The Korean Communications
in Statistics Vol. 12 No. 2, 2005
pp. 395-411

A Study on Noninformative Priors of Intraclass Correlation Coefficients in Familial Data

Bong Soo Jin[1] and Byung Hwee Kim[2]

Abstract

In this paper, we develop the Jeffreys' prior, reference prior and the the probability matching priors for the difference of intraclass correlation coefficients in familial data. We prove the sufficient condition for propriety of posterior distributions. Using marginal posterior distributions under those noninformative priors, we compare posterior quantiles and frequentist coverage probability.

Keywords : Jeffrey's prior, reference prior, probability matching priors, intraclass correlation coefficients, reparametrization, posterior distribution

1. Introduction

The intraclass correlation coefficient ρ is frequently used to measure the degree of intrafamily resemblance with respect to characteristics such as blood pressure, cholesterol, weight, height, stature, lung capacity, and so forth. Statistical inference concerning ρ for a single sample problem based on a normal distribution has been studied by several authors (Rao (1973), Rosner and Donner (1977), Donner and Koval (1980), Donner and Bull (1983), Srivastava (1984), Srivastava and Kaptapa (1986), Young and Bhandary (1998)). Surprisingly, its extension to multisample problems based on several multivariate normal distribution has recevied very little attention. There is a considerable study of a statistical inference for intraclass correlation coefficient from familial data by several authors. However nothing is known about approach of Bayesian inference except for Kim, Kang, and Lee (2001). But none of the above authors considered any Bayesian inference for unequal family sizes. In practice, we come across families with unequal numbers of children and hence, this is a very important practical problem to consider as a estimation for intraclass correlation coefficient under unequal family sizes. In this paper, we consider Bayesian estimation problem for the difference of two intraclass correlation coefficients based on two independent multinormal samples under

1) Graduate, Department of Mathematics Hanyang University Seoul, 133-791, Korea
E-mail : bsjin7@ihanyang.ac.kr
2) Professor, Department of Mathematics Hanyang University Seoul, 133-791, Korea

unequal family sizes. The most frequently used noninformative prior is Jeffreys' prior. But, in spite of its success in one parameter, Jeffreys' prior frequently runs into serious difficulties in the presence of nuisance parameters. To overcome these deficiencies of Jeffreys' prior, Berger and Bernardo(1989,1992) expounded the reference prior approach of Bernardo(1979) for deriving noninformative priors in multiparameter situations by dividing the parameters into parameters of interest and nuisance parameters. This approach is very successful in various practical problems. As an alternative, we use the method of Peers(1965) to find priors which require the frequentist coverage probability of the posterior region of a real-valued parametric function to match the nominal level with a remainder of $o(n^{-\frac{1}{2}})$. These priors, as usually referred to as the first order matching priors. In this paper, we consider the problem of inferencing the difference of ρ_1 and ρ_2 using noninformative priors in the following situation :

Suppose we have a sample of measurements from k_1, k_2 families and let X^*_i, $i=1,2,\cdots,k_1$, and X^*_j, $j=1,2,\cdots,k_2$, represent measurements from the i^{th} family and j^{th} family, respectively , where

$$X^*_i=(X_{i1}, X_{i2},\cdots, X_{ip_1})' \text{ and } Y^*_j=(Y_{j1}, Y_{j2},\cdots, Y_{jp_2})', \text{ where } p_1, p_2, k_1, k_2 \geq 2 .$$

