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# 퍼지 랜덤 집합에 대한 중심극한정리

# Central limit theorems for fuzzy random sets

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## 요 약

이 논문에서는 서로 독립이고 동일한 분포를 갖는 집합치 랜덤 변수의 합에 대한 중심극한정리의 일반화로서, 수준연속인 퍼지 집합치 랜덤 변수의 합에 대한 중심극한정리를 연구하였다.

## Abstract

The present paper establishes the improved version of central limit theorem for sums of level-continuous fuzzy set-valued random variables as a generalization of central limit theorem for sums of independent and identically distributed set-valued random variables.

**Key words** : level-continuous fuzzy sets, set-valued random variables, random sets, fuzzy random variables, fuzzy random sets, central limit theorems

## 1. Introduction

The concept of a fuzzy random set as a natural generalization of a set-valued random variable was introduced by Puri and Ralescu [15] in order to represent the relationships between the outcomes of random experiment and inexact data due to the human subjectivity. There they [15] used a term "fuzzy random variable" instead of "fuzzy random set". Here we distinguish between a fuzzy random variable and a fuzzy random set; a fuzzy random variable is a random element taking valued in the space of fuzzy numbers, where as a fuzzy random set is a random element taking valued in the space of more general fuzzy sets without fuzzy convexity.

Statistical analysis for fuzzy probability models led to the need for central limit theorems for fuzzy random sets. As the first result related to this problem, Klement et al. [12] provided a good insight about the central limit theorem for fuzzy random variables assuming Lipschitz-condition. Wu [18] studied a completely different point of view from that by introducing the concept and strong convergence in fuzzy distribution for fuzzy random variables. Kräschmer [13] established central limit theorem for fuzzy random variables by using different metric from that of Klement et al. [12].

In this paper, we formulate the improved version of the above works for fuzzy random sets. It is expected that our results have considerable potential usefulness to statistical analysis for imprecise data.

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## 2. Preliminaries

Let $\mathscr{K}(R^p)$ be the family of all non-empty compact subsets of $R^p$. Then $\mathscr{K}(R^p)$ is metrizable by the Hausdorff metric $h$ defined by

$$h(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\}.$$

A norm of $A\in \mathscr{K}(R^p)$ is defined by

$$\|A\| = h(A,\{0\}) = \sup_{a\in A}|a|.$$

It is well known that the metric space $(\mathscr{K}(R^p),h)$ is complete and separable (See Debreu [3]).

The addition and scalar multiplication in $\mathscr{K}(R^p)$ are defined as usual;

$$A \oplus B = \{a+b \mid a \in A, b \in B\}, \ \lambda A = \{\lambda a \mid a \in A\}.$$

We denote by $K(R^p)$ the family of convex $A \in \mathscr{K}(R^p)$. A support function $A \in K(R^p)$ is defined by

$$s_A : S^{p-1} \to R, \ s_A(x) = \sup_{y \in A} \langle x, y \rangle,$$

where $S^{p-1}$ denotes the unit sphere of $R^p$ and $\langle \ ; \ \rangle$ is the usual scalar product.

Then it is well-known that $s_A$ is continuous, i.e., $s_A \in C(S^{p-1})$ and that

$$s_{A \oplus B} = s_A + s_B, \ s_{\lambda A} = \lambda s_A \text{ for } \lambda \geq 0$$

and

$$h(A,B) = \| s_A - s_B \| = sup_{x \in S^{p-1}} |s_A(x) - s_B(x)|.$$

Let $\mathcal{F}(R^p)$ denote the space the family of all fuzzy sets $\tilde{u} : R^p \to [0,1]$ wih the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;
(2) $\tilde{u}$ is upper-semicontinuous;
(3) $supp\ \tilde{u} = cl\ \{x \in R^p : \tilde{u}(x) > 0\}$ is compact, where $cl$ denotes the closure.

