.** Let \((X, \tau)\) be an \(L\)-fuzzy topological space, \(G\) be a fuzzy filter on \(L^X\), \(\Delta\) be the collection of all subfilters of \(G\), \(e \in \text{Pt}(L^X)\). Then we have

(i) \( G \rightarrow^l e \Rightarrow l = \bigwedge_{H \in \Delta} \{r \in L; H \rightarrow^r e\} \),

(ii) \( G \otimes^l e \Rightarrow l = \bigvee_{H \in \Delta} \{r \in L; H \otimes^r e\} \),

(iii) \( G \otimes(l) e \Rightarrow l = \bigvee_{H \in \Delta} \{r \in L; H \rightarrow (r)e\} \), if \(L\) is a chain,

(iv) \( G \otimes(l) e \Rightarrow \exists\) a subfilter \(H\) of \(G\) such that \(H \rightarrow (l)e\) if \(L\) is a chain,

(v) \( G \rightarrow_l e \Rightarrow l = \bigwedge_{H \in \Delta} \{r \in L; H \rightarrow^r e\} \), and

(vi) \( G \otimes_l e \Rightarrow l = \bigvee_{H \in \Delta} \{r \in L; H \otimes^r e\} \).

**Proof.** (i) For, any \(H \in \Delta\), \(H \rightarrow^r e\) and \(G \rightarrow^l e\) implies \(r \geq l\), so

\[
 l \leq \bigwedge_{H \in \Delta} \{r \in L; H \rightarrow^r e\}.
\]

Again as a particular case taking \(H = G\) we get \(l \geq \bigwedge_{H \in \Delta} \{r \in L; H \rightarrow^r e\}\). Hence the proof follows.

(ii) Similar to (i).

(iii) Let \(H\) be a subfilter of \(G\) such that \(H \rightarrow (r)e\). Then, for every \(s > r\), \(\tilde{Q}_s(e) \subseteq H\). So, \(U \in \tilde{Q}_s(e)\) and \(V \in G\) implies \(U, V \in H\) since \(\tilde{Q}_s(e), G \subseteq H\). This implies \(U \cap V \neq \emptyset\).

So, for some \(l \geq r\), \(G \otimes(l) e\). Again as \(H\) is any subfilter of \(G\), so \(G \otimes(l) e \Rightarrow l \geq \bigvee_{H \in \Delta} \{r \in L; H \rightarrow (r)e\}\).
Next let $G \infty(l) e$ in $(X, \tau)$ and let $B = G \cup \left( \bigcup_{m > l'} \tilde{Q}_m(e) \right)$. Then $U_1, U_2 \in \bigcup_{m > l'} \tilde{Q}_m(e)$ implies that there exists a $m_1, m_2 \in L_0$ such that $m_1, m_2 > l'$ and $U_1 \in \tilde{Q}_{m_1}(e)$ and $U_2 \in \tilde{Q}_{m_2}(e)$.

Without loss of generality let $m_1 > m_2$ then $U_1, U_2 \in \tilde{Q}_{m_2}$ (as $\tilde{Q}_{m_1}(e) \subseteq \tilde{Q}_{m_2}(e)$). So,

$$U_1 \cap U_2 \in \tilde{Q}_{m_2}(e) \Rightarrow U_1 \cap U_2 \in \bigcup_{m > l'} \tilde{Q}_m(e),$$

i.e., $\bigcup_{m > l'} \tilde{Q}_m(e)$ has the finite intersection property. $G$ being a fuzzy filter, also has the finite intersection property.

Again $G \infty(l) e$ implies that, for all $m > l'$, $U \in G$ and $V \in \tilde{Q}_m(e)$ means $U \cap V \neq \emptyset$. Therefore $B = G \cup \left( \bigcup_{m > l'} \tilde{Q}_m(e) \right)$ has the finite intersection property. As $\emptyset \not\in G$ and $\emptyset \not\in \bigcup_{m > l'} \tilde{Q}_m(e)$, so $\emptyset \not\in B$. Denote the filter generated by $B$ by $\uparrow B$.

So, $\uparrow B$ is a subfilter of $G$. Let $\mathcal{H} = \uparrow B$ then, for all $m > l'$, $\tilde{Q}_m(e) \subseteq \mathcal{H}$. This implies that, for some $r \geq l, \mathcal{H} \rightarrow (r)e$.

The proofs of (iv)-(vi) are straightforward. \qed

**Definition 2.12.** Let $(X, \tau)$ be an $L$-fuzzy topological space, $S$ be a molecule net on $L^X$, $G$ be a fuzzy filter on $L^X$. For $S$ and $G$ we define the fuzzy filter associated with the net $S$ as the family $G(S)$ of all fuzzy subsets of $X$ with which the net $S$ eventually quasi-coincides. For $G$, let $D(G) = \{(e, A) \in M(L^X) \times G; e \leq A\}$ and equip $D(G)$ with the relation $\leq$ on it as $\forall (e, A), (d, B) \in D(G), (e, A) \leq (d, B) \iff A \geq B$. Define the molecule net associated with the fuzzy filter $G$ as the mapping $S(G) : D(G) \rightarrow M(L^X)$, defined by $S(G)(e, A) = e \forall (e, A) \in D(G)$.

**Definition 2.13 (Mondal & Samanta [10]).** Let $(X, \tau)$ be an $L$-fuzzy topological space and $e \in \text{Pt}(L^X)$. Let $D$ be any directed set and $S : D \rightarrow \text{Pt}(L^X)$ be any fuzzy net. For $U \in L^X$ if $\exists m \in D$ such that $S(n) \leq U \forall n \geq m$ holds then we say $S \leq U$ eventually;

if, for every $m \in D$, there exists $n \in D$ such that $n \geq m$ and $S(n) \leq U$ then we say $S \leq U$ frequently. Call $e$ a cluster point with upper grade $l$, denoted by $S \infty e^l$ (respectively, a cluster point with lower grade $k$, denoted by $S \infty e^k$) of a fuzzy net $S : D \rightarrow \text{Pt}(L^X)$, if

$$l' = \bigwedge\{r \in L_0; \forall U \in \tilde{Q}_r(e), U \leq S \text{ frequently}\}$$

(respectively, if $k' = \bigvee\{r \in L_0; \exists V \in \tilde{Q}_r(e) \text{ such that } V \geq S \text{ eventually}\}$). Call $e$ a limit point of upper grade $l$ of $S$, denoted by $S \rightarrow e^l$ (respectively, a limit point of
lower grade $k$ of $S$, denoted by $S \rightarrow_k e$) if

$$l' = \bigwedge \{r \in L_0; \forall U \in \tilde{Q}_r(e), \ U \not\subseteq S \text{ eventually} \}$$

(respectively, $k' = \bigvee \{r \in L_0; \exists V \in \tilde{Q}_r(e) \text{ such that } V \not\varsubsetneq S \text{ frequently} \} ).$

**Proposition 2.14.** Let $(X, \tau)$ be an $L$-fuzzy topological space, $\mathcal{G}$ be a fuzzy filter on $L^X$, $S$ be a molecule net in $L^X$, $e \in \text{Pt}(L^X)$. Then, for $k \in L$, we have the following properties.

