경로적분법에 의한 원공크랙이 있는 직교이방성 탄성평판의 응력 확대계수 계산

The Calculation of Stress Intensity Factors in the Orthotropic Elastic Plate with the Cracked Circular-hole using a Contour Integral Method

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요 약

특이응력해석을 위한 일반화된 가역상반일 경계적분식이 섬유강화복합재를 모형화한 직교 이방성 크랙평판의 수치해를 위하여 발전시켰다. 이 적분방정식은 평판경계에서의 탄성변위와 트랙션의 변수로 구성된 경계적분식의 형태로 하중이 없다는 두 크랙면의 경계조건과 유한의 탄성변형에너지의 개념에서 경계적분식에 필요한 특성해를 규정하고 대응되는 보조해를 계산하였다. 대칭모우드 I형의 중앙원공크랙평판 및 복합모우드형의 반원편측크랙 일단고정평판의 응력확대계수가 임의의 섬유방향각에 따라서 계산되었다.

주요기술용어(주제어) : Contour Integral Fomulation(경로적분방정식 구성), Stress Intensity Factors(SIF 응력확대계수), Cracked Orthotropic Plate(직교이방성 크랙평판), Symmetric Mode and Mixed Mode(대칭 및 복합모우드)

1. Introduction

The crack tip singularity in rectilinearly anisotropic materials, as observed by Sih et. al.[1] can be completely characterized in the same manner as in the isotropic case by properly defining anisotropic stress intensity factors (SIF). Determination of the SIF is thus an important design aspect for assessing fracture strength of anisotropic composite structural components. Within the framework of plane, linear elastic fracture mechanics, a problem of continuing interest is the calculation of the SIF in the cracked orthotropic plates modelling the fiber-reinforced composites. Many different numerical methods developed for isotropic materials have been extended to treat some problems involving anisotropic materials including the mapping - collocation method of Bowie and Freeze[2], the displacement hybrid element of Atluri et. al.[3], and a conservation integral method introduced recently by Wang, Yau and Corten[4~6]. Among these a contour integral method proposed by Stern[7] is attractive one due to its numerical simplicity and computational efficiency. This method is based on the Betti's reciprocal work

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theorem from which the singular stress intensities at the crack tip can be evaluated in terms of an path independent integral involving the tractions and displacements on a contour boundary remote from the crack tip. Thus this method is easily incorporated into most existing finite element stress analysis programs which have no provision for treating singular stress states. The terms representing the singular intensities can be obtained explicitly by a routine evaluation of the reciprocal work of the product functions between the characteristic singular solutions and corresponding complementary elastic states defined on the neighborhood of the singular point. With identification of such singular functions there have been successful applications of the method to the various types of singularity problems in isotropic materials[8,9] as well as in dissimilar media[10]. For the orthotropic composite materials Soni and Stern[11] also developed the desired characteristic singular functions by using two complex potentials in anisotropic elasticity theory of Green and Zerna[12], and England's singularity representation[13]. Demonstration of the extension, however, was still restricted to illustrate the general orthotropic singular behavior in view of dealing orthotropic configurations where the crack axis is always parallel to one of the principal directions of material symmetry corresponding to the fiber direction.

The purpose of present work is to extend the contour integral method to treat the general orthotropic plates with cracks emanating from a circular hole. While the basic method follows closely in spirit the ideas outlined in [7], the formulation of the problem is different in the use of new characteristic singular solutions. In the next section the contour integral representation is outlined and the singular solutions required can be identified. The corresponding complementary elastic states are constructed on the basis of the traction free boundary conditions of crack edge and the concept of finite reciprocal work on the contour enclosing the crack tip. The computational procedures newly adopted for the SIF are described in Section 3. Finally the results of numerical computation for two cracked orthotroic elastic plates are presented in Section 4: symmetric mode I type of the tension plate with the cracks emanating from a circular hole and mixed mode type of cantilever plate with a single edge crack containing stress concentration.

