The Korean Communications
in Statistics Vol. 12 No. 2, 2005
pp. 275-283

Convergence in Probability for Weighted Sums of Fuzzy Random Variables[1)]

Sang Yeol Joo[2)] and Young Nam Hyun[3)]

Abstract

In this paper, we give a sufficient condition for convergence in probability of weighted sums of convex-compactly uniformly integrable fuzzy random variables. As a result, we obtain weak law of large numbers for weighted sums of convexly tight fuzzy random variables.

Keywords : Fuzzy random variables, Convergence in probability, Tightness, Weighted sums.

1. Introduction

The concept of fuzzy random variables introduced by Puri and Ralescu(1986) has been extensively studied and applied in statistics and probability areas in recent years. Statistical inference for fuzzy probability models led to the need for laws of large numbers in order to ensure consistency in estimation problems. Strong laws of large numbers for independent fuzzy random variables have been studied by Klement, Puri and Ralescu (1986), Inoue (1991), Uemura (1993), Taylor and Inoue (1997), Molchanov (1999), Joo (2002), Proske and Puri (2002) etc. Recently, some results on weak laws of large numbers for fuzzy random variables were obtained by Taylor et al. (2001) and Joo (2004).

It is one of important problems how we can generalize laws of large numbers for sums of fuzzy random variables to the case of weighted sums. Related to this problem, Guan and Li (2004) established strong law of large numbers for weighted sums of fuzzy random variables under restrictive assumption. Joo et al.(to appear) obtained some results on strong convergence different from that of Guan and Li (2004). Kim (2004b) presented weak convergence for weighted sums of level-continuous fuzzy random variables. But weak convergence for weighted sums of general fuzzy random variables have not been studied until now.

---

1) This study is supported by Kangwon National University.

2) Professor, Department of Statistics, Kangwon National University, Chunchon 200-701, Korea
E-mail : syjoo@kangwon.ac.kr

3) Complete the doctor's course, Department of Statistics, Kangwon National University, Chunchon 200-701, Korea.

In this paper, we give some results on convergence in probability of weighted sums of convex-compactly uniformly integrable fuzzy random variables. This provide a generalization of weak laws of large numbers for fuzzy random variables.

2. Preliminaries

Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties ;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semicontinuous.

(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is convex, i.e. $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \text{supp } \tilde{u}, & \alpha = 0, \end{cases}$$

then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the closed intervals $L_\alpha \tilde{u} = [u^l(\alpha), u^r(\alpha)]$. That is, we can identify a fuzzy number $\tilde{u}$ with the family of closed intervals $\{[u^l(\alpha), u^r(\alpha)] : 0 \leq \alpha \leq 1\}$.

Theorem 2.1. For $\tilde{u} \in F(R)$, the followings hold;

(1) u^l is a bounded increasing function on $[0,1]$.

(2) u^r is a bounded decreasing function on $[0,1]$.

(3) $u^l(1) \leq u^r(1)$.

(4) u^l and u^r are left continuous on $[0,1]$ and right continuous at 0.

(5) If v^l and v^r satisfy above (1)-(4), then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v^l(\alpha), v^r(\alpha)]$.

Proof. see Goetschel and Voxman (1986).

In the above theorem, if u^l and u^r are continuous on $[0,1]$, $\tilde{u}$ is called level-continuous.

We denote by $F_C(R)$ the family of all level-continuous fuzzy numbers.

The linear structure on $F(R)$ is defined as usual ;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}(z), & \lambda = 0, \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$.

Then it is well-known that if $\tilde{u} = \{[u^l(\alpha), u^r(\alpha)] | 0 \le \alpha \le 1\}$ and $\tilde{v} = \{[v^l(\alpha), v^r(\alpha)] | 0 \le \alpha \le 1\}$, then

$$\tilde{u} \oplus \tilde{v} = \{[u^l(\alpha) + v^l(\alpha), u^r(\alpha) + v^r(\alpha)] | 0 \le \alpha \le 1\},$$

and

$$\lambda \tilde{u} = \{[\lambda u^l(\alpha), \lambda u^r(\alpha)] | 0 \le \alpha \le 1\}.$$

Now, we define the metric d_∞ on $F(R)$ by

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} \max(| u^l(\alpha) - v^l(\alpha) |, | u^r(\alpha) - v^r(\alpha) |).$$

The norm of $\tilde{u} \in F(R)$ is defined by

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \max(| u^l(0) |, | u^r(0) |).$$