For a fuzzy set $\tilde{u}$ in $R^p$, we define the $\alpha$- level set of $\tilde{u}$ by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ supp\ \tilde{u}, & \alpha = 0. \end{cases}$$

Then it follows that $\tilde{u} \in \mathcal{F}(R^p)$ if and only if

$$L_\alpha \tilde{u} \in \mathscr{K}(R^p) \text{ for each } \alpha \in [0,1].$$

Also, if we denote

$$L_{\alpha^+} \tilde{u} = cl\ \{x \in R^p : \tilde{u}(x) > \alpha\},$$

then by Lemma 2.2 of Joo and Kim [9],

$$\lim_{\beta \downarrow \alpha} h(L_\beta \tilde{u}, L_{\alpha^+} \tilde{u}) = 0.$$

A fuzzy set $\tilde{u}$ is called level-continuous if $L_\alpha \tilde{u} = L_{\alpha^+} \tilde{u}$ for all $\alpha \in [0,1]$ and the family of all level-continuous $\tilde{u} \in \mathcal{F}(R^p)$ will be denoted by $\mathcal{F}_C(R^p)$.

The linear structure on $\mathcal{F}(R^p)$ are defined as usual;

$$(\tilde{u} \oplus \tilde{v})(x) = \sup_{y+z=x} \min(\tilde{u}(y), \tilde{v}(z)),$$

$$(\lambda \tilde{u})(x) = \begin{cases} \tilde{u}(x/\lambda), & \text{if } \lambda \neq 0, \\ \tilde{0}(x), & \text{if } \lambda = 0, \end{cases}$$

for $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$.

Then it is well-known that for each $\alpha \in [0,1]$,

$$L_\alpha(\tilde{u} \oplus \tilde{v}) = L_\alpha \tilde{u} \oplus L_\alpha \tilde{v}$$

and $L_\alpha(\lambda \tilde{u}) = \lambda L_\alpha \tilde{u}$.

Recall that a fuzzy subset $\tilde{u}$ of $R^p$ is said to be convex if for $x, y \in R^p$ and $\lambda \in [0,1]$,

$$\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y)).$$

The convex hull of $\tilde{u}$ is defined by

$$co(\tilde{u}) = \inf\{\tilde{v} \mid \tilde{v} \text{ is convex and } \tilde{v} \geq \tilde{u}\}.$$

Then for each $\alpha \in [0,1]$,

$$L_\alpha co(\tilde{u}) = co(L_\alpha \tilde{u}).$$

(For details, see Lowen [14]).

We denote by $F(R^p)$ (resp. $F_C(R^p)$) the family of all fuzzy convex $\tilde{u} \in \mathcal{F}(R^p)$ (resp. $\tilde{u} \in \mathcal{F}_C(R^p)$) and an element of $F(R^p)$ is called a fuzzy number. Then it follows that $\tilde{u} \in F(R^p)$ if and only if $L_\alpha \tilde{u} \in K(R^p)$ for each $\alpha \in [0,1]$.

Now, we can define the uniform metric $d_\infty$ on $\mathcal{F}(R^p)$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}).$$

Also, the norm of $u \in \mathcal{F}(R^p)$ is defined as

$$\| \tilde{u} \| = d_\infty(\tilde{u}, \tilde{0}) = sup_{x \in L_0 \tilde{u}} |x|.$$

Then it is well-known that $(\mathcal{F}(R^p)\ d_\infty)$ is complete, but is not separable. (See Diamond and Kloeden [4]). However, $(\mathcal{F}_C(R^p)\ d_\infty)$ is complete and separable.