2. Contour Integral Formulation

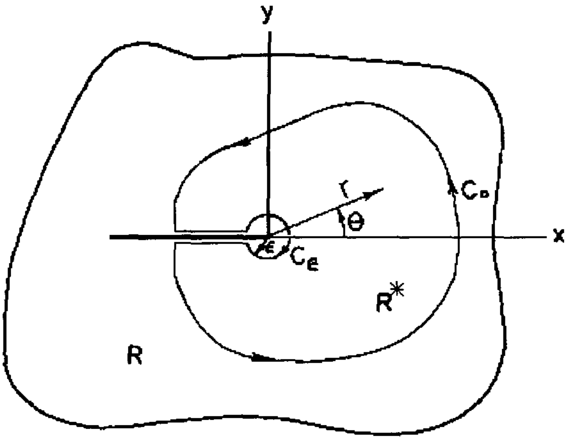
The integral representation outlined in [7] can be started from the Betti's reciprocal work theorem for plane elastic states with vanishing body forces:

$$\int_{\partial R^*} (\underset{\sim}{T} \cdot \underset{\sim}{u}^c - \underset{\sim}{T}^c \cdot \underset{\sim}{u}) \, ds = 0 \qquad (1)$$

Here $\underset{\sim}{u}$ is the displacement field and $\underset{\sim}{T}$ the traction vector on the boundary ∂R^* of a simply connected and bounded region R^*, corresponding to the solution of any particular equilibrium problem ; $\underset{\sim}{u}^c$ and $\underset{\sim}{T}^c$ are the displacement and traction, respectively, corresponding to any such problem.

In order to apply the reciprocal work theorem to a body with an edge crack, we first delete the body points within a circle of radius ε centered at the crack tip as indicated in Fig. 1.

The boundary of the remaining body is decomposed into two parts: the circular boundary of the deleted region, denoted C_ε, and the remaining boundary denoted C_o. Then Eq. (1) becomes



[Fig. 1] Deleted region for a singularity

$$I_\varepsilon = -\int_{C_\varepsilon} (\underset{\sim}{T} \cdot \underset{\sim}{u}^c - \underset{\sim}{T}^c \cdot \underset{\sim}{u})\, ds = \int_{C_0} (\underset{\sim}{T} \cdot \underset{\sim}{u}^c - \underset{\sim}{T}^c \cdot \underset{\sim}{u})\, ds \qquad (2)$$

and the objective now is to evaluate these integrals for arbitrarily small ε and suitable choice of the complementary equilibrium state.

We suppose that for the equilibrium stress state of interest, the crack faces are free of traction and on the remainder of the boundary either the displacement or the traction components in the normal and tangential directions are prescribed so that we have a well posed problem. The stresses and displacements in the neighborhood of the crack tip in rectilinearly anisotropic media are then known to have the following forms[14] :

$$u_x^s = \sqrt{\frac{2r}{\pi}} \mathrm{Re}\{\frac{1}{\gamma_2-\gamma_1}[(\gamma_2 p_1\sqrt{\cos\theta-\gamma_1\sin\theta} - \gamma_1 p_2\sqrt{\cos\theta-\gamma_2\sin\theta})K_I + (p_2\sqrt{\cos\theta-\gamma_2\sin\theta} - p_1\sqrt{\cos\theta-\gamma_1\sin\theta})K_{II}]\}$$

$$u_y^s = \sqrt{\frac{2r}{\pi}} \mathrm{Re}\{\frac{1}{\gamma_2-\gamma_1}[(\gamma_2 q_1\sqrt{\cos\theta-\gamma_1\sin\theta} - \gamma_1 q_2\sqrt{\cos\theta-\gamma_2\sin\theta})K_I + (q_2\sqrt{\cos\theta-\gamma_2\sin\theta} - q_1\sqrt{\cos\theta-\gamma_1\sin\theta})K_{II}]\}$$

$$\sigma_x^s = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}[\frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1}(\frac{\gamma_1}{\sqrt{\cos\theta-\gamma_1\sin\theta}} - \frac{\gamma_2}{\sqrt{\cos\theta-\gamma_2\sin\theta}})] + \frac{K_{II}}{\sqrt{2\pi r}} \mathrm{Re}[\frac{1}{\gamma_2-\gamma_1}(\frac{\gamma_2^2}{\sqrt{\cos\theta-\gamma_2\sin\theta}} - \frac{\gamma_1^2}{\sqrt{\cos\theta-\gamma_1\sin\theta}})] \qquad (3)$$

$$\sigma_y^s = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}[\frac{1}{\gamma_2-\gamma_1}(\frac{\gamma_2}{\sqrt{\cos\theta-\gamma_1\sin\theta}} - \frac{\gamma_1}{\sqrt{\cos\theta-\gamma_2\sin\theta}})] + \frac{K_{II}}{\sqrt{2\pi r}} \mathrm{Re}[\frac{1}{\gamma_2-\gamma_1}(\frac{1}{\sqrt{\cos\theta-\gamma_2\sin\theta}} - \frac{1}{\sqrt{\cos\theta-\gamma_1\sin\theta}})]$$