The structure of the mean vector and the covariance matrix for the familial data is given by Rao(1973) as

$$\mu^*_1=\mu_1 1_{p_1},\ \ \Sigma^*_1=\sigma^2(1-\rho_1)I_{p_1}+\rho_1 J_{p_1} \text{ and } \mu^*_2=\mu_2 1_{p_2},\ \ \Sigma^*_2=\sigma^2(1-\rho_2)I_{p_2}+\rho_2 J_{p_2},$$

where 1_{p_1} is a $p_1\times 1$ vector of $1's$, $p_2\times 1$ vector of $1's$ and J_{p_1} is the $p_1\times p_1$ matrix containg only ones , J_{p_2} is the $p_2\times p_2$ matrix containg only ones, and I is the identity matirx , and $\mu_1, \mu_2 (\in R^1)$ are the common mean of X_i^* , Y_j^* , respectively, $\sigma^2 (0 < \sigma^2)$ is the common variance of members of the family, and ρ_1, ρ_2 called the intraclass correlation coefficient, which are the coefficient of correlations among the members of the family and $-\frac{1}{p_1-1} < \rho_1 < 1$, $-\frac{1}{p_2-1} < \rho_2 < 1$, respectively. It is assumed that

$$X^*_i \sim N_{p_1}(\ \mu^*_1,\ \Sigma^*_1), i=1,2,\cdots,k_1, \text{ and } Y^*_j \sim N_{p_2}(\ \mu^*_2,\ \Sigma^*_2), j=1,2,\cdots,k_2,$$

where N_p represents a p-variate normal distribution. In Section 2, we treat the reparametrization $(\rho_1, \rho_2, \sigma^2, \mu_1, \mu_2)$ to $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$. In Section 3, we derive, using this transformation , Jeffreys's prior, reference prior , and probability matching priors for $\eta=(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$ when η_1 is the parameter of interest and $(\eta_2, \eta_3, \eta_4, \eta_5)$ is a nuisance parameter vector where $\eta_1=\theta_1-\theta_2$, $\eta_2=\theta_2$, $\eta_3=\theta_3$, $\eta_4=\theta_4$, $\eta_5=\theta_5$. The sufficient condition for propriety of posterior distribution of η and marginal posterior densities for η_1 under these priors are given in Section 4. In Section 5, using marginal posterior distributions under those noninformative priors, we compare posterior quantiles and frequentist coverage probability.

2. Fisher Information Matrices

Let $X_i=(X_{i1}, X_{i2},\cdots, X_{ip_1})'=Q\,X^*_i$ and $Y_j=(Y_{j1}, Y_{j2},\cdots, Y_{jp_2})'=Q\,Y^*_j$ where Q is a Helmert orthogonal matrix. Under these orthogonal transformations, it is obvious that

$$X_i\sim N_{p_1}(\,\boldsymbol{\mu}_1,\ \Sigma_1),\ i=1,2,\cdots,k_1,\quad Y_j\sim N_{p_2}(\,\boldsymbol{\mu}_2,\ \Sigma_2),\ j=1,2,\cdots,k_2,$$

where

$$\boldsymbol{\mu}_1=(\sqrt{p_1}\mu_1, 0,\cdots,0)',\quad \Sigma_1=\sigma^2 Diag[1+(p_1-1)\rho_1, 1-\rho_1,\cdots,1-\rho_1],$$
$$\boldsymbol{\mu}_2=(\sqrt{p_2}\mu_2, 0,\cdots,0)',\quad \Sigma_2=\sigma^2 Diag[1+(p_2-1)\rho_2, 1-\rho_2,\cdots,1-\rho_2].$$

Then the likelihood function of $(\rho_1,\rho_2,\sigma^2,\mu_1,\mu_2)$ is $l(\rho_1,\rho_2,\sigma^2,\mu_1,\mu_2|\,\boldsymbol{x},y)$

$$= (2\pi)^{-\frac{k}{2}}\sigma^{-p_1k_1}(1+(p_1-1)\rho_1)^{-\frac{k_1}{2}}$$
$$\times(1-\rho_1)^{-\frac{(p_1-1)k_1}{2}}\sigma^{-p_2k_2}(1+(p_2-1)\rho_2)^{-\frac{k_2}{2}}(1-\rho_2)^{-\frac{(p_2-1)k_2}{2}}$$
$$\times\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\mu_1)^2}{1+(p_1-1)\rho_1}+\frac{1}{1-\rho_1}\sum_{m=2}^{p_1}x^2_{im}\right)\right]\right\}$$
$$\times\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\mu_2)^2}{1+(p_2-1)\rho_2}+\frac{1}{1-\rho_2}\sum_{n=2}^{p_2}y^2_{jn}\right)\right]\right\},\ \text{where}\ k=k_1p_1+k_2p_2\,.$$