A support function of a fuzzy number $\tilde{u} \in F(R^p)$ is defined by

$$s_{\tilde{u}} : [0,1] \times S^{p-1} \to R, \ s_{\tilde{u}}(\alpha, x) = s_{L_\alpha \tilde{u}}(x) = sup_{y \in L_\alpha \tilde{u}} \langle x, y \rangle.$$

Then it is well-known that

$$s_{\tilde{u} \oplus \tilde{v}} = s_{\tilde{u}} + s_{\tilde{v}}, \quad s_{\lambda \tilde{u}} = \lambda s_{\tilde{u}} \text{ for } \lambda \geq 0,$$

and

$$d_\infty(\tilde{u}, \tilde{v}) = \| s_{\tilde{u}} - s_{\tilde{v}} \| = sup_{(\alpha,x) \in [0,1] \times S^{p-1}} |s_{\tilde{u}}(\alpha, x) - s_{\tilde{v}}(\alpha, x)|.$$

Also, $s_{\tilde{u}} \in C([0,1] \times S^{p-1})$ if and only if $\tilde{u} \in F_C(R^p)$ (For details, see Roman-Flores and Rojas-Medar [16]).

## 3. Main Results

Throughout this paper. let $(\Omega, \mathcal{B}, P)$ be a probability space. A set-valued function $X : \Omega \to (\mathscr{K}(R^p), h)$ is called a random set if it is measurable. A random set $X$ is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set $X$ is defined by

$$E(X) = \{E(\xi) \mid \xi \in L(\Omega, R^p) \text{ and } \xi(\omega) \in X(\omega) \text{ a.s.}\},$$

where $L(\Omega, R^p)$ denotes the class of all $R^p$-valued random variables $\xi$ such that $E|\xi| < \infty$.

The central limit theorem for random sets was first given by Cressie [2] in a particular case. The general central limit theorem for random sets appeared in Weil [17], and independently in Gine et al. [5].

**Theorem 3.1.** Let $\{X_n\}$ be independent and identically distributed random sets. If $E\|X_1\|^2 < \infty$, then

$$\sqrt{n}\, h(\frac{1}{n} \oplus_{i=1}^{n} X_i, co(EX_1)) \Rightarrow \|Z\|,$$

where $Z$ is a centered Gaussian random element in $C(S^{p-1})$ and $\Rightarrow$ denotes the convergence in distribution.

A fuzzy valued function $\tilde{X}: \Omega \to \mathcal{F}(R^p)$ is called a fuzzy random set if for each $\alpha \in [0,1]$, $L_\alpha \tilde{X}$ is a random set. This definition was introduced by Puri and Ralescu [15] as a natural generalization of a random set. If $\tilde{X}$ is $F(R^p)$-valued, then it is called a fuzzy random variable. If $\tilde{X}$ is a random element of the metric space $(\mathcal{F}(R^p)\ d_\infty)$, then it is a fuzzy random set. But the converse is not true (See Kim [11]). However, if $\tilde{X}$ is a $\mathcal{F}_C(R^p)$-valued, then $\tilde{X}$ is a fuzzy random set if and only if it is a random element of the metric space $(\mathcal{F}_C(R^p)\ d_\infty)$ (See Joo et al. [10]).

A fuzzy random set $\tilde{X}$ is called integrably bounded if $E|\tilde{X}| < \infty$. The expectation of integrably bounded fuzzy random set $\tilde{X}$ is a fuzzy set defined by

$$E(\tilde{X})(x) = sup\{\alpha \in [0,1] : x \in E(L_\alpha \tilde{X})\}.$$

It is well-known that if $\tilde{X}, \tilde{Y}$ are integrably bounded fuzzy random sets, then

(1) $E(\tilde{X}) \in \mathcal{F}(R^p)$, and if $\tilde{X}$ is $\mathcal{F}_C(R^p)$ -valued, then $E(\tilde{X}) \in \mathcal{F}_C(R^p)$.
(2) $L_\alpha E(\tilde{X}) = E(L_\alpha \tilde{X})$ for all $\alpha \in [0,1]$.
(3) $E(\tilde{X} \oplus \tilde{Y}) = E(\tilde{X}) \oplus E(\tilde{Y})$.
(4) $E(\lambda \tilde{X}) = \lambda E(\tilde{X})$.
(5) If $\tilde{X}$ is $F(R^p)$ (resp. $F_C(R^p)$)-valued, then $E(\tilde{X}) \in F(R^p)$ (resp. $F_C(R^p)$).