$$\tau_{xy}^s = \frac{K_I}{\sqrt{2\pi r}} \mathrm{Re}[\frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1}(\frac{1}{\sqrt{\cos\theta-\gamma_1\sin\theta}} - \frac{1}{\sqrt{\cos\theta-\gamma_2\sin\theta}})] + \frac{K_{II}}{\sqrt{2\pi r}} \mathrm{Re}[\frac{1}{\gamma_2-\gamma_1}(\frac{\gamma_2}{\sqrt{\cos\theta-\gamma_2\sin\theta}} - \frac{\gamma_1}{\sqrt{\cos\theta-\gamma_1\sin\theta}})]$$

where γ_1, γ_2 are the roots of the following characteristic equation:

$$C_{11}\gamma^4 - 2C_{16}\gamma^3 + (2C_{12}+C_{66})\gamma^2 - 2C_{26}\gamma + C_{22} = 0 \qquad (4)$$

C_{ij}, i, j = 1, 2, 6 are symmetric elastic compliance coefficients. Note that the root of Eq. (4) are always complex or purely imaginary and occur in conjugate pairs as γ_1, $\tilde{\gamma}_1$ and γ_2, $\tilde{\gamma}_2$. Here also

$$p_k = C_{11}\gamma_k^{\,2} + C_{12} + C_{16}\gamma_k$$
$$q_k = -C_{12}\gamma_k - C_{22}/\gamma_k - C_{26}\ ,\ k = 1,2 \tag{5}$$

The anisotropic stress intensity factor can be introduced as

$$K_{\rm I} = \lim_{r\to 0} \sqrt{2\pi z}\,\sigma_y\Big|_{\theta=0}$$
$$K_{\rm II} = \lim_{r\to 0} \sqrt{2\pi z}\,\tau_{xy}\Big|_{\theta=0} \tag{6}$$

where $z = re^{i\theta}$ complex variable.

The complementary elastic state to be used in the reciprocal work relation can be derived in the Appendix and has the form:

$$u_x^{\,s} = \frac{2}{\sqrt{r}}\mathrm{Re}\{\frac{1}{\gamma_2-\gamma_1}[(\frac{\gamma_1 p_2}{\sqrt{\cos\theta-\gamma_2\sin\theta}} - \frac{\gamma_2 p_1}{\sqrt{\cos\theta-\gamma_1\sin\theta}})C_1 + (\frac{p_1}{\sqrt{\cos\theta-\gamma_1\sin\theta}} - \frac{p_2}{\sqrt{\cos\theta-\gamma_2\sin\theta}})C_2]\}$$

$$u_y^{\,s} = \frac{2}{\sqrt{r}}\mathrm{Re}\{\frac{1}{\gamma_2-\gamma_1}[(\frac{\gamma_1 q_2}{\sqrt{\cos\theta-\gamma_2\sin\theta}} - \frac{\gamma_2 q_1}{\sqrt{\cos\theta-\gamma_1\sin\theta}})C_1 + (\frac{q_1}{\sqrt{\cos\theta-\gamma_1\sin\theta}} - \frac{q_2}{\sqrt{\cos\theta-\gamma_2\sin\theta}})C_2]\}$$

$$\sigma_x^{\,c} = \frac{1}{r^{3/2}}\mathrm{Re}\{\frac{1}{\gamma_2-\gamma_1}[\gamma_2\gamma_1(\frac{\gamma_1}{(\cos\theta-\gamma_1\sin\theta)^{3/2}} - \frac{\gamma_2}{(\cos\theta-\gamma_2\sin\theta)^{3/2}})C_1 + (\frac{\gamma_2^{\,2}}{(\cos\theta-\gamma_1\sin\theta)^{3/2}} - \frac{\gamma_1^{\,2}}{(\cos\theta-\gamma_1\sin\theta)^{3/2}})C_2]\} \tag{7}$$