Lemma 2.1 (Original Parametrization)
The Fisher information matrix of $(\rho_1,\rho_2,\sigma^2,\mu_1,\mu_2)$ is given by

$$I_1(\rho_1,\rho_2,\sigma^2,\mu_1,\mu_2)=\begin{pmatrix} A & 0 & F & 0 & 0\\ 0 & B & G & 0 & 0\\ F & G & E & 0 & 0\\ 0 & 0 & 0 & C & 0\\ 0 & 0 & 0 & 0 & D\end{pmatrix}, \tag{2.1}$$

where

$$A=\frac{k_1p_1(p_1-1)(1+(p_1-1)\rho^2_1)}{2(1-\rho_1)^2(1+(p_1-1)\rho_1)^2},\quad B=\frac{k_2p_2(p_2-1)(1+(p_2-1)\rho^2_2)}{2(1-\rho_2)^2(1+(p_2-1)\rho_2)^2},\quad C=\frac{p_1k_1}{\sigma^2(1+(p_1-1)\rho_1)},$$

$$D=\frac{p_2k_2}{\sigma^2(1+(p_2-1)\rho_2)},\quad E=\frac{k_1p_1+k_2p_2}{2\sigma^4},\quad F=-\frac{p_1k_1(p_1-1)\rho_1}{2\sigma^2(1-\rho_1)(1+(p_1-1)\rho_1)},$$

and

$$G=-\frac{p_2k_2(p_2-1)\rho_2}{2\sigma^2(1-\rho_2)(1+(p_2-1)\rho_2)}\,.$$

Consider the following transform from $(\rho_1,\rho_2,\sigma^2,\mu_1,\mu_2)$ to $(\theta_1,\theta_2,\theta_3,\theta_3,\theta_4,\theta_5)$ where $\rho_1=\theta_1$, $\rho_2=\theta_2$,

$$\sigma^2=(1-\theta_1)^{-\frac{(p_1-1)k_1}{k}}(1+(p_1-1)\theta_1)^{-\frac{k_1}{k}}(1-\theta_2)^{-\frac{(p_2-1)k_2}{k}}(1+(p_2-1)\theta_2))^{\frac{k_2}{k}}\theta_3\ ,$$

$\mu_1=\theta_4$, and $\mu_2=\theta_5$, where $k=k_1p_1+k_2p_2$.

Lemma 2.2 (Reparametrization)
The Fisher information matrix is $(\theta_1,\theta_2,\theta_3,\theta_3,\theta_4,\theta_5)$

$$I_2(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)=\begin{pmatrix} I_{11} & I_{12} & 0 & 0 & 0 \\ I_{21} & I_{22} & 0 & 0 & 0 \\ 0 & 0 & I_{33} & 0 & 0 \\ 0 & 0 & 0 & I_{44} & 0 \\ 0 & 0 & 0 & 0 & I_{55} \end{pmatrix}, \tag{2.2}$$

where

$$I_{11}=\frac{k_1p_1(p_1-1)[k+k_1p_1(p_1-1)\ \theta^2{}_1]}{2k(1-\theta_1)^2[1+(p_1-1)\theta_1]^2}\ ,\ I_{22}=\frac{k_2p_2(p_2-1)[k+k_2p_2(p_2-1)\ \theta^2{}_2]}{2k(1-\theta_2)^2[1+(p_2-1)\theta_2]^2}\ ,$$