(i) $S \rightarrow_k e \iff \mathcal{G}(S) \rightarrow_k e$.

(ii) $\mathcal{G} \rightarrow_k e \iff S(\mathcal{G}) \rightarrow_k e$.

(iii) $\mathcal{G} \infty_k e \iff S(\mathcal{G}) \infty_k e$.

(iv) $S \infty_k e \Rightarrow \mathcal{G}(S) \infty_l e$ for some $l \geq k$.

**Proof.** (i) $S \rightarrow_k e \iff k' = \bigwedge \{r \in L_0; \forall U \in \tilde{Q}_r(e), \ S \not\subseteq U \text{ eventually} \}$

$\iff k' = \bigwedge \{r \in L_0; \forall U \in \tilde{Q}_r(e), \ U \in \mathcal{G}(S) \}$

$\iff k' \infty \bigwedge \{r \in L_0; \tilde{Q}_r(e) \subseteq \mathcal{G}(S) \}$

$\iff \mathcal{G}(S) \rightarrow_k e$.

Similarly, we can prove the other results. □

**Proposition 2.15.** Let $(X, \tau)$ be an $L$-fuzzy topological space, $\mathcal{G}$ be a fuzzy filter on $L^X$, $S$ be a molecule net in $L^X$, $e \in \text{Pt}(L^X)$. Then, for $k \in L$, we have the following properties.

(i) $S \rightarrow_k e \iff \mathcal{G}(S) \rightarrow_k e$.

(ii) $\mathcal{G} \rightarrow_k e \iff S(\mathcal{G}) \rightarrow_k e$.

(iii) $\mathcal{G} \infty_k e \iff S(\mathcal{G}) \infty_k e$.

(iv) $S \infty_k e \Rightarrow \mathcal{G}(S) \infty_l e$ for some $l \geq k$.

**Proof.** (i) $S \rightarrow_k e \iff k' = \bigvee \{r \in L_0; \exists U \in \tilde{Q}_r(e) \text{ such that } S \not\varsubsetneq U \text{ frequently} \}$

$\iff k' = \bigvee \{r \in L_0; \exists U \in \tilde{Q}_r(e) \text{ such that } U \not\subseteq \mathcal{G}(S) \}$

$\iff k' = \bigwedge \{r \in L_0; \tilde{Q}_r(e) \not\subseteq \mathcal{G}(S) \} \iff \mathcal{G}(S) \rightarrow_k e$.

(ii) $\mathcal{G} \rightarrow_k e \Rightarrow k' = \bigvee \{r \in L_0; \tilde{Q}_r(e) \not\subseteq \mathcal{G} \}$.

Now $\tilde{Q}_r(e) \not\subseteq \mathcal{G}$ \Rightarrow $U \notin \mathcal{Q}_r(e)$ such that $U \not\subseteq \mathcal{G}$. Then, for every $(f, V) \in D(\mathcal{G})$, $U \not\supset V$.

Now $U \not\supset V \Rightarrow U' \not\subseteq V' \Rightarrow \exists x \in X$ such that $U'(x) \not\subseteq V'(x)$. As $M(L)$ is a join generating subset of $L$ so there exists $k \in M(L)$ such that $U''(x) \geq k \not\subseteq V''(x)$ \Rightarrow $k_{x} \in M(L^X)$ and $k_{x} \subseteq U'$ but $k_{x} \not\subseteq V'$ \Rightarrow $k_{x} \not\subseteq V$ but $k_{x} \not\subseteq U \Rightarrow (k_{x}, V) \in D(\mathcal{G})$.

Again $(k_{x}, V) \geq (f, V)$ but $S(\mathcal{G})(k_{x}, V) = k_{x} \not\subseteq U \Rightarrow S(\mathcal{G}) \not\subseteq U$ frequently.
Conversely, if \( U \in \mathcal{G} \) then for all \((f, V), (g, U) \in D(\mathcal{G})\) with \((f, V) \geq (g, U)\) we have \( V \subseteq U \). Now \( f \not\preceq V \) and hence \( f \not\preceq U \). So, \([S(\mathcal{G})(f, V)] \not\preceq U\) i.e., \( S(\mathcal{G}) \not\preceq U \) eventually. Hence \( S(\mathcal{G}) \not\preceq U \) frequently \( \Rightarrow U \not\in \mathcal{G} \Rightarrow \tilde{Q}_r(e) \not\in \mathcal{G} \). So, \( U \in \tilde{Q}_r(e) \) and \( S(\mathcal{G}) \not\preceq U \) frequently \( \Rightarrow \tilde{Q}_r(e) \not\subseteq \mathcal{G} \). So, we can say now that

\[
\tilde{Q}_r(e) \not\subseteq \mathcal{G} \iff \exists U \in \tilde{Q}_r(e) \text{ such that } S(\mathcal{G}) \not\preceq U \text{ frequently}.
\]

Hence,

\[
\bigvee \{r \in L_0; \tilde{Q}_r(e) \not\subseteq \mathcal{G}\} = \bigvee \{r \in L_0; \exists U \in \tilde{Q}_r(e) \text{ such that } S(\mathcal{G}) \not\preceq U \text{ frequently}\},
\]

i.e., \( \mathcal{G} \to_k e \iff S(\mathcal{G}) \to_k e \).

(iii) Let, for some \( r \in L_0 \) there exists \( U \in \tilde{Q}_r(e) \) and \( V \in \mathcal{G} \) such that \( U \cap V = \emptyset \). Take \((f, V) \in D(\mathcal{G})\). We shall show that for all \((g, W) \in D(\mathcal{G})\) if \((g, W) \geq (f, V)\) then \( S(\mathcal{G})(g, W) \not\preceq U \). Suppose \( S(\mathcal{G})(g, W) \not\preceq U \), then \( g \not\preceq U \). Again

\[
(g, W) \geq (f, V) \Rightarrow g \not\preceq U \subseteq V,
\]

so \( g \not\preceq V \). Therefore \( g \in M(L^X) \), \( g \not\preceq U \) and \( g \not\preceq V \Rightarrow g \not\preceq (U \cap V) \Rightarrow U \cap V \neq \emptyset \), a contradiction.

Next let for some \( r \in L_0 \) \( \exists U \in \tilde{Q}_r(e) \) such that \( S(\mathcal{G}) \not\preceq U \) eventually. Then, there exists \((f, V) \in D(\mathcal{G})\) such that, for all \((g, W) \in D(\mathcal{G})\), \((g, W) \geq (f, V)\). This implies \( S(\mathcal{G})(g, W) \not\preceq U \), i.e., \( g \not\preceq U \).