The purpose of this paper is to generalize Theorem 3.1 to the case of fuzzy random sets. Klement et al. [12] provided a generalization of Theorem 3.1 for fuzzy random variables taking values in the space $F_L(R^p)$ of fuzzy sets $u \in F(R^p)$ such that $\alpha \mapsto L_\alpha u$ is Lipschitz, i.e., there exists a constant $M > 0$ such that

$$h(L_\alpha u, L_\beta u) \le M|\alpha - \beta| \quad \text{for all } \alpha, \beta \in [0,1].$$

Kräschmer [13] formulated central limit theorems for fuzzy random variables in a more general setting by using different metric

$$\rho_r(u, v) = \{\int_0^1 \int_{S^{p-1}} |s_{L_\alpha u} - s_{L_\alpha v}|^r d\mu\, d\alpha\}^{\frac{1}{r}}, \qquad r \ge 1,$$

where $\mu$ denotes the unit Lebesgue measure on $S^{p-1}$.

Here we give an improvement of the result of Klement et al. [12] by considering $\mathcal{F}_C(R^p)$-valued fuzzy random sets.

**Theorem 3.2.** Let $\{\tilde{X}_n\}$ be independent and identically distributed $F_C(R^p)$-valued fuzzy random variables with $E\|\tilde{X}_1\|^2 < \infty$. If there exists a non-negative random variable $\xi$ with $E(\xi^2) < \infty$ such that for all $w \in \Omega$,

$$h(L_\alpha \tilde{X}_1(w), L_\beta \tilde{X}_1(w)) \le \xi(w)|\alpha - \beta|,$$

then there exists a centered Gaussian random element $Z$ in $C([0,1] \times S^{p-1})$ such that

$$\frac{\sum_{i=1}^{n} s_{\tilde{X}_i} - n s_{E(\tilde{X}_1)}}{\sqrt{n}} \Rightarrow Z,$$

where $\Rightarrow$ denotes the convergence in distribution.

**Proof:** By hypothesis, $\{s_{\tilde{X}_n}\}$ is independent and identically distributed $C([0,1] \times S^{p-1})$-valued random variables and $E\|s_{\tilde{X}_1}\|^2 = E\|\tilde{X}_1\|^2 < \infty$. In order to apply the CLT in $C([0,1] \times S^{p-1})$ (Corollary 7.17 of Arujo and Gine [1] or Theorem 1 of Jain and Marcus [8]), we must check that two conditions is satisfied. First, we have to check a Lipschitz condition for $\{s_{\tilde{X}_1} - E s_{\tilde{X}_1}\}$. In deed,

$$\begin{aligned}
& |(s_{\tilde{X}_1}(\alpha, x) - E s_{\tilde{X}_1}(\alpha, x)) - (s_{\tilde{X}_1}(\beta, y) - E s_{\tilde{X}_1}(\beta, y))| \\
\le\ & |(s_{L_\alpha \tilde{X}_1}(x) - s_{L_\beta \tilde{X}_1}(y)| + |s_{L_\alpha E\tilde{X}_1}(x) - s_{L_\beta E\tilde{X}_1}(y)| \\
\le\ & |(s_{L_\alpha \tilde{X}_1}(x) - s_{L_\beta \tilde{X}_1}(x)| + |(s_{L_\beta \tilde{X}_1}(x) - s_{L_\beta \tilde{X}_1}(y)| \\
& + |s_{L_\alpha E\tilde{X}_1}(x) - s_{L_\beta E\tilde{X}_1}(x)| + |s_{L_\beta E\tilde{X}_1}(x) - s_{L_\beta E\tilde{X}_1}(y)| \\
\le\ & \xi|\alpha - \beta| + \|s_{L_\beta \tilde{X}_1}\|\, \|x - y\| + E(\xi)|\alpha - \beta| \\
& + \|s_{L_\beta E(\tilde{X}_1)}\|\, \|x - y\| \\
\le\ & \max(\xi + E(\xi),\ \|s_{L_\beta \tilde{X}_1}\| + \|s_{L_\beta E(\tilde{X}_1)}\|) \\
& (|\alpha - \beta| + \|x - y\|) \\
\le\ & \max(\xi + E(\xi),\ \|\tilde{X}_1\| + \|E(\tilde{X}_1)\|) \\
& (|\alpha - \beta| + \|x - y\|).
\end{aligned}$$