Then it is well known that $F(R)$ is complete but is not separable with respect to the metric d_∞. Joo and Kim (2000) introduced a metric d_s on $F(R)$ which makes it a separable metric space as follows :

Definition 2.2. Let T denote the class of strictly increasing, continuous mapping of $[0,1]$ onto itself. For $\tilde{u}, \tilde{v} \in F(R)$, we define

$$d_s(\tilde{u}, \tilde{v}) = \inf\{\epsilon > 0 : \text{there exist a } t \in T \text{ such that} \sup_{0 \le \alpha \le 1} | t(\alpha) - \alpha | \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

It follows immediately that d_s is a metric on $F(R)$ and $d_s(\tilde{u}, \tilde{v}) \le d_\infty(\tilde{u}, \tilde{v})$. The metric d_s will be called the Skorokhod metric. It is well-known that $(F(R), d_s)$ is separable and topologically complete (For details, see Joo and Kim (2000)).

Now, for $\tilde{u} \in F(R)$ and a positive integer m, we denote $A_k = [u^l(\frac{k}{m}), u^r(\frac{k}{m})]$, $k = 0, 1, \cdots, m$, and if we define,

$$f_m(\tilde{u})(x) = \sum_{k=1}^{m} \frac{k-1}{m} I_{A_{k-1} \setminus A_k}(x) + I_{A_m}(x),$$

then it follows that

$$L_\alpha f_m(\tilde{u}) = \begin{cases} A_1, & \text{if } 0 \le \alpha \le \frac{1}{m}, \\ A_k, & \text{if } \frac{k-1}{m} < \alpha \le \frac{k}{m}, \ k = 2, \cdots, m. \end{cases}$$

It is obvious that

$$f_m(\tilde{u} \oplus \tilde{v}) = f_m(\tilde{u}) \oplus f_m(\tilde{v}) \text{ and } f_m(\lambda \tilde{u}) = \lambda f_m(\tilde{u}).$$

The following lemmas were obtained by Joo (2002).

Lemma 2.3. If K is a relatively compact subset of $F(R)$ in the d_s-topology, then

$$\lim_{m \to \infty} \sup_{\tilde{u} \in K} d_s(\tilde{u}, f_m(\tilde{u})) = 0 .$$

Lemma 2.4. For each $\tilde{u}, \tilde{v}, \tilde{a}, \tilde{b} \in F(R)$, we have

$$d_s(\tilde{u} \oplus \tilde{a}, \tilde{v} \oplus \tilde{b}) \le d_s(\tilde{u}, \tilde{v}) + \|\tilde{a}\| + \|\tilde{b}\|.$$

3. Main Results

Throughout this paper, let (Ω, A, P) be a probability space. A fuzzy number valued function $\tilde{X} : \Omega \to F(R)$, $\tilde{X} = \{[X^l(\alpha), X^r(\alpha)] : 0 \le \alpha \le 1\}$ is called a fuzzy random variable if for each $\alpha \in [0,1]$, $X^l(\alpha)$, and $X^r(\alpha)$ are random variable in the usual sense. It is well-known that $\tilde{X}$ is a fuzzy random variable if and only if $\tilde{X} : \Omega \to (F(R), d_s)$ is measurable. So throughout the remainder of this paper, we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated.

A fuzzy random variable $\tilde{X}$ is called integrable if $E\|\tilde{X}\| < \infty$. The expectation of integrable fuzzy random variable $\tilde{X}$ is a fuzzy number defined by

$$E(\tilde{X}) = \{[EX^l(\alpha), EX^r(\alpha)] | 0 \le \alpha \le 1\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variable and $\{\lambda_{ni}\}$ be a double array of real numbers such that

$$\sum_{i=1}^{\infty} |\lambda_{ni}| \le C \text{ for all } n \text{ and for some positive constant } C. \tag{3.1}$$

The problem that will be considered in this section is to obtain a sufficient condition for

$$d_s(\oplus_{i=1}^{n}\lambda_{ni}\widetilde{X}_i, \oplus_{i=1}^{n}\lambda_{ni}E(\widetilde{X}_i)) \rightarrow 0 \text{ in probability.}$$

Similar problem was handled by Kim (2004b) in case of $F_C(R)$-valued random elements. But in order to obtain some results $F(R)$-valued random elements which is in a more general setting, we need the following concepts.