$$\sigma_y^{\,c} = \frac{1}{r^{3/2}}\mathrm{Re}\{\frac{1}{\gamma_2-\gamma_1}[(\frac{\gamma_2}{(\cos\theta-\gamma_1\sin\theta)^{3/2}} - \frac{\gamma_1}{(\cos\theta-\gamma_2\sin\theta)^{3/2}})C_1 + (\frac{1}{(\cos\theta-\gamma_1\sin\theta)^{3/2}} - \frac{1}{(\cos\theta-\gamma_1\sin\theta)^{3/2}})C_2]\}$$

$$\tau^c_{\ xy} = \frac{1}{r^{3/2}}\mathrm{Re}\{\frac{1}{\gamma_2-\gamma_1}[\gamma_2\gamma_1(\frac{1}{(\cos\theta-\gamma_1\sin\theta)^{3/2}} - \frac{1}{(\cos\theta-\gamma_2\sin\theta)^{3/2}})C_1 + (\frac{\gamma_2}{(\cos\theta-\gamma_2\sin\theta)^{3/2}} - \frac{\gamma_1}{(\cos\theta-\gamma_1\sin\theta)^{3/2}})C_2]\}$$

where C_1 and C_2 are arbitrary constants.

Now on the inner circular boundary C_ε the evaluation of the contour integral in terms of the traction and displacement components(writing in polar coordinate system)takes the form

$$I_\varepsilon = -\int_{C_\varepsilon}(\underset{\sim}{T}\cdot\underset{\sim}{u}^c - \underset{\sim}{T}^c\cdot\underset{\sim}{u})ds$$
$$= \int_{-\pi}^{\pi}(\sigma_r^{\,c}u_r^{\,s} + \tau^c_{\ r\theta}u_\theta^{\,s} - \sigma_r^{\,s}u_r^{\,c} + \tau^s_{\ r\theta}u_\theta^{\,c})rd\theta \tag{8}$$

Upon substitution from the foregoing singular solutions (3) and complementary ones (7), a routine evaluation or an appropriate numerical quadrature of the integral produces symbolically

$$I_\varepsilon = (J_1{}^{\mathrm{I}} C_1 + J_2{}^{\mathrm{I}} C_2) K_{\mathrm{I}} + (J_1{}^{\mathrm{II}} C_1 + J_2{}^{\mathrm{II}} C_2) K_{\mathrm{II}} \quad (9)$$

where the quantities J_i^{I}, J_i^{II} i = 1,2 can be computed using Eqs. (3) and (7) once the elastic properties of the material are given.

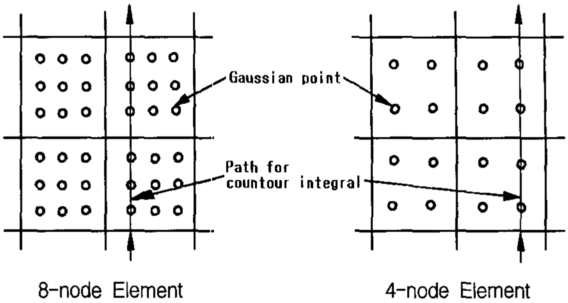
Consequently, for arbitrarily small ε Eq. (2) yields the representation formula for the SIF:

$$(J_1{}^{\mathrm{I}} C_1 + J_2{}^{\mathrm{I}} C_2) K_{\mathrm{I}} + (J_1{}^{\mathrm{II}} C_1 + J_2{}^{\mathrm{II}} C_2) K_{\mathrm{II}} = \int_{C_0} (\underset{\sim}{T} \cdot \underset{\sim}{u}^c - \underset{\sim}{T}^c \cdot \underset{\sim}{u}) ds \quad (10)$$

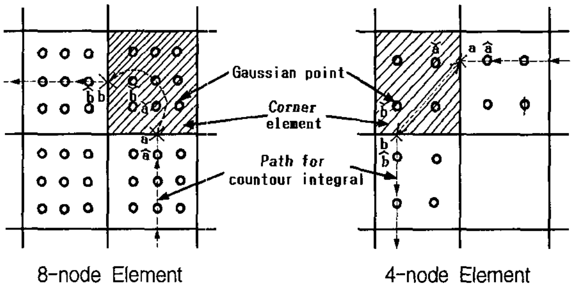
where it is important to note that the contour C_o involves only the outer boundary since both $\underset{\sim}{T}$ and $\underset{\sim}{T}^c$ necessarily vanish on the crack faces. It remains only to obtain $\underset{\sim}{u}$ and $\underset{\sim}{T}$ on the outer boundary from the prescribed data so that the contour integral may be evaluated as a linear combination of C_1 and C_2, the coefficients of which are the desired stress intensity factors, K_I and K_{II} .