Here, $d((\alpha, x), (\beta, y)) = |\alpha - \beta| + \|x - y\|$ is a continuous metric with respect to the usual metric on $[0,1] \times S^{p-1}$ and

$$M = \max(\xi + E(\xi),\ \|\tilde{X}_1\| + \|E(\tilde{X}_1)\|)$$

is a non-negative random variable with $E(M^2) < \infty$ and the above inequality implies that
$| (s_{\tilde{X}_1}(\alpha, x) - Es_{\tilde{X}_1}(\alpha, x)) - (s_{\tilde{X}_1}(\beta, y) - Es_{\tilde{X}_1}(\beta, y)) |$
$\leq M\, d((\alpha, x), (\beta, y))$
for all $(\alpha, x), (\beta, y) \in [0,1] \times S^{p-1}$.

Thus, a Lipschitz condition for $\{s_{\tilde{X}_1} - Es_{\tilde{X}_1}\}$ is satisfied.

Secondly, we have to check metric entropy condition

$$\int_0^1 H^{1/2}(\epsilon)\, d\epsilon < \infty,$$

where $H$ is the metric entropy of $[0,1] \times S^{p-1}$ defined by $H(\epsilon) = \log N(\epsilon)$ with $N(\epsilon)$ the minimal number of balls of radius $\epsilon > 0$ which cover $[0,1] \times S^{p-1}$. But this is identical with the work of Klement et al. [12]. In deed, $N(\epsilon) \leq C_p \epsilon^{-p}$

with $C_p$ a constant depending only on dimension $p$. Thus, the metric entropy condition is satisfied. Q.E.D.

**Corollary 3.3.** Let $\{\tilde{X}_n\}$ be independent and identically distributed $\mathcal{F}_C(R^p)$-valued fuzzy random sets with $E|\tilde{X}_1|^2 < \infty$. If there exists a non-negative random variable $\xi$ with $E(\xi^2) < \infty$ such that for all $w \in \Omega$,

$$h(L_\alpha \tilde{X}_1(w), L_\beta \tilde{X}_1(w)) \leq \xi(w) |\alpha - \beta|,$$

then there exists a centered Gaussian random element $Z$ in $C([0,1] \times S^{p-1})$ such that

$$(1)\quad \frac{\sum_{i=1}^n s_{co(\tilde{X}_i)} - n s_{co(E\tilde{X}_1)}}{\sqrt{n}} \Rightarrow Z,$$

$$(2)\quad \sqrt{n}\, d_\infty\left(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, co(E\tilde{X}_1)\right) \Rightarrow |Z\|$$

**Proof.** (1): By hypothesis, $\{co(\tilde{X}_n)\}$ is independent and identically distributed $F_C(R^p)$-valued fuzzy random sets and

$$E\|co(\tilde{X}_1)\|^2 = E\|\tilde{X}_1\|^2 < \infty.$$

Also, by convexity inequality, we have

$$h(L_\alpha co\tilde{X}_1(w), L_\beta co\tilde{X}_1(w)) \leq h(L_\alpha \tilde{X}_1(w), L_\beta \tilde{X}_1(w)) \leq \xi(w)|\alpha - \beta|.$$

Therefore (1) follows immediately from Theorem 3.2.