Definition 3.1. Let $\{\widetilde{X}_n\}$ be a sequence of fuzzy random variables.

(1) $\{\widetilde{X}_n\}$ is said to be compactly uniformly integrable if for each $\epsilon > 0$, there exists a compact subset K of $F(R)$ such that

$$E(I_{\{\widetilde{X}_n \not\in K\}}\|\widetilde{X}_n\|) < \epsilon \text{ for all } n.$$

If K is convex and compact, then $\{\widetilde{X}_n\}$ is said to be convex-compactly uniformly integrable.

(2) $\{\widetilde{X}_n\}$ is said to be tight if for each $\epsilon > 0$, there exists a compact subset K of $F(R)$ such that

$$P(\widetilde{X}_n \not\in K) < \epsilon \text{ for all } n.$$

If K is convex and compact, then $\{\widetilde{X}_n\}$ is said to be convexly tight.

It is trivial that convex-compactly uniform integrability implies convex-compactly uniform integrability. But the converse is not true because there exists a compact subset K of $F(R)$ such that its convex hull coK is not compact(For details, see Kim (2001)). However on the space $F_C(R)$, the concept of convex-compactly uniform integrability is equivalent to that of compactly uniform integrability. Similar statements can be applied between convex tightness and tightness.

Our main result is as follows ;

Theorem 3.2. Let $\{\widetilde{X}_n\}$ be a sequence of convex-compactly uniformly integrable fuzzy random variables and $\{\lambda_{ni}\}$ be a double array of real numbers satisfying (3.1). If for each $j = l, r$ and for all $\alpha \in [0,1]$,

$$\sum_{i=1}^{n}\lambda_{ni}(X_i^j(\alpha) - E(X_i^j(\alpha))) \rightarrow 0 \text{ in probability,}$$

then

$$d_s(\oplus_{i=1}^{n}\lambda_{ni}\widetilde{X}_i, \oplus_{i=1}^{n}\lambda_{ni}E(\widetilde{X}_i)) \rightarrow 0 \text{ in probability.}$$

Proof. We will proceed by similar arguments in Kim (2004), Without loss of generality, we may assume $C=1$. Let $\epsilon > 0$ and $0 < \delta < 1$ be given. By the convex-compactly uniform integrability of $\{\tilde{X}_n\}$, we can choose a convex compact subset K of $F(R)$ such that

$$\int_{\{\tilde{X}_n \not\in K_1\}} \|\tilde{X}_n\| dP \le \epsilon\delta/12 \quad \text{for all } n. \tag{3.2}$$

Since the convex hull $co(K_1 \cup \{\tilde{0}\})$ is compact, we may assume that $\tilde{0} \in K_1$. If we let

$$K = K_1 \oplus (-1)K_1 = \{\tilde{u} \oplus (-1)\tilde{v} : \tilde{u}, \tilde{v} \in K_1\},$$

then by Theorem 3.3 of Kim (2004a), K is convex and compact. By lemma 2.3, we can choose m such that

$$d_s(\tilde{u}, f_m(\tilde{u})) < \epsilon/6 \text{ for all } \tilde{u} \in K. \tag{3.3}$$

First we note that since K is convex compact and $\tilde{0} \in K$, we have

$$E(I_{\{\tilde{X}_i \in K\}}\tilde{X}_i) \in K \text{ and } \oplus_{i=1}^n \lambda_{ni} E(I_{\{\tilde{X}_i \in K\}}\tilde{X}_i) \in K.$$

Thus, by (3.3)

$$d_s(\oplus_{i=1}^n \lambda_{ni} E(I_{\{\tilde{X}_i \in K\}}\tilde{X}_i), f_m(\oplus_{i=1}^n \lambda_{ni} E(I_{\{\tilde{X}_i \in K\}}\tilde{X}_i)) < \epsilon/6 . \tag{3.4}$$

By lemma 2.4 and (3.4), we have

$$\begin{aligned} & d_s(\oplus_{i=1}^n \lambda_{ni} E(\tilde{X}_i), \oplus_{i=1}^n \lambda_{ni} f_m(E(\tilde{X}_i))) \\ & \le \left\|\oplus_{i=1}^n \lambda_{ni} E(I_{\{\tilde{X}_i \not\in K\}}\tilde{X}_i)\right\| + \left\|\oplus_{i=1}^n \lambda_{ni} f_m(E(I_{\{\tilde{X}_i \not\in K\}}\tilde{X}_i))\right\| + \epsilon/6 \\ & \le 2\sum_{i=1}^n \lambda_{ni} \left\|E(I_{\{\tilde{X}_i \not\in K\}}\tilde{X}_i)\right\| + \epsilon/6 \\ & \le \epsilon\delta/6 + \epsilon/6 \le \epsilon/3 . \end{aligned}$$