Therefore for all \( g \not\preceq V \) as \((g, V) \geq (f, V)\) so \( S(\mathcal{G})(g, V) \not\preceq U \), i.e., \( g \not\preceq U \), i.e., \( \forall g \not\preceq V, g \not\preceq U \). So, \( U \cap V = \emptyset \). Thus (iii) is proved.

(iv) Let \( U \in \tilde{Q}_r(e) \) and \( V \in \mathcal{G}(S) \) be such that \( U \cap V = \emptyset \). Now \( V \in \mathcal{G}(S) \Rightarrow S \not\preceq V \) eventually \( \Rightarrow \exists m \in D \) such that \( \forall n \geq m, S(n) \not\preceq V \). We shall show that \( S \not\preceq U \) eventually. Suppose \( S \not\preceq U \) frequently, then \( \exists p \in D \) such that \( p \geq m \) and \( S(p) \not\preceq U \).

Now \( S(p) \not\preceq V \), \( S(p) \not\preceq U \) and \( S(p) \in M(L^X) \Rightarrow S(p)(U \cap V) \Rightarrow U \cap V \neq \emptyset \), a contradiction.

Thus for \( U \in \tilde{Q}_r(e) \), \( V \in \mathcal{G}(S) \) if \( U \cap V = \emptyset \) then \( S \not\preceq U \) eventually. So,

\[
l' = \bigvee \{r \in L_0; \exists U \in \tilde{Q}_r(e), \exists V \in \mathcal{G}(S); U \cap V = \emptyset\}
\leq \bigvee \{r \in L_0; \exists U \in \tilde{Q}_r(e) \text{ such that } S \not\preceq U \text{ eventually}\} = k',
\]

i.e., \( \mathcal{G}(S) \not\preceq_k e \Rightarrow S \not\preceq_k e \text{ for some } k \leq l \).
Definition 2.16 (Mondal & Samanta [10]). Let \((X, \mathcal{F})\) be an \(L\)-fuzzy co-topological space with \(\mathcal{F}\) as a \(GC\) on \(X\). For each \(A \in L^X\) we define
\[
cl(A, r) = \bigwedge \{D \in L^X; A \subseteq D; D \in \mathcal{F}_r\}
\]
where \(\mathcal{F}_r = \{C \in L^X; \mathcal{F}(C) \geq r\}\). The operator \(cl\) is said to be \(L\)-fuzzy closure operator in \((X, \mathcal{F})\).

Definition 2.17 (Mondal & Samanta [10]). In an \(L\)-fuzzy topological space \((X, \tau)\),
\[
p_x \in cl(A, m)
\]
if and only if, for all \(U \in \tau_m\), \(p_x \sqcup U \Rightarrow U \sqcup A\).

Proposition 2.18. Let \((X, \tau)\) be an \(L\)-fuzzy topological space and \(A \in L^X; e \in M(L^X)\). Then \(e \in cl(A, k')\) implies that there exists a fuzzy filter \(G\) on \(L^X\) such that \(A' \notin G\) and for some \(l \geq k\), \(G \rightarrow^l e\).

Proof. Let \(e \in cl(A, k')\). Then for every \(U \in \hat{\mathcal{Q}}_{k'}(e)\), \(U \sqcup A\) (by Proposition 2.17), \(i.e.,\) for every \(U \in \hat{\mathcal{Q}}_{k'}(e)\) \(\exists x^u \in X\) such that \(U(x^u) \not\leq A'(x^u) \Rightarrow A(x^u) \not< U'(x^u)\).

As \(M(L)\) is a join generating subset of \(L\) so \(\exists p^u \in M(L)\) such that \(A(x^u) \geq p^u \not< U'(x^u) \Rightarrow p_{x^u}^p \in M(L^X)\) and \(p_{x^u}^p \sqcup U\) and \(p_{x^u}^p \in A\).

As \(e \in M(L^X)\) so \(\hat{\mathcal{Q}}_{k'}(e)\) is a directed set with respect to the relation \(\geq\) defined by \(\forall U, V \in \hat{\mathcal{Q}}_{k'}(e); U \geq V \iff U \subseteq V\). So we define a molecule net \(S: \hat{\mathcal{Q}}_{k'}(e) \rightarrow M(L^X)\) by \(S(U) = p_{x^u}^p\). Then \(S\) is a molecule net in \(A\) and as \(\forall U \in \hat{\mathcal{Q}}_{k'}(e), U \sqcup A\) so \(\forall U \in \hat{\mathcal{Q}}_{k'}(e), U \sqcup S\) eventually, which implies \(\bigwedge\{s \in L_0; \forall U \in \hat{\mathcal{Q}}_s(e), U \sqcup S\} \leq k' \Rightarrow S \rightarrow^l e\) for some \(l \geq k\).

Now, for the associated filter \(G(S)\), by (i) of Proposition 2.14, \(G(S) \rightarrow^l e\). If \(A' \in G(S)\) then \(S\) eventually quasi-coincides with \(A'\) (\(i.e.,\) \(S\) is eventually not in \(A\)), this contradicts the fact that \(S\) is a fuzzy net in \(A\). So, \(A' \notin G(S)\).

Definition 2.19. Let \(X\) be nonempty crisp set. A nonempty subfamily \(A \subseteq L^X\) is called a filter base on \(L^X\), if \(\emptyset \notin A\) and \(A\) is closed under finite intersection. For a filter base \(A\) on \(L^X\), denote the filter generated by \(A\) as \(\uparrow A\).

Definition 2.20. Let \((X, \tau)\) be an \(L\)-fuzzy topological space, \(A\) a filter base on \(L^X\). An \(L\)-fuzzy point \(e \in \text{Pt}(L^X)\) is called a cluster point of \(A\) with upper grade \(k\), denoted by \(A \inf^k e\) (respectively, a cluster point of \(A\) with lower grade \(l\), denoted by \(A \inf_l e\)) if \(\uparrow A \inf^k e\) (respectively, if \(\uparrow A \inf_l e\); \(e\) is called a limit point of \(A\) with upper and lower grades \(m\) and \(n\), denoted by \(A \rightarrow^m e\) and \(A \rightarrow^n e\) respectively if \(\uparrow A \rightarrow^m e\) and \(\uparrow A \rightarrow^n e\).
Proposition 2.21. Let \((X, \tau)\) and \((Y, \delta)\) be any two \(L\)-fuzzy topological spaces and let \(f : (X, \tau) \to (Y, \delta)\) be a gp-map then for any filter base \(A\) in \((X, \tau)\) and \(\forall e \in \Pt(L^X), \ A \to^k e \Rightarrow f[A] \to^k f(e)\) for some \(k \geq l\).