$$I_{12}=I_{21}=-\frac{p_1k_1p_2k_2(p_1-1)(p_2-1)\theta_1\theta_2}{2k(1-\theta_1)(1-\theta_2)[1+(p_1-1)\theta_1][1+(p_2-1)\theta_2]}\ ,\ I_{33}=\frac{k}{2\theta_3^2}\ ,$$

$$I_{44}=p_1k_1\theta_3^{-1}(1-\theta_1)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)\theta_1]^{\frac{k_1}{k}-1}(1-\theta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{\frac{k_2}{k}}\ ,$$

and

$$I_{55}=p_2k_2\,\theta^{-1}{}_3(1-\theta_1)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)\theta_1]^{\frac{k_1}{k}}(1-\theta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{\frac{k_2}{k}-1}.$$

Proof

The result follow, after lengthy calculation using (2.1), from $I_2=J^T{}_1I_1J_1$ where $J_1=(J_{ij})$, $i=1,2,3,4,5$, $j=1,2,3,4,5$, with

$$J_1=\begin{pmatrix} \frac{\partial\rho_1}{\partial\theta_1} & \frac{\partial\rho_1}{\partial\theta_2} & \frac{\partial\rho_1}{\partial\theta_3} & \frac{\partial\rho_1}{\partial\theta_4} & \frac{\partial\rho_1}{\partial\theta_5} \\ \frac{\partial\rho_2}{\partial\theta_1} & \frac{\partial\rho_2}{\partial\theta_2} & \frac{\partial\rho_2}{\partial\theta_3} & \frac{\partial\rho_2}{\partial\theta_4} & \frac{\partial\rho_2}{\partial\theta_5} \\ \frac{\partial\sigma^2}{\partial\theta_1} & \frac{\partial\sigma^2}{\partial\theta_2} & \frac{\partial\sigma^2}{\partial\theta_3} & \frac{\partial\sigma^2}{\partial\theta_4} & \frac{\partial\sigma^2}{\partial\theta_5} \\ \frac{\partial\mu_1}{\partial\theta_1} & \frac{\partial\mu_1}{\partial\theta_2} & \frac{\partial\mu_1}{\partial\theta_3} & \frac{\partial\mu_1}{\partial\theta_4} & \frac{\partial\mu_1}{\partial\theta_5} \\ \frac{\partial\mu_2}{\partial\theta_1} & \frac{\partial\mu_2}{\partial\theta_2} & \frac{\partial\mu_2}{\partial\theta_3} & \frac{\partial\mu_2}{\partial\theta_4} & \frac{\partial\mu_2}{\partial\theta_5} \end{pmatrix},$$

where

$$J_{12}=J_{13}=J_{14}=J_{15}=J_{21}=J_{23}=J_{24}=J_{25}=J_{34}=J_{35}=0\ ,$$

$$J_{41}=J_{42}=J_{43}=J_{45}=J_{51}=J_{52}=J_{53}=J_{54}=0\ ,\ J_{11}=J_{22}=J_{44}=J_{55}=1\ ,\ J_{33}=\frac{\sigma^2}{\theta_3}\ ,$$

$$J_{31}=(p_1-1)k_1k^{-1}[(1-\theta_1)^{-1}-[1+(p_1-1)\theta_1]^{-1}](1-\theta_1)^{-\frac{(p_1-1)k_1}{k}}$$
$$\times[1+(p_1-1)\theta_1]^{-\frac{k_1}{k}}(1-\theta_2)^{-\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{-\frac{k_2}{k}}\theta_3,$$

and

$$J_{32}=(p_2-1)k_2k^{-1}[(1-\theta_2)^{-1}-[1+(p_2-1)\theta_2]^{-1}](1-\theta_1)^{-\frac{(p_1-1)k_1}{k}}$$
$$\times[1+(p_1-1)\theta_1]^{-\frac{k_1}{k}}(1-\theta_2)^{-\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{-\frac{k_2}{k}}\theta_3.$$

3. Noninformative Priors

In this section, we first provide, using (2.2), Jeffreys' and reference prior for $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ and then gives Jeffreys' and reference prior for $(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)$ with $\eta_1=\theta_1-\theta_2$, parameter of interest, and $\eta_2=\theta_2$, $\eta_3=\theta_3$, $\eta_4=\theta_4$, $\eta_5=\theta_5$.