(2): Since the norm $|\cdot|$ on $C([0,1] \times S^{p-1})$ is continuous and

$$\left\| \frac{\sum_{i=1}^n s_{co(\tilde{X}_i)} - n s_{co(E\tilde{X}_1)}}{\sqrt{n}} \right\|$$
$$= \sqrt{n} \| s_{\frac{1}{n}\oplus_{i=1}^n co(\tilde{X}_i)} - s_{co(E\tilde{X}_1)} \|$$
$$= \sqrt{n}\, d_\infty\left(\frac{1}{n} \oplus_{i=1}^n co(\tilde{X}_i), co(E\tilde{X}_1)\right),$$

we have

$$\sqrt{n}\, d_\infty\left(\frac{1}{n} \oplus_{i=1}^n co(\tilde{X}_i), co(E\tilde{X}_1)\right) \Rightarrow \|Z\|.$$

Now, by Shapley-Folkman inequality for fuzzy sets, we obtain

$$\left| \sqrt{n} d_\infty\left(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, co(E\tilde{X}_1)\right) - \sqrt{n} d_\infty\left(\frac{1}{n} \oplus_{i=1}^n co(\tilde{X}_i), co(E\tilde{X}_1)\right) \right|$$
$$\leq \sqrt{n} d_\infty\left(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^n co(\tilde{X}_i)\right)$$
$$= \frac{1}{\sqrt{n}} d_\infty\left(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n co(\tilde{X}_i)\right)$$
$$= \sqrt{\frac{p}{n}} \max_{1 \leq i \leq n} \|\tilde{X}_i\|.$$

Since $\{\|\tilde{X}_i\|\}$ is i.i.d. real valued random variables with $E\|\tilde{X}_1\|^2 < \infty$, the CLT for real-valued random variables implies that $\max_{1 \leq i \leq n} \|\tilde{X}_i\| \Rightarrow 0$.

Therefore, we have

$$\sqrt{n}\, d_\infty\left(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, co(E\tilde{X}_1)\right) \Rightarrow \|Z\|. \text{ Q.E.D.}$$

## 4. one-dimensional case

If we consider one-dimensional fuzzy random variables as a special case, then we can obtain further more interesting results. First we recall that $\tilde{u} \in F(R)$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_\alpha^l, u_\alpha^r]$ by the following theorem (see Goetschel and Voxman [6]).

**Theorem 4.1.** For $\tilde{u} \in F(R)$, we consider $u^l$ and $u^r$ as functions of $\alpha \in [0,1]$. Then

(1) $u^l$ is a bounded increasing function on $[0,1]$.
(2) $u^r$ is a bounded decreasing function on $[0,1]$.
(3) $u_1^l \leq u_1^r$.
(4) $u^l$ and $u^r$ are left continuous on $[0,1]$ and right continuous at 0.
(5) If $v^l$ and $v^r$ satisfy above (1)-(4), then there exists a unique $\tilde{v} \in F(R)$ such that

$$L_\alpha \tilde{v} = [v_\alpha^l, v_\alpha^r] \text{ for all } \alpha \in [0,1].$$

By the above Theorem, we can identify a fuzzy number $\tilde{u} \in F(R)$ with the parameterized representation

$$\{(u_\alpha^l, u_\alpha^r) \mid 0 \leq \alpha \leq 1\},$$

where $u^l$ and $u^r$ satisfy the conditions (1)-(4) of Theorem 4.1. Also, it follows that $\tilde{u} \in F_C(R)$ if and only if $u^l$ and $u^r$ are continuous on $[0,1]$.

If $\tilde{u}, \tilde{v} \in F(R)$ and

$$\tilde{u} = \{(u^l_\alpha, u^r_\alpha) \mid 0 \le \alpha \le 1\},$$
$$\tilde{v} = \{(v^l_\alpha, v^r_\alpha) \mid 0 \le \alpha \le 1\},$$

then

$$\tilde{u} \oplus \tilde{v} = \{(u^l_\alpha + v^l_\alpha, u^r_\alpha + v^r_\alpha) \mid 0 \le \alpha \le 1\},$$
$$\lambda \tilde{u} = \{(\lambda u^l_\alpha, \lambda u^r_\alpha) \mid 0 \le \alpha \le 1\},\ \lambda \ge 0.$$