Proof. Let \(\tilde{Q}_r(e)\) and \(\tilde{Q}_r(f(e))\) be the \(q\)-neighborhood systems of \(e\) and \(f(e)\) with respect to the Chang fuzzy topologies \(\tau_r\) and \(\delta_r\) respectively. Suppose \(A\) is a filter base in \((X, \tau)\), \(e \in \Pt(L^X)\) and \(A \to^l e\). Let \(\tilde{Q}_r(e) \subseteq A\). Then \(\forall V \in \tilde{Q}_r(f(e))\), since \(f\) is a gp-map, \(f^{-1}(V) \in \tilde{Q}_r(e) \Rightarrow \exists A \in A\) such that \(f^{-1}(V) \geq A\).

Therefore \(V \supseteq ff^{-1}(V) \supseteq f(A) \in f[A] \Rightarrow V \in f[A]\). \(\square\)

Proposition 2.22. Let \(f : (X, \tau) \to (Y, \delta)\) be a mapping where \((X, \tau)\) and \((Y, \delta)\) be any two \(L\)-Fuzzy topological spaces. If, for any fuzzy filter base \(A\) and for any \(e \in M(L^X)\),

\[ A \to^k e \Rightarrow f[A] \to^l f(e) \text{ for some } l \geq k, \]

then \(f\) is a gp-map.

Proof. Suppose \(f\) be not a gp-map, then \(\exists V \in L^Y\) such that \(\tau(f^{-1}(V)) \not\supseteq \delta(V)\). Therefore from the order dense property of \(L\) we get \(k_1, k_2 \in L\) such that \(\tau(f^{-1}(V)) \not\supseteq k_1 < k_2 < \delta(V)\).

Now we have

\[ \tau(f^{-1}(V)) \not\supseteq k_1 \]

\[ \Rightarrow \bigwedge\{Q(e, f^{-1}(V)); e \in M(L^X)\} \not\supseteq k_1 \]

\[ \Rightarrow \exists e^0 \in M(L^X) \text{ such that } e^0 q f^{-1}(V) \text{ and } Q(e^0, f^{-1}(V)) \not\supseteq k_1 \]

\[ \Rightarrow \bigvee\{\tau(U); e^0 q U \subseteq f^{-1}(V)\} \not\supseteq k_1 \]

\[ \Rightarrow \forall U \in L^X \text{ with } \tau(U) \geq k_1 \text{ and } e^0 q U, U \not\subseteq f^{-1}(V) \]

\[ \Rightarrow f^{-1}(V) \not\subseteq \tilde{Q}_{k_1}(e^0) \]

\[ \Rightarrow ff^{-1}(V) \not\subseteq f(\tilde{Q}_{k_1}(e^0)). \]

For, if \(f^{-1}(V) \not\subseteq \tilde{Q}_{k_1}(e^0)\) but \(ff^{-1}(V) \in f(\tilde{Q}_{k_1}(e^0))\) then \(\exists W \in \tilde{Q}_{k_1}(e^0)\) such that \(ff^{-1}(V) = f(W)\),

\[ V \supseteq ff^{-1}(V) = f(W) \]

\[ \Rightarrow f^{-1}(V) \supseteq W \Rightarrow f^{-1}(V) \in \tilde{Q}_{k_1}(e^0) \text{ (as } e^0 \in M(L^X) \Rightarrow \tilde{Q}_{k_1}(e^0) \text{ is a fuzzy filter),} \]

a contradiction. So, \(f^{-1}(V) \not\subseteq \tilde{Q}_{k_1}(e^0)\).

Hence

\[ V \supseteq ff^{-1}(V) \not\subseteq f(\tilde{Q}_{k_1}(e^0)) \] (1)
Again $e^0 \mathcal{Q} f^{-1}(V) \Rightarrow f(e^0) \mathcal{Q} V$ and we have $\delta(V) > k_2$. So,

$$V \in \mathcal{Q}_{k_2}'(f(e^0)).$$

(2)

So, by (1) and (2), we have $f(\mathcal{Q}_{k_1}(e^0)) \not\supseteq \mathcal{Q}_{k_2}'(f(e^0))$. This means if

$$f[\mathcal{Q}_{k_1}(e^0)] \rightarrow f(e^0)$$

then $l' \geq k_2$. But from the definition of convergence we have if $\mathcal{Q}_{k_1}(e^0) \rightarrow k e^0$ then $k \geq k_1'$. This implies $k' \leq k_1$.

Therefore $l' \geq k_2 > k_1 \geq k' \Rightarrow l < k$, a contradiction to the given condition. Hence $f$ is a gp-map. □

3. Lattice Valued Generalized Filter

Definition 3.1 (Burton, Muraleetharan & Gutiérrez [1]). Let $\mathcal{G} : L^X \rightarrow L$ be a mapping satisfying

(GF1) $\mathcal{G}(\emptyset) = 0; \mathcal{G}(\mathbb{1}) = 1,$

(GF2) $\forall A_1, A_2 \in L^X, \mathcal{G}(A_1 \land A_2) \geq \mathcal{G}(A_1) \land \mathcal{G}(A_2)$, and

(GF3) $\forall A, B \in L^X, \mathcal{G}(B) \geq \mathcal{G}(A)$ if $A \subset B$,

then $\mathcal{G}$ is said to be a generalized filter (g-filter) on $L^X$.

Example 3.2. Let $Q$ be the gradation of $q$-neighborhoodness in an $L$-fuzzy topological space $(X, \tau)$, $e \in M(L^X)$. We define a mapping $Q_e : L^X \rightarrow L$ by

$$Q_e(U) = Q(e, U), \forall U \in L^X.$$ 

Then $Q_e$ is a g-filter on $L^X$.

Example 3.3. Similarly the mapping $N_e : L^X \rightarrow L$ for a particular $e \in \text{Pt}(L^X)$, defined by $N_e(U) = N(e, U), \forall U \in L^X$ is a g-filter on $L^X$ where $N$ is the gradation of neighborhoodness in an $L$-fuzzy topological space $(X, \tau)$.

Example 3.4. Let $X$ be an infinite crisp set and let $\mathcal{G} : L^X \rightarrow L$ be defined by $\mathcal{G}(A) = \bigvee \{r \in L_0; A'[r'] = \text{finite}\}$ where $A'[r']$ is an $r'$-cut of $A'$, then $\mathcal{G}$ is a g-filter on $L^X$.