Therefore, we have

$$\begin{aligned} & d_s(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} E(\tilde{X}_i)) \\ & \le d_s(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} f_m(\tilde{X}_i)) \\ & \qquad + d_s(\oplus_{i=1}^n \lambda_{ni} f_m(\tilde{X}_i), \oplus_{i=1}^n \lambda_{ni} f_m(E\tilde{X}_i)) > \epsilon/3 . \end{aligned}$$

This implies that

$$\begin{aligned} & P(d_s(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} E(\tilde{X}_i)) > \epsilon) \\ & \le P(d_s(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} f_m(\tilde{X}_i)) > \epsilon/3) \\ & \qquad + P(d_s(\oplus_{i=1}^n \lambda_{ni} f_m(\tilde{X}_i), \oplus_{i=1}^n \lambda_{ni} f_m(E\tilde{X}_i)) > \epsilon/3) \\ & = (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

For (I), by using (3.2), (3.3) and lemma 2.4, we have

$$\begin{aligned} (\mathrm{I}) & \le P(\left\|\oplus_{i=1}^n \lambda_{ni} I_{\{\tilde{X}_i \not\in K\}}\tilde{X}_i\right\| + \left\|\oplus_{i=1}^n \lambda_{ni} f_m(I_{\{\tilde{X}_i \not\in K\}}\tilde{X}_i)\right\| > \epsilon/6) \\ & \le P(2\sum_{i=1}^n \lambda_{ni} \left\|I_{\{\tilde{X}_i \not\in K\}}\tilde{X}_i\right\| > \epsilon/6) \end{aligned}$$

$$\leq \frac{12}{\epsilon}\sum_{i=1}^{n}\lambda_{ni}E\left\|I_{\{\widetilde{X}_i \not\in K\}}\widetilde{X}_i\right\| \leq \frac{12}{\epsilon}\frac{\epsilon\delta}{12} = \delta .$$

Now for (II),

$$(\mathrm{II}) = P[d_s(\oplus_{i=1}^{n}\lambda_{ni}f_m(\widetilde{X}_i), \oplus_{i=1}^{n}\lambda_{ni}f_m(E\widetilde{X}_i)) > \epsilon/3]$$

$$\leq P[d_\infty(\oplus_{i=1}^{n}\lambda_{ni}f_m(\widetilde{X}_i), \oplus_{i=1}^{n}\lambda_{ni}f_m(E\widetilde{X}_i)) > \epsilon/3]$$

$$\leq \sum_{k=1}^{m}P[\sum_{i=1}^{n}|\lambda_{ni}(X_i^l(k/m) - EX_i^l(k/m))| > \epsilon/3m]$$

$$+ \sum_{k=1}^{m}P[\sum_{i=1}^{n}|\lambda_{ni}(X_i^r(k/m) - EX_i^r(k/m))| > \epsilon/3m]$$

$\leq \delta$ for sufficiently large n from the assumption.

This completes the proof.

Corollary 3.3. Let $\{\widetilde{X}_n\}$ be a sequence of convexly tight fuzzy random variables such that $\sup_n\|\widetilde{X}_n\|^p = M < \infty$ for some $p > 1$ and $\{\lambda_{ni}\}$ be a double array of real numbers satisfying (3.1). If for each $j = 1, r$ and for all $\alpha \in [0,1]$,

$$\sum_{i=1}^{n}\lambda_{ni}(X_i^j(\alpha) - E(X_i^j(\alpha))) \to 0 \text{ in probability,}$$

then

$$d_s(\oplus_{i=1}^{n}\lambda_{ni}\widetilde{X}_i, \oplus_{i=1}^{n}\lambda_{ni}E(\widetilde{X}_i)) \to 0 \quad \text{in probability.}$$

Proof. This follows immediately from the fact that convex tightness and p-th moment condition $(p > 1)$ of $\{\widetilde{X}_n\}$ implies convex-compactly uniform integrability of $\{\widetilde{X}_n\}$. In deed, let $\epsilon > 0$ be given and K be a convex compact subset of $F(R)$ such that

$$P\{\widetilde{X}_n \not\in K\} < \epsilon^{p/(p-1)}M^{-1/(p-1)} \text{ for all } n.$$