3. Computational Procedures

The evaluation of the integral on the right-hand side of (10) involves the complementary functions(7) and the tractions and displacements along the contour C_o. A finite element method (FEM) including 4, 8 node quadrilateral elements in plane stress analysis was used to obtain the traction and displacement data on the boundary C_o. The contour C_o was chosen to pass through Gaussian quadrature rule integration points in the interior of each element as shown in Fig. 2(a) so



[Fig. 2(a)] Gaussian points and integration path for a boundary integral



[Fig. 2(b)] Integration path through the corner element

that the stress components generated by the FEM solutions can be directly used for an appropriate numerical quadrature on C_o. A linearly (or quadratically for 8 node quadrilateral element) varying displacement furnished from the FEM solutions can be evaluated at the integration points on C_o. Fig. 2(b) depicts a special treatment of a corner element which might be a part of contour. Both traction vectors and displacements at a and b is first determined by interpolating the values at ($\hat{a}$, $\breve{a}$) and ($\hat{a}$, $\breve{a}$) respectively of two adjacent elements.

Then use these values at a and b for interpolating again, with the tractions and displacements to be evaluated at the corresponding integration points of the corner elements.

With the tractions and displacements furnished by the finite element analysis and the complementary solutions($\underset{\sim}{T}^c, \underset{\sim}{u}^c$) known to be within multiplicative constants C_1 and C_2 from (7), the reciprocal work can be computed for each interior of elements forming the contour C_o and these values accumulated. The resulting quantities thus obtained are denoted by I_1 and I_2 Then we have finally

$$(J_1{}^{I}C_1 + J_2{}^{I}C_2)K_I + (J_1{}^{II}C_1 + J_2{}^{II}C_2)K_{II} = I_1C_1 + I_2C_2 \tag{11}$$

The two SIF in the equations are then determined by choosing the constants C_1 and C_2 (for example, C_1 = 1 and C_2 = 0 for symmetric mode I case) so that the complementary solutions pick up only the symmetric or skew symmetric components of the actual elastic field.

4. Numerical Results

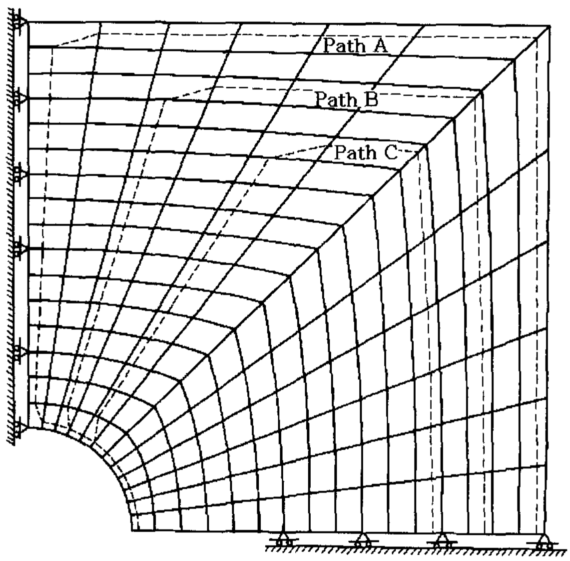
For illustrative numerical examples, two cracked orthotropic plates with circular hole modelling bidirectional fiber-reinforced composites have been treated; symmetric mode I type of the tension plate with the cracks emanating from a circular hole and mixed mode type of cantilever plate with a single edge crack containing stress concentration. The bidirectional fiber orientations are arranged at $\pm\alpha$ to the principal material direction 1, where α is the angle between the crack axis and the fiber direction. It should be noted here the elastic compliance coefficients in Eq.(4) can be calculated by the same transformation rule as the off-axis unidirectional fibe composites [15] except the vanishing terms of C_{16} and C_{26}. The same argument is also applicable to establish the elastic stiffness matrix. This implies, in other word, that no coupling terms appear between normal stresses and shear stresses. The elastic moduli data used in both cases are:

$$E_{11} = 21\times10^6\,psi$$
$$E_{22} = 1.7\times10^6\,psi$$
$$G_{12} = 1.4\times10^6\,psi$$
$$\nu_{12} = 0.21$$

which is representative of fiber-reinforced graphite/epoxy composites. Note that subscript 1 indicated the direction parallel to fiber orientation.