Definition 3.5. Let $\mathcal{G}$ and $\mathcal{H}$ be any two g-filters on $L^X$. We say $\mathcal{G}$ is coarser than $\mathcal{H}$ or $\mathcal{H}$ is finer than $\mathcal{G}$ if $\mathcal{G} \leq \mathcal{H}$. In this case $\mathcal{H}$ is also called a subfilter of $\mathcal{G}$.
Definition 3.6. Let $\mathcal{G}$ be a $g$-filter in an $L$-fuzzy topological space $(X, \tau)$ and $e \in \text{Pt}(L^X)$. Call $e$ a limit point of $\mathcal{G}$, denoted by $\mathcal{G} \to e$ if $Q(e, U) \leq \mathcal{G}(U) \ \forall \ U \in L^X$, where $Q$ is the gradation of $q$-neighborhoodness in $(X, \tau)$. Denote the join of all limit points of $\mathcal{G}$ by $\lim \mathcal{G}$. Call $e$ a cluster point of $\mathcal{G}$, denoted by $\mathcal{G} \triangleleft e$ if $\mathcal{G}(A) \ngeq Q'(e, U) \Rightarrow A \cap U \neq \emptyset, \ \forall \ A, U \in L^X$, where $Q$ is the gradation of $q$-neighborhoodness on $(X, \tau)$. Denote the join of all cluster points of $\mathcal{G}$ by $\text{clu} \mathcal{G}$.

Proposition 3.7. In an $L$-fuzzy topological space $(X, \tau)$ for any $g$-filter $\mathcal{G}$ and, for $e, f \in \text{Pt}(L^X)$, we have

(i) $\mathcal{G} \to e \Rightarrow \mathcal{G} \triangleleft f$; if $L$ is complemented,

(ii) $\mathcal{G} \triangleleft e \geq f \Rightarrow \mathcal{G} \triangleleft f$, and

(iii) $\mathcal{G} \to e \geq f \Rightarrow \mathcal{G} \to f$.

Proof. (i) Let $\mathcal{G} \to e$ and let $\mathcal{G}(A) \leq Q'(e, U)$ for some $A, U \in L^X$. Then from the order dense property of $L$ $\exists \ k \in M(L)$ such that $\mathcal{G}(A) \geq k \leq Q'(e, U) \Rightarrow \mathcal{G}(A) \geq k$ and $Q(e, U) \leq k'$.

Again $Q(e, U) \leq k' \Rightarrow \exists \ l \in M(L)$ such that $Q(e, U) \geq l \leq k'$. So,

$$\mathcal{G}(A \cap U) \geq \mathcal{G}(A) \wedge \mathcal{G}(U) \ \text{by (GF2)}$$

$$\geq \mathcal{G}(A) \wedge Q(e, U) \ (\text{as } \mathcal{G} \to e)$$

$$\geq k \wedge l.$$ 

Now $l \leq 1 = k \vee k'$ (as $L$ is complemented) $\Rightarrow l \leq k$ or $l \leq k'$ (as $l \in M(L)$) $\Rightarrow l \leq k$ (as $l \leq k'$ is assumed). So, $k \wedge l = l > 0$ (as $l \leq k'$ $\Rightarrow l > 0$) $\Rightarrow \mathcal{G}(A \cap U) > 0 \Rightarrow A \cap U \neq \emptyset$, by (GF1) $\Rightarrow \mathcal{G} \triangleleft e$.

(ii) Let $\mathcal{G} \triangleleft e \geq f$ and let $\mathcal{G}(A) \leq Q'(f, U)$ for some $A, U \in L^X$. Then as $e \geq f \Rightarrow Q(e, U) \geq Q(f, U)$ and $Q'(f, U) \geq Q'(e, U) \Rightarrow \mathcal{G}(A) \leq Q'e, U) \Rightarrow A \cap U \neq \emptyset$ (as $\mathcal{G} \triangleleft e$) $\Rightarrow \mathcal{G} \triangleleft f$.

(iii) The proof is straightforward. □

Proposition 3.8. In an $L$-fuzzy topological space $(X, \tau)$ if $\mathcal{H}$ is finer than $\mathcal{G}$ and $p_x \in \text{Pt}(L^X)$ then, we have

(i) $\lim \mathcal{G} \leq \text{clu} \mathcal{G}$ if $L$ is complemented,

(ii) $p_x \in \text{clu} \mathcal{G} \iff \mathcal{G} \triangleleft p_x$, if $L$ is a chain,

(iii) $p_x \in \lim \mathcal{G} \iff \mathcal{G} \to p_x$, if $L$ is a chain,

(iv) $\mathcal{H} \triangleleft p_x \Rightarrow \mathcal{G} \triangleleft p_x$,

(v) $\text{clu} \mathcal{G} \geq \text{clu} \mathcal{H}$, and
(vi) \( \lim \mathcal{G} \leq \lim \mathcal{H} \).

Proof. (i) is clear.

(ii) \( \mathcal{G} \cap p_x \Rightarrow p_x \in \text{cl} u \mathcal{G} \) is clear.

Let \( p_x \in \text{clu} \mathcal{G} \) and suppose \( \mathcal{G} \not\supseteq p_x \). Then \( \exists A, U \in L^X \) such that \( \mathcal{G}(A) \not\subseteq Q'(p_x, U) \) but \( A \cap U = \tilde{0} \).

Now \( \mathcal{G}(A) \not\subseteq Q'(p_x, U) \Rightarrow \mathcal{G}'(A) \not\subseteq Q(p_x, U) \mathcal{G}'(A) \not\subseteq \bigvee \{ r \in L_0; U \in \tilde{Q}_r(p_x) \} \Rightarrow \exists s \in L_0 \) such that \( s \not\subseteq \mathcal{G}'(A) \) but \( U \in \tilde{Q}_s(p_x) \).

Now \( U \in \tilde{Q}_s(p_x) \Rightarrow \exists V \in \tau_s \) such that \( p_x \sqcup V \subseteq U \). Again \( p_x \sqcup V \Rightarrow p \not\subseteq V'(x) \Rightarrow \exists t \in L_0 \) such that \( p > t \not\subseteq V'(x) \) (since \( L \) is order dense), i.e., \( t_x \sqcup V \subseteq U \Rightarrow U \in \tilde{Q}_s(t_x) \Rightarrow Q(t_x, U) \geq s \). As \( L \) is a chain so \( \mathcal{G} \cap t_x \) from the definition of \( \text{clu} \mathcal{G} \). Now \( \mathcal{G}'(A) \not\subseteq s \Rightarrow \mathcal{G}(A) \not\subseteq s' \geq Q'(t_x, U) \Rightarrow \mathcal{G}(A) \not\subseteq Q'(t_x, U) \Rightarrow A \cap U \neq \tilde{0} \), a contradiction.

(iii) Similar to (ii).

Proofs of (iv)-(vi) are straightforward.

Proposition 3.9. Let \( \mathcal{G} \) be a g-filter on \( L^X \) and let \( \mathcal{G}_r = \{ U \in L^X; \mathcal{G}(U) \geq r \} \) then

1. for every \( r \in L_0, \mathcal{G}_r \) is a fuzzy filter on \( L^X \),
2. \( \forall r, s \in L_0, \mathcal{G}_r \subseteq \mathcal{G}_s \) if \( r \geq s \), and
3. \( \bigcap_{i \in \Delta} \mathcal{G}_{r_i} = \mathcal{G}_{V_{i \in \Delta} r_i} \).