Then

$$\int_{\{\widetilde{X}_n \not\in K_1\}}\|\widetilde{X}_n\|dP \leq (E\|\widetilde{X}_n\|^p)^{1/p}P[\widetilde{X}_n \not\in K]^{(p-1)/p}$$

$$\leq M^{1/p}P[\widetilde{X}_n \not\in K]^{(p-1)/p}$$

$$< \epsilon \text{ for all } n.$$

Corollary 3.4. Let $\{\widetilde{X}_n\}$ be a sequence of independent and convexly tight fuzzy random variables such that $\sup_n\|\widetilde{X}_n\|^p = M < \infty$ for some $p > 1$. If $\{\lambda_{ni}\}$ is a double array of real numbers satisfying (3.1) and

$$\max_{1 \le i \le n} |\lambda_{ni}| = O(n^{-r}), r > \frac{1}{p-1},$$

then

$$d_s(\oplus_{i=1}^{n} \lambda_{ni}\widetilde{X}_i, \oplus_{i=1}^{n} \lambda_{ni}E(\widetilde{X}_i)) \to 0 \text{ in probability.}$$

References

[1] Goetschel, R. and Voxman, W.(1986). Elementary fuzzy calculus, *Fuzzy Sets and Systems,* Vol. 18, 31-43.

[2] Guan L. and Li, S.(2004). Laws of large numbers for weighted sums of fuzzy set-valued random variables, *International Journal of Uncertainty, Fuzziness and Knowledge Based Systems,* Vol. 12, 811-825.

[3] Inoue, H.(1991) A strong law of large numbers for fuzzy random sets, *Fuzzy Sets and Systems,* Vol. 41, 285-291.

[4] Joo, S. Y.(2002). Strong law of large numbers for tight fuzzy random variables, *Journal of the Korean Statistical Society,* Vol. 31, 129-140.

[5] Joo, S. Y.(2004). Weak law of large numbers for fuzzy random variables, *Fuzzy Sets and Systems,* Vol. 147, 453-464.

[6] Joo, S. Y. and Kim, Y. K.(2000). The Skorokhod topology on space of fuzzy numbers, *Fuzzy sets and Systems,* Vol. 111, 497-501.

[7] Joo, S. Y. and Kim, Y. K. and Kwon, J. S.(to appear). Strong convergence for weighted sums of fuzzy random sets, *Information Sciences*.

[8] Kim, Y. K,(2001). Compactness and convexity on the space of fuzzy sets, *Journal of Mathematical Analysis and Applications,* Vol. 264, 122-132.

[9] Kim, Y. K.(2004). Compactness and convexity on the space of fuzzy sets II, *Nonlinear Analysis*. Vol. 57, 639-653.

[10] Kim, Y. K.(2004). Weak convergence for weighted sums of level-continuous fuzzy random variables, *Journal of Fuzzy Logic and Intelligent Systems,* Vol. 14, 852-856.

[11] Klement, E. P. Puri, M. L. and Ralescu, D. A.(1986). Limit theorems for fuzzy random variables, *Proc. Roy. Soc. Lond. Ser. A,* Vol. 407, 171-182.

[12] Molchanov, I.(1999). On strong laws of large numbers for random upper semicontinuous, *Journal of Mathematical Analysis and Applications*. Vol. 235, 349-355.

[13] Proske, F. and Puri, M. L.(2002). Strong laws of large numbers for Banach space valued fuzzy random variables, *Journal of Theoretical Probability*. Vol. 15, 543-551.

[14] Puri, M. L. and Ralescu, D. A.(1986). Fuzzy random variables, *Journal of Mathematical Analysis and Applications*. 114 , 402-422.

[15] Taylor, R. L. and Inoue, H.(1997). Laws of large numbers for random sets, *Random sets: Theory and Applications, IMA* Vol. 97, Springer, New York, 347-366.

[16] Taylor, R. L. Seymour, L. and Chen, Y.(2001). Weak laws of large numbers for fuzzy random sets, *Nonlinear Analysis,* Vol. 47, 1245-1256.

[17] Uemura, T.(1993). A law of large numbers for random sets, *Fuzzy Sets and Systems,* Vol. 59, 181-188.

[Received November 2004, Accepted March 2005]