For the first problem, the tension plate with the cracks emanating from a circular hole, the FEM grid for a quadrant of the plate is shown in Fig. 3 which also depicts the integration contours used for evaluation by dotted lines. The normalized SIF for various fiber angles were computed and tabulated in Table 1.

Fig. 4 gives the variation of the SIF with fiber orientations. For comparative purposes, the corresponding isotropic solution can be obtained

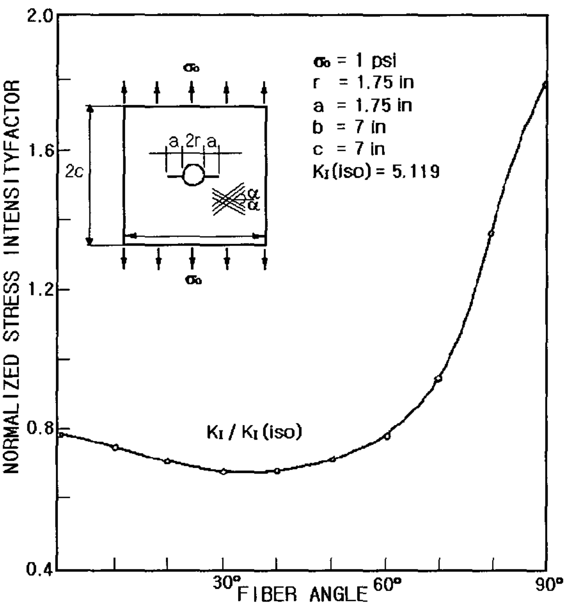


[Fig. 3] Finite element grid and integration contour for a quadrant

[Table 1] The normalized SIF in tension plate with cracks emanating from a circular hole

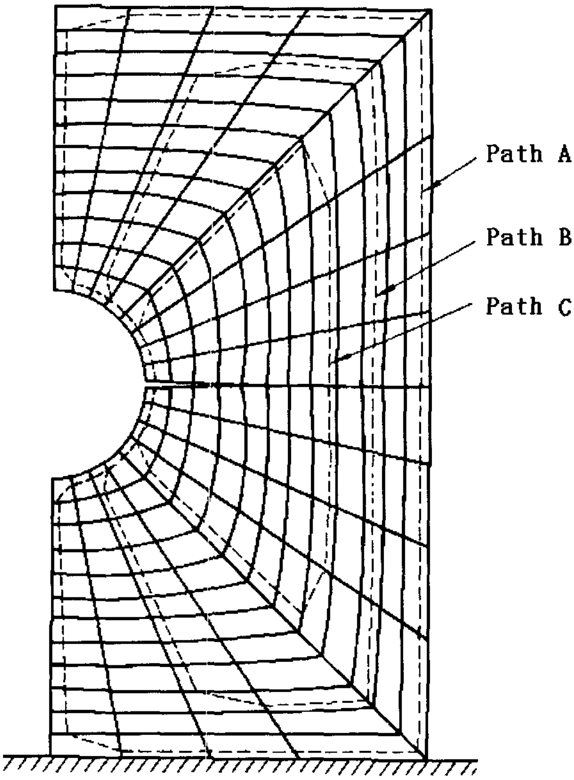
Fiber angle	$K_I / \sigma_0\sqrt{\pi a}$		
	Path A	Path B	Path C
0	1.729	1.727	1.726
10	1.656	1.650	1.639
20	1.565	1.552	1.526
30	1.532	1.505	1.467
40	1.552	1.505	1.449
50	1.635	1.579	1.510
60	1.765	1.721	1.657
70	2.103	2.074	2.004
80	3.014	2.986	2.863
90	3.960	3.918	3.727
*	2.215	2.180	2.156

* corresponding isotropic solution



[Fig. 4] Effect of fiber angle in the tension plate with cracks emanating from a circular hole

using present method. Results show that K_I gradually decreases to a minimum as α



[Fig. 5] FEM model for single edge cracked cantilever plate

approaches 40° and then increases with the increase of the fiber angle. The value of K_I at α=70° is approximately equal to the value of isotropic case.