Proof. (1) (i) We have \( \mathcal{G}(\bar{1}) = 1 \Rightarrow \mathcal{G}_r \neq \emptyset \forall r \in L_0 \).

(ii) \( \mathcal{G}(\bar{0}) = 0 \Rightarrow \bar{0} \not\in \mathcal{G}_r \forall r \in L_0 \).

(iii) \( U_1, U_2 \in \mathcal{G}_r \Rightarrow \mathcal{G}(U_i) \geq r; \ i = 1, 2 \)

\( \Rightarrow \mathcal{G}(U_1 \cap U_2) \geq \mathcal{G}(U_1) \wedge \mathcal{G}(U_2) \), by (GF2)

\( \geq r \)

\( \Rightarrow U_1 \wedge U_2 \in \mathcal{G}_r, \forall r \in L_0 \).

(iv) Let \( U \in \mathcal{G}_r \) and \( U \subseteq V \) then \( \mathcal{G}(V) \geq \mathcal{G}(U) \), by (GF3)

\( \geq r \)

\( \Rightarrow V \in \mathcal{G}_r, \forall r \in L_0 \).

Hence \( \mathcal{G}_r \) is a fuzzy filter on \( L^X \).

(2) The proof is straightforward.

(3) \( A \in \bigcap_{i \in \Delta} \mathcal{G}_{r_i} \iff \forall i \in \Delta, A \in \mathcal{G}_{r_i} \iff \forall i \in \Delta, \mathcal{G}(A) \geq r_i \iff \mathcal{G}(A) \geq V_{i \in \Delta} r_i \iff A \in \mathcal{G}_{V_{i \in \Delta} r_i} \). So, \( \bigcap_{i \in \Delta} \mathcal{G}_{r_i} = \mathcal{G}_{V_{i \in \Delta} r_i} \).
Proposition 3.10. Let for each \( r \in L_0 \), \( \mathcal{G}_r \) be a collection of \( L \)-fuzzy subsets of \( X \) satisfying conditions

1. \( \mathcal{G}_r \) is a fuzzy filter on \( L^X \) for each \( r \in L_0 \), and
2. For all \( r, s \in L_0 \), \( \mathcal{G}_r \subseteq \mathcal{G}_s \) if \( r \geq s \),

then the mapping \( \tilde{\mathcal{G}} : L^X \to L \), defined by \( \tilde{\mathcal{G}}(A) = \bigvee \{ r \in L_0; A \in \mathcal{G}_r \} \) is a g-filter on \( L^X \). If further \( \{ \mathcal{G}_r \}_{r \in L_0} \) satisfies Condition (3) of Proposition 3.9, then, for all \( r \in L_0 \), \( \mathcal{G}_r = \tilde{\mathcal{G}}_r = \{ U \in L^X; \tilde{\mathcal{G}}(U) \geq r \} \).

Proof. (1) (i) Since \( \forall r \in L_0 \), \( \mathcal{G}_r \) is a fuzzy filter on \( L^X \), it follows that \( \tilde{\mathcal{G}}(0) = 0 \) and \( \forall r \in L_0 \), \( \mathcal{G}_r \neq \phi \). So, \( \tilde{\mathcal{G}}(1) = 1 \).

(ii) \( A_1 \in \mathcal{G}_{r_1}, A_2 \in \mathcal{G}_{r_2} \Rightarrow A_1, A_2 \in \mathcal{G}_{r_1 \wedge r_2} \) (by (2)) \( \Rightarrow A_1 \cap A_2 \in \mathcal{G}_{r_1 \wedge r_2} \Rightarrow \tilde{\mathcal{G}}(A_1 \cap A_2) \geq r_1 \wedge r_2 \). As \( L \) is completely distributive so

\[
\tilde{\mathcal{G}}(A_1 \cap A_2) \geq \tilde{\mathcal{G}}(A_1) \wedge \tilde{\mathcal{G}}(A_2).
\]

(iii) Let \( A \subseteq B \). Then for \( r \in L_0 \), \( A \in \mathcal{G}_r \Rightarrow B \in \mathcal{G}_r \). So, \( \tilde{\mathcal{G}}(B) \geq \tilde{\mathcal{G}}(A) \).

(2) Now we shall show that \( \forall r \in L_0 \), \( \mathcal{G}_r = \tilde{\mathcal{G}}_r \). In fact \( A \in \mathcal{G}_r \Rightarrow \bigvee \{ k; A \in \mathcal{G}_k \} \geq r \Rightarrow \tilde{\mathcal{G}}(A) \geq r \Rightarrow A \in \tilde{\mathcal{G}}_r \). So, for all \( r \in L_0 \), \( \mathcal{G}_r \subseteq \tilde{\mathcal{G}}_r \). Again \( B \in \tilde{\mathcal{G}}_r \Rightarrow \tilde{\mathcal{G}}(B) \geq r \Rightarrow \bigvee \{ k \in L_0; B \in \mathcal{G}_k \} \geq r \). Let \( S = \{ k \in L_0; B \in \mathcal{G}_k \} \) then for all \( k \in S \), \( B \in \mathcal{G}_k \). So, \( B \in \bigcap_{k \in S} \mathcal{G}_k = \mathcal{G}_{\bigvee_{k \in S} k} = \mathcal{G}_{k'} \), where \( k' \geq r \subseteq \mathcal{G}_r \). So, \( B \in \mathcal{G}_r \Rightarrow B \in \tilde{\mathcal{G}}_r \subseteq \mathcal{G}_r \). \( \square \)

Proposition 3.11. Let \( \mathcal{G} \) be a g-filter on an \( L \)-fuzzy topological space \( (X, \tau) \) and let \( e \in \text{Pt}(L^X) \) then

\[
\forall r \in L_0, \mathcal{G} \rightarrow e \Rightarrow \mathcal{G}_r \rightarrow^{l} e \text{ for some } l \geq r'.
\]

Proof. \( \mathcal{G} \rightarrow e \Rightarrow \mathcal{G}(U) \geq Q(e, U) \forall U \in L^X \Rightarrow \mathcal{G}_r \supseteq \tilde{\mathcal{G}}_r(e) \Rightarrow \]

\[
l' = \bigwedge \{ s \in L_0; \tilde{\mathcal{G}}_s(e) \subseteq \mathcal{G}_r \} \leq r.
\]

Therefore \( \mathcal{G}_r \rightarrow^{l} e \) for some \( l \geq r' \). \( \square \)

Proposition 3.12. Let \( \mathcal{G} \) be a g-filter on an \( L \)-fuzzy topological space \( (X, \tau) \) and \( e \in \text{Pt}(L^X) \). If for every \( r \in L_0 \), \( \mathcal{G}_r \rightarrow_k e \) for some \( k \geq r' \) then \( \mathcal{G} \rightarrow e \).