And then,

$$d_\infty(\tilde{u}, \tilde{v}) = sup_{0 \le \alpha \le 1} \max(|u^l_\alpha - v^l_\alpha|, |u^r_\alpha - v^r_\alpha|)$$

and

$$\|\tilde{u}\| = \max(|u^l_0|, |u^r_0|).$$

Furthermore, if $\tilde{X} = \{(X^l_\alpha, X^r_\alpha) \mid 0 \le \alpha \le 1\}$ is a $F(R)$-valued function, then the following three statements are equivalent;

(1) $\tilde{X}$ is a fuzzy random variable.

(2) For each $\alpha \in [0,1]$, $X^l_\alpha$ and $X^r_\alpha$ are real-valued random variables.

(3) $X^l$ and $X^r$ are $C[0,1]$-valued random variables.

In this case, $E(\tilde{X}) = \{(E(X^l_\alpha), E(X^r_\alpha)) \mid 0 \le \alpha \le 1\}$.

Thus, we can obtain the following results by applying Theorem 3.2, Corollary 3.3 and the result of Hahn [7].

**Theorem 4.2.** Let

$$\tilde{X}_n = \{(X^l_{n,\alpha}, X^r_{n,\alpha}) \mid 0 \le \alpha \le 1\}$$

be independent and identically distributed $F_C(R)$-valued fuzzy random variables with $E|X^l_{1,0}|^2 < \infty$ and $E|X^r_{1,0}|^2 < \infty$. If there exists a non-negative random variable $\xi$ with $E(\xi^2) < \infty$ such that for all $w \in \Omega$,

$$\max(|X^l_{1,\alpha}(w) - X^r_{1,\beta}(w)|, |X^r_{1,\alpha}(w) - X^r_{1,\beta}(w)|) \le \xi(w)|\alpha - \beta|,$$

then there exists a centered Gaussian random element $Z = (Z_1, Z_2)$ in $C[0,1] \times C[0,1]$ such that the following hold;

(1) for each $\alpha \in [0,1]$,

$$\frac{\sum_{i=1}^n X^l_{i,\alpha} - nEX^l_{1,\alpha}}{\sigma^l_\alpha \sqrt{n}} \Rightarrow N(0,1)$$

and

$$\frac{\sum_{i=1}^n X^r_{i,\alpha} - nEX^r_{1,\alpha}}{\sigma^r_\alpha \sqrt{n}} \Rightarrow N(0,1),$$

where $\sigma^l_\alpha$ and $\sigma^r_\alpha$ are the standard deviation of $X^l_{1,\alpha}$ and $X^r_{1,\alpha}$ respectively.

(2) $$\frac{\sum_{i=1}^n X^l_i - nEX^l_1}{\sqrt{n}} \Rightarrow Z_1$$

and

$$\frac{\sum_{i=1}^n X^r_i - nEX^r_1}{\sqrt{n}} \Rightarrow Z_2.$$

(3) $$\frac{\|\sum_{i=1}^n X^l_i - nEX^l_1\|}{\sqrt{n}} \Rightarrow |Z_1|$$

and

$$\frac{\|\sum_{i=1}^n X^r_i - nEX^r_1\|}{\sqrt{n}} \Rightarrow |Z_2|.$$

(4) $$\sqrt{n}\, d_\infty\left(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, E(\tilde{X}_1)\right) = max\left(\frac{\|\sum_{i=1}^n X^l_i - nEX^l_1\|}{\sqrt{n}}, \frac{\|\sum_{i=1}^n X^r_i - nEX^r_1\|}{\sqrt{n}}\right)$$

$$\Rightarrow \|Z\| = \max(\|Z_1\|, \|Z_2\|).$$

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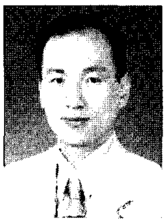


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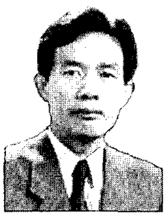


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