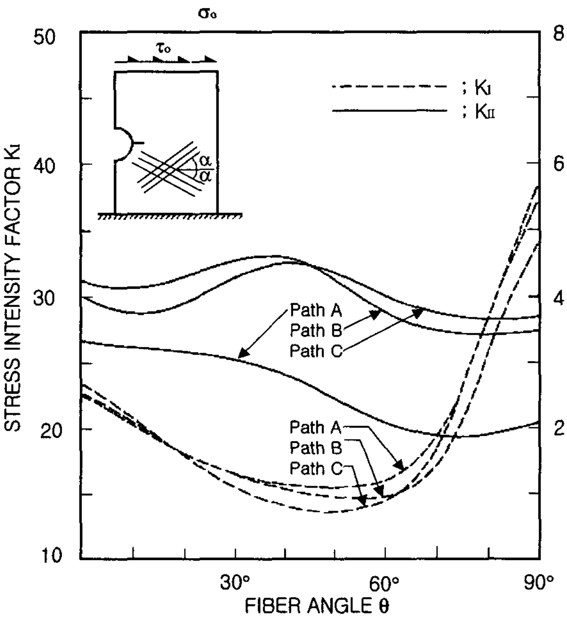
The second problem, a mixed mode type of cantilever plate with a single edge crack subjected to end shear, was analysed with 192 elements over full length of the plate as shown in Fig. 5. The resulting SIF are plotted in Fig. 6 and listed in Table 2.

As can be seen from Fig. 6, the calculation is stable with regard to contour selection except the value of K_{II} of the exterior contour. K_I curve indicates a minimum value at α = 50° and the value at α = 80° approximately reaches to the value of isotropic case. Although the values of

[Table 2] The mixed mode SIF for the cantilever plate with single edge crack containing stress concentration

Path / SIF / Angle	A		B		C	
	K_I	K_{II}	K_I	K_{II}	K_I	K_{II}
0	23.32	3.334	22.76	4.230	22.38	4.032
10	20.84	3.234	20.57	4.132	20.32	3.756
20	17.67	3.160	17.94	4.251	17.98	3.858
30	15.57	3.066	16.41	4.509	16.60	4.252
40	14.14	2.843	15.43	4.5774	15.75	4.485
50	13.68	2.461	14.86	.239	15.54	4.318
60	14.33	2.100	14.71	3.794	15.97	3.997
70	18.20	1.868	17.32	3.509	19.32	3.749
80	28.17	1.899	25.51	3.405	28.21	3.630
90	38.14	2.069	34.08	3.478	37.18	3.691
*	28.08	3.585	27.04	4.998	26.94	4.902

* corresponding isotropic solution



[Fig. 6] Effect of fiber angle in the cantilever plate with single edge crack containing stress concentration

K_{II} differ from those of the forgoing symmetric mode I type of the tension plate with circular hole cracks, the shape of the curve shows qualitatively same behavior with variation of fiber angle. The effect of fiber angle on the antisymmetric SIF, K_{II} is somewhat different.

K_{II} curve moves down monotonically with increase of fiber angle being always smaller values than those for isotropic case. In this mixed mode analysis there appear the values of K_{II} of outer contour to be significant discrepancy from the results calculated on path B and C. The source of this discrepancy has not yet been identified.

5. Conclusions

An existing boundary integral technique for calculating the SIF can be extended to treat the cracked orthoropic elastic plates involving stress concentration. The characteristic singular functions governing in the neighborhood of crack tip can be identified and the corresponding complementary elastic states are constructed on the basis of traction free boundary conditions of crack edge and the concept of finite reciprocal elastic work on the contour enclosing the crack tip.

The effect of fiber orientation on the SIF in bidirectional fiber-reinforced composite media is investigated by treating both mode I type of the tension plate with the cracks emanating from a circular hole and mixed mode type of cantilever plate with a single edge crack containing stress concentration subjected to end shear. While the numerical results of the examples above appear to be generally correct, they are far from satisfactory. In particular the results for the mixed mode problem are not numerically independent of choice of the integration contour, although the contour integral is theoretically path independent. This might be a consequence of the approximate nature of the numerical computation using the data provided by FEM solutions. Furthermore the crude approximation for corner element(4 elements in case of mixed mode) can be possibly associated with the resulting SIF solutions. To obtain more reliable results further study is needed to identify and correct, or at least to minimize the effect of the major error mechanism. Numerical experiments with different crack geometries, mesh pattens and interpolation schemes could be useful.