Proof. Let the given condition be satisfied. To show

\[
\mathcal{G}(U) \geq Q(e, U), \forall U \in L^X,
\]

suppose, for some \( U \in L^X, \mathcal{G}(U) \not\supseteq Q(e, U) \), i.e., \( \mathcal{G}(U) \not\supseteq \bigvee \{ \tau(V); e \in V \subseteq U \} \)

\[
\Rightarrow \not\exists V \in L^X \text{ such that } e \in V \subseteq U \text{ and } \tau(V) \leq \mathcal{G}(U)
\]
\[ \Rightarrow \exists \alpha, \beta \in L_0 \text{ such that } G(U) \not\leq \alpha < \beta < \tau(V) \quad \text{(since } L \text{ is order dense)} \]
\[ \Rightarrow U \not\in G_\alpha \text{ but } \tau(V) > \beta \text{ means } U \in \tilde{Q}_\beta(e). \]
Therefore
\[ \tilde{Q}_\beta(e) \not\in G_\alpha \quad (\ast) \]
Now according to the given condition \( G_\alpha \rightarrow_k e \) for some \( k \geq \alpha' \) where
\[ k' = \bigvee \{ s \in L_0; \tilde{Q}_s(e) \not\subset G_\alpha \}. \]
\[ \Rightarrow k' \geq \beta; \text{ by } (\ast) \Rightarrow k' \geq \beta > \alpha \Rightarrow k < \alpha', \text{ a contradiction.} \]

**Proposition 3.13.** Let \( G \) be a g-filter on an L-fuzzy topological space \((X, \tau)\) and let \( e \in \Pt(L^X) \) then \( \forall r \in L_0, \ G \bowtie e \Rightarrow G_r \bowtie_k e \) for some \( k \geq r \).

**Proof.** Let \( G \bowtie e \) and suppose \( \exists r \in L_0 \) such that for no \( k \geq r \), \( G_r \bowtie_k e \). Then
\[ G_r \bowtie_k e \Rightarrow k \not\leq r \Rightarrow k' \not\leq r' \Rightarrow \]
\[ \bigvee \{ s \in L_0; \exists U \in \tilde{Q}_s(e), \exists A \in G_r; A \cap U = \tilde{0} \} \not\leq r' \]
\[ \Rightarrow \exists s \in L_0 \text{ such that } s \not\leq r' \text{ and } \exists U \in \tilde{Q}_s(e), \exists A \in G_r \text{ such that } A \cap U = \tilde{0}. \]
Now \( U \in \tilde{Q}_s(e) \) and \( A \in G_r \Rightarrow Q(e, U) \geq s \) and \( G(A) \geq r \Rightarrow Q'(e, U) \leq s' \) and \( G(A) \geq r \). Therefore, \( s \not\leq r' \Rightarrow s' \not\leq r \Rightarrow Q'(e, U) \not\leq G(A) \) but \( A \cap U = \tilde{0} \), a contradiction to the fact \( G \bowtie e \). \]

**Proposition 3.14.** Let \( G \) be a g-filter on an L-fuzzy topological space \((X, \tau)\) and \( e \in \Pt(L^X) \). If for every \( r \in L_0, \ G_r \bowtie_k e \) for some \( k \geq r \) then \( G \bowtie e \).

**Proof.** Let the given condition be satisfied. To show \( G \bowtie e \), suppose \( G \not\bowtie e \). Then
\[ \exists A, U \in L^X \text{ such that } G(A) \not\leq Q'(e, U) \text{ but } A \cap U = \tilde{0}. \]
Therefore
\[ G(A) \not\leq Q'(e, U). \]
\[ \Rightarrow \exists \alpha \in L_0 \text{ such that } G(A) > \alpha \not\leq Q'(e, U) \text{ (from the order dense property of } L) \]
\[ \Rightarrow G(A) > \alpha \text{ and } Q(e, U) \not\leq \alpha'. \]
Now
\[ G(A) > \alpha \Rightarrow A \in G_\alpha \text{ and } Q(e, U) \not\leq \alpha' \]
\[ \Rightarrow \bigvee \{ r \in L_0; U \in \tilde{Q}_r(e) \} \not\leq \alpha' \text{ (from the definition of } Q(e, U) \)} \]
\[ \Rightarrow \exists \beta \in L_0 \text{ such that } \beta \not\leq \alpha' \text{ and } U \in \tilde{Q}_\beta(e). \]
Therefore,

\[ A \in \mathcal{G}_\alpha, \ U \in \hat{Q}_\beta(e) \text{ but } A \cap U = \tilde{0} \text{ and } \beta \not\leq \alpha' \Rightarrow \bigvee \{ r \in L_0; \exists U \in \hat{Q}_r(e), \exists A \in \mathcal{G}_\alpha; \ A \cap U = \tilde{0} \} \geq \beta \not\leq \alpha'. \]

Let \( k' = \bigvee \{ r \in L_0; \exists U \in \hat{Q}_r(e), \exists A \in \mathcal{G}_\alpha; \ A \cap U = \tilde{0} \} \) then \( \mathcal{G}_\alpha \propto_k e \) where \( k' \geq \beta \not\leq \alpha' \Rightarrow \mathcal{G}_\alpha \propto_k e \) where \( k' \not\leq \alpha' \) i.e., \( k \not\leq \alpha \) which is contradictory to the given condition.

\[ \square \]

**Lemma 3.15.** Let \((X, \tau)\) and \((Y, \delta)\) be any two \(L\)-fuzzy topological spaces and \( f : (X, \tau) \to (Y, \delta) \) be any mapping then \( f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \).

**Proof.** For all \( x \in X \),

\[ f^{-1}(B_1 \cap B_2)(x) = (B_1 \cap B_2)f(x) \]
\[ = B_1(f(x)) \land B_2(f(x)) \]
\[ = [f^{-1}(B_1)(x)] \land [f^{-1}(B_2)(x)] \]
\[ = [f^{-1}(B_1) \cap f^{-1}(B_2)](x). \]

Hence the proof. \[ \square \]

**Lemma 3.16.** Let \((X, \tau)\) and \((Y, \delta)\) be any two \(L\)-fuzzy topological spaces and let \( Q, \hat{Q} \) be the gradation of \( q \)-neighborhoodness in \((X, \tau)\) and \((Y, \delta)\) respectively. A mapping \( f : (X, \tau) \to (Y, \delta) \) is a \( gp \)-map if and only if

\[ \forall e \in M(L^X) \text{ and } \forall V \in L^Y, \ Q(e, f^{-1}(V)) \geq \hat{Q}(f(e), V). \]

