

# A remark to a Constrained OWA Aggregation

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## Abstract

The problem of maximizing an OWA aggregation of a group of variables that are interrelated and constrained by a collection of linear inequalities is considered by Yager[Fuzzy Sets and Systems, 81(1996) 89-101]. He obtained how this problem can be modelled as a mixed integer linear programming problem. Recently, Carlsson et al. [Fuzzy Sets and Systems, 139(2003) 543-546] obtained a simple algorithm for exact computation of optimal solutions to a constrained OWA aggregation problem with a single constraint on the sum of all decision variables. In this note, we introduce anew approach to the same problem as Carlsson et al. considered. Indeed, it is a direct consequence of a known result of the linear programming problem.

**Key words** : OWA operators, Constrained optimization, Linear programming.

## 1. Introduction

An Ordered Weighted Averaging(OWA) is a mapping  $F: R^n \rightarrow R$  that has an associated weighting vector  $\mathbf{w} = (w_1, \dots, w_n)^T$  of having the properties  $w_1 + \dots + w_n = 1$ ,  $0 \leq w_i \leq 1$ ,  $i = 1, \dots, n$  and such that

$$F(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_{(i)},$$

where  $x_{(j)}$  is the  $j$ th largest element of the bag  $\{x_1, \dots, x_n\}$ .

The constrained OWA aggregation problem [3] can be expressed as the following mathematical programming problem:

$$\max F(x_1, \dots, x_n) \text{ subjects to } \{A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}.$$

where

$$F(x_1, \dots, x_n) = \mathbf{w}^T \mathbf{x}_{(.))} = w_1 x_{(1)} + \dots + w_n x_{(n)}.$$

Recently, Carlsson et al. [1] obtained a simple algorithm for solving the following (nonlinear) constrained OWA aggregation problem

$$\begin{aligned} & \max \mathbf{w}^T \mathbf{x}_{(.))} \\ & \text{subjects to } \{x_1 + \dots + x_n, x_i \geq 0\} \end{aligned} \quad (1)$$

We revisit this problem.

## 2. Result

We first see the following known result.

접수일자 : 2005년 1월 17일  
완료일자 : 2005년 4월 15일

**The Linear programming problem** : Maximize the objective function

$$f(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

subject to the condition that  $(x_1, \dots, x_n)$  be in

$$F = \{ \mathbf{x} \in R_+^n : x_1, \dots, x_n \text{ satisfy a finite set of linear constraints } \}.$$

**Theorem [2, p241]**. If the objective function is bounded above on  $F \neq \emptyset$  then there are optimal solutions to the linear programming problem, at least one of which is an extreme point of  $F$ . Using above theorem, we directly prove the OWA aggregation problem (1).

**Proof.** Let  $H = \{h: \{1, \dots, n\} \rightarrow \{1, \dots, n\}, \text{ a bijective function } \}$ . Then we have

$$\begin{aligned} & \max \{ \mathbf{w}^T \mathbf{x}_{(.))} \mid x_1 + \dots + x_n \leq 1, x_i \geq 0 \} \\ & = \max_{h \in H} \max \{ \mathbf{w}^T \mathbf{x}_h = w_1 x_{h(1)} + \dots + w_n x_{h(n)} \mid \\ & \quad x_1 + \dots + x_n \leq 1, x_{h(1)} \geq \dots \geq x_{h(n)} \geq 0 \}. \end{aligned}$$

We note that

the set of extreme point of

$$\{x_1 + \dots + x_n \leq 1, x_{h(1)} \geq \dots \geq x_{h(n)} \geq 0\}$$

$$= \{x_{h(1)} = 1, x_{h(i)} = 0, i \neq 1\}$$

$$\cup \{x_{h(1)} = x_{h(2)} = \frac{1}{2}, x_{h(i)} = 0, i \neq 1, 2\} \cup \dots$$

$$\cup \{x_{h(1)} = \dots = x_{h(n)} = \frac{1}{n}\} \cup \{x_{h(1)} = \dots = x_{h(n)} = 0\}.$$

For example,

the set of extreme point of

$$\{x_1 + x_2 + x_3 + x_4 \leq 1, x_3 \geq x_1 \geq x_4 \geq x_2 = 0\}$$

$$= \left\{ (0, 0, 1, 0), \left(-\frac{1}{2}, 0, \frac{1}{2}, 0\right), \left(-\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right), \right.$$

$$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (0, 0, 0, 0)\}.$$

And by above theorem, we have that

$$\begin{aligned} & \max \{ \mathbf{w}^T \mathbf{x}_h \mid x_1 + \dots + x_n \leq 1, x_{h(1)} \geq \dots \geq x_{h(n)} \geq 0 \} \\ &= \max \left\{ w_1, \frac{w_1 + w_2}{2}, \dots, \frac{w_1 + \dots + w_n}{n} \right\}. \end{aligned}$$

Now, we define  $h^*: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  where  $w_{h^*(i)} = i$ th largest element of the bag  $\{w_1, \dots, w_n\}$ .

Then, clearly, we have that  $\mathbf{w}^T \mathbf{x}_{(.)} \leq \mathbf{w}^T \mathbf{x}_h$ . Therefore, we have

$$\begin{aligned} & \max \{ \mathbf{w}^T \mathbf{x}_{(.)} \mid x_1 + \dots + x_n \leq 1, x \geq 0 \} \\ &= \max \{ \mathbf{w}^T \mathbf{x}_h \mid x_1 + \dots + x_n \leq 1, x_{h^*(1)} \geq \dots \geq x_{h^*(n)} \geq 0 \} \\ &= \max \left\{ w_1, \frac{w_1 + w_2}{2}, \dots, \frac{w_1 + \dots + w_n}{n} \right\}. \end{aligned}$$

If the maximum value is  $(w_1 + \dots + w_m)/m$ , an optimal solution to problem (1) will be  $x_{h^*(1)} = \dots = x_{h^*(m)} = 1/m$  with  $F(\mathbf{x}_h^*) = (w_1 + \dots + w_m)/m$ .

**Example.** We consider the following five-dimensional constrained OWA aggregation problem with  $w_3 \geq w_1 \geq w_4 \geq w_2 \geq w_5 \geq 0$ .

$$\begin{aligned} & \max F(x_1, x_2, x_3, x_4, x_5) \text{ subject to} \\ & \{x_1 + \dots + x_5 \leq 1, x \geq 0\}. \end{aligned} \quad (2)$$

The set of all possible optimal value is considered as

$$D = \left\{ w_1, \frac{w_1 + w_2}{2}, \dots, \frac{w_1 + w_2 + w_3 + w_4 + w_5}{5} \right\}$$

and, the corresponding optimal solutions are, for example, if  $\max D = (w_1 + w_2 + w_3)/3$ , the optimal solution to problem (2) will be  $x_3 = x_1 = x_4 = 1/3$  with  $F(\mathbf{x}_h) = (w_1 + w_2 + w_3)/3$ .

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