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Appendix

In terms of complex variable $z_k = x - \gamma_k y$, $k = 1, 2$ the problems of two-dimensional anisotropic elasticity for equilibrium configurations have solutions with the following representation in terms of the analytic functions $\phi_k(z_k)$ (for example, see ref. [14]) :

$$
\begin{aligned}
u_x &= 2\mathrm{Re}[p_1\phi_1'(z_1) + p_2\phi_2'(z_2)] \\
u_y &= 2\mathrm{Re}[q_1\phi_1'(z_1) + q_2\phi_2'(z_2)] \\
\sigma_x &= 2\mathrm{Re}[\gamma_1^2\phi_1''(z_1) + \gamma_2^2\phi_2''(z_2)] \\
\sigma_y &= 2\mathrm{Re}[\phi_1''(z_1) + \phi_2''(z_2)] \\
\tau_{xy} &= 2\mathrm{Re}[\gamma_1\phi_1''(z_1) + \gamma_2\phi_2''(z_2)]
\end{aligned} \qquad \text{(A1)}
$$

where $\gamma_k, k = 1, 2$ are the roots of Eq. (4) and p_k, q_k are the coefficients defined by Eq. (5). Substitution of the following traction free conditions of crack edge,

into (A1) yields

$$
\begin{aligned}
2\mathrm{Re}[\phi_1''(z_1) + \phi_2''(z_2)] &= 0 \\
2\mathrm{Re}[\gamma_1\phi_1''(z_1) + \gamma_2\phi_2''(z_2)] &= 0
\end{aligned} \qquad \text{(A2)}
$$

Note that here $z = z_1 = z_2$ since we only consider the negative real axis.

The stress intensity factors in terms of ϕ_1 and ϕ_2 can be written as

$$
\begin{aligned}
K_{\mathrm{I}} &= \lim_{r\to 0}\sqrt{2\pi z}\,\sigma_y\Big|_{\theta=0} \\
&= \sqrt{2\pi z}\cdot 2[\phi_1''(z) + \phi_2''(z)] \\
K_{\mathrm{II}} &= \lim_{r\to 0}\sqrt{2\pi z}\,\tau_{xy}\Big|_{\theta=0} \\
&= \sqrt{2\pi z}\cdot 2[\gamma_1\phi_1''(z) + \gamma_2\phi_2''(z)]
\end{aligned} \qquad \text{(A3)}
$$

Now in view of the expressions of Eqs. (A2) and (A3), and the concept of bounded strain energy in an arbitrarily small neighborhood of the crack tip, we may establish the relation between the functions ϕ_1, ϕ_2 and two parameters C_1, C_2 associated with the complementary elastic states

$$
\begin{aligned}
C_1 &= 2z^{3/2}[\phi_1''(z) + \phi_2''(z)] \\
C_2 &= 2z^{3/2}[\gamma_1\phi_1''(z) + \gamma_2\phi_2''(z)]
\end{aligned} \qquad \text{(A4)}
$$

The order of $z^{3/2}$ can be induced from the fact that the integrand of (2) on the inner boundary appears that a finite contribution will result if $\varepsilon^{1/2}\left|\underset{\sim}{u}^c\right|$ and $\varepsilon^{3/2}\left|\underset{\sim}{T}^c\right|$ are finite on C_ε

Inverting Eq. (A4), then we have the following explicit functional forms of ϕ_1, ϕ_2 as

$$
\begin{aligned}
\phi_1''(z_1) &= \frac{\gamma_2 C_1 - C_2}{(\gamma_2-\gamma_1)\sqrt{\gamma^3(\cos\theta-\gamma_1\sin\theta)^3}} \\
\phi_2''(z_2) &= \frac{-\gamma_1 C_1 + C_2}{(\gamma_2-\gamma_1)\sqrt{\gamma^3(\cos\theta-\gamma_2\sin\theta)^3}}
\end{aligned} \qquad \text{(A5)}
$$

Integration of the above Eq. (A5) produces finally

$$
\begin{aligned}
\phi_1'(z_1) &= \frac{-\gamma_2 C_1 + C_2}{(\gamma_2-\gamma_1)\sqrt{\gamma(\cos\theta-\gamma_1\sin\theta)}} \\
\phi_2'(z_2) &= \frac{\gamma_1 C_1 - C_2}{(\gamma_2-\gamma_1)\sqrt{\gamma(\cos\theta-\gamma_2\sin\theta)}}
\end{aligned} \qquad \text{(A6)}
$$

and substitution of these functions, ϕ_1, ϕ_2 into (A1) yields the desired complementary elastic solutions of Eq. (7).