**Proof.** We have \( \hat{Q}(f(e), V) = \bigvee \{ \delta(W); \ f(e) \downarrow W \subseteq V \} \). Now

\[ f(e) \downarrow W \subseteq V \Rightarrow e \downarrow f^{-1}(W) \subseteq f^{-1}(V) \text{ and } \tau(f^{-1}(W)) \geq \delta(W), \]

as \( f \) is a \( gp \)-map. So, \( \bigvee \{ \tau(U); \ e \downarrow U \subseteq f^{-1}(V) \} \geq \bigvee \{ \delta(W); \ f(e) \downarrow W \subseteq V \} \). So,

\[ Q(e, f^{-1}(V)) \geq \hat{Q}(f(e), V), \forall e \in M(L^X) \text{ and } \forall V \in L^Y. \]

Conversely, let \( Q(e, f^{-1}(U)) \geq \hat{Q}(f(e), U), \forall e \in M(L^X) \text{ and } \forall U \in L^Y \) and suppose \( f \) be not a \( gp \)-map. Then \( \exists \) at least one \( U \in L^Y \) such that \( \tau(f^{-1}(U)) \not\leq \delta(U) \). Therefore, by Propositions 1.5, 1.6 and 1.7, we have

\[ \bigwedge \{ Q(p_x, f^{-1}(U)); \ p_x \in M(L^X) \text{ and } p_x \downarrow f^{-1}(U) \} \]
\[ \not\leq \bigwedge \{ \hat{Q}(r_y, U); \ r_y \in M(L^Y) \text{ and } r_y \downarrow U \} \]
\[ \Rightarrow \exists p_x \in M(L^X) \text{ such that } p_x \not\in Q(f^{-1}(U)) \text{ and } Q(p_x, f^{-1}(U)) \]
\[ \not\leq \bigwedge \{ \hat{Q}(r_y, U); r_y \in M(L^Y) \text{ and } r_y \not\in Q(U) \} \]
\[ \Rightarrow Q(p_x, f^{-1}(U)) \not\leq \hat{Q}(r_y, U), \forall r_y \in M(L^Y) \text{ with } r_y \not\in Q(U). \]

This implies
\[ Q(p_x, f^{-1}(U)) \not\leq \hat{Q}(f(p_x), U) \]
(since \( p_x \in M(L^X) \) and \( p_x \not\in Q(U) \Rightarrow f(p_x) \in M(L^Y) \) and \( f(p_x) \not\in Q(U) \), which is a contradiction. Hence the proof. \( \square \)

**Definition 3.17.** Let \((X, \tau)\) and \((Y, \delta)\) be any two \(L\)-fuzzy topological spaces and \(f : (X, \tau) \rightarrow (Y, \delta)\) be any mapping and \(\mathcal{G}\) be any \(g\)-filter on \(X\), we define
\[ f[\mathcal{G}](B) = \mathcal{G}(f^{-1}(B)), \forall B \in L^Y. \]

**Proposition 3.18.** Let \((X, \tau)\) and \((Y, \delta)\) be any two \(L\)-fuzzy topological spaces and \(f : (X, \tau) \rightarrow (Y, \delta)\) be any mapping then for any \(g\)-filter \(\mathcal{G}\) on \((X, \tau)\), \(f[\mathcal{G}]\) is a \(g\)-filter on \((Y, \delta)\).

**Proof.** (1) As we know \(f^{-1}(\tilde{0}_Y) = \tilde{0}_X\) and \(f^{-1}(\tilde{1}_Y) = \tilde{1}_X\) so,
\[ f[\mathcal{G}](\tilde{0}_Y) = \mathcal{G}(f^{-1}(\tilde{0}_Y)) = \mathcal{G}(\tilde{0}_X) = 0 \text{ and } f[\mathcal{G}](\tilde{1}_Y) = \mathcal{G}(f^{-1}(\tilde{1}_Y)) = \mathcal{G}(\tilde{1}_X) = 1. \]

(2) \( f[\mathcal{G}](B_1 \cap B_2) = \mathcal{G}(f^{-1}(B_1 \cap B_2)) \geq \mathcal{G}(f^{-1}(B_1) \cap f^{-1}(B_2)) \)
by Lemma 3.15 and (GF3). Therefore, by (GF2),
\[ f[\mathcal{G}](B_1 \cap B_2) \geq \mathcal{G}(f^{-1}(B_1)) \land \mathcal{G}(f^{-1}(B_2)) = f[\mathcal{G}](B_1) \land f[\mathcal{G}](B_2). \]

(3) \( B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2), \forall B_1, B_2 \in L^Y. \) Then, by (GF3),
\[ f[\mathcal{G}](B_2) = \mathcal{G}(f^{-1}(B_2)) \geq \mathcal{G}(f^{-1}(B_1)) = f[\mathcal{G}](B_1). \]

Hence \(f[\mathcal{G}]\) is a \(g\)-filter on \((Y, \delta)\). \( \square \)

**Proposition 3.19.** Let \((X, \tau)\) and \((Y, \delta)\) be any two \(L\)-fuzzy topological spaces and \(f : (X, \tau) \rightarrow (Y, \delta)\) be a \(gp\)-map then, for any \(g\)-filter \(\mathcal{G}\) and for any \(e \in \text{Pt}(L^X)\),
\[ \mathcal{G} \rightarrow e \Rightarrow f[\mathcal{G}] \rightarrow f(e). \]

**Proof.** Let \(Q\) and \(\hat{Q}\) be the gradations of \(Q\)-neighborhoodness in \((X, \tau)\) and \((Y, \delta)\) respectively and let \(B \in L^Y\), then
\[ \mathcal{G}(f^{-1}(B)) \geq Q(e, f^{-1}(B)) \quad \text{[as } \mathcal{G} \rightarrow e]\]
\[ \geq \hat{Q}(f(e), B), \]
by Lemma 3.16. This implies \(f[\mathcal{G}](B) \geq \hat{Q}(f(e), B). \) \( \square \)
Proposition 3.20. Let $f : (X, \tau) \to (Y, \delta)$ be a mapping where $(X, \tau)$ and $(Y, \delta)$ be any two $L$-fuzzy topological spaces. If, for any g-filter $\mathcal{G}$ and for any $e \in M(L^X)$,

$$\mathcal{G} \to e \Rightarrow f[\mathcal{G}] \to f(e)$$

then $f$ is a gp-map.

Proof. Let $Q$ and $\hat{Q}$ be the gradations of $\hat{q}$-neighborhoodness in $(X, \tau)$ and $(Y, \delta)$ respectively. As $e \in M(L^X)$ so the mapping $Q_e : L^X \to L$ given by $Q_e(U) = Q(e, U)$ is a $g$-filter on $L^X$ and $Q_e \to e$. So, according to the given condition $f[Q_e] \to f(e)$. So, $\forall V \in L^Y$, $f[Q_e](V) \geq \hat{Q}(f(e), V) \Rightarrow Q_e(f^{-1}(V)) \geq \hat{Q}(f(e), V) \Rightarrow Q(e, f^{-1}(V)) \geq \hat{Q}(f(e), V)$.

Hence, by Lemma 3.16, $f$ is a gp-map. \qed

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