Theorem 3.1

The Jeffreys's prior for $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ is

$$\pi_J\propto \theta^{-2}_3(1-\theta_1)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)\theta_1]^{\frac{k_1}{k}-\frac{1}{2}}(1-\theta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\theta_2]^{\frac{k_2}{k}-\frac{1}{2}}B(\theta_1,\theta_2),$$

where $B(\theta_1,\theta_2)=[k_1p_1[1+(p_2-1)\theta_2^2]+k_2p_2[1+(p_1-1)\theta_1^2]]^{\frac{1}{2}}$
$$\times[1+(p_1-1)\theta_1]^{-1}[1+(p_2-1)\theta_2]^{-1}(1-\theta_1)^{-1}(1-\theta_2)^{-1}.$$

And the Jeffreys' prior for is $(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)$

$$\pi_J(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\propto(1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}-\frac{1}{2}}$$
$$\times(1-\eta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\eta_2]^{\frac{k_2}{k}-\frac{1}{2}}\eta^{-2}_3B(\eta_1,\eta_2). \tag{3.1}$$

From Berger and Bernardo (1989), we have the reference prior for $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ when (θ_1,θ_2) is the parameter of interest and $(\theta_3,\theta_4,\theta_5)$ is the nuisance parameter vector.

Theorem 3.2

The reference prior for $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ is given by $\pi_R\propto\theta^{-2}_3B(\theta_1,\theta_2)$. Also,

$$\pi_R(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\propto\eta^{-2}_3B(\eta_1,\eta_2). \tag{3.2}$$

Next, we first derive matching priors for η_1. Following Datta and Ghosh (1995) with (2.2), we have matching priors for $\eta_1=t(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)=\theta_1-\theta_2$ which are the solutions π to the differential equation

$$\sum_{i=1}^{5}\frac{\partial}{\partial\theta_i}\left\{\frac{\rho_i' I^{-1}{}_2(\theta)\nabla_t(\theta)}{\sqrt{\nabla_t'(\theta)I^{-1}{}_2(\theta)\nabla_t(\theta)}}\pi(\theta)\right\}=0, \tag{3.3}$$

where $I^{-1}{}_2(\theta)=((I^{ij}))$ is the inverse matrix of $I_2(\theta)$, $i,j=1,2,3,4,5$ and ρ_i is the i^{th} unit column 5-vector, and $\nabla_t(\theta)$ is the gradient of $t(\theta)$.

Theorem 3.3

The matching prior for η_1 is given by

$$\begin{aligned}\pi_M(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5) &\propto (1-\eta_1-\eta_2)(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)][1+(p_2-1)\eta_2] \\ &\quad\times D(\eta_1,\eta_2)B(\eta_1,\eta_2)g(\eta_3,\eta_4,\eta_5),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}B(\eta_1,\eta_2) &= [k_1p_1[1+(p_2-1)\eta_2^2]+k_2p_2[1+(p_1-1)(\eta_1+\eta_2)^2]]^{-\frac{1}{2}} \\ &\quad\times[1+(p_1-1)(\eta_1+\eta_2)]^{-1}[1+(p_2-1)\eta_2]^{-1}(1-\eta_1-\eta_2)^{-1}(1-\eta_2)^{-1}\end{aligned} \tag{3.5}$$

and g is any positive differentiable function.

Proof

Let $t(\theta)=\theta_1-\theta_2$ in (3.3). Then $\nabla_t'(\theta)=(1\ \ -1\ \ 0\ \ 0\ \ 0)$.

Hence we can obtain following results :

$$\nabla_t'(\theta)I^{-1}{}_2(\theta)\nabla_t(\theta)=I^{11}-2I^{21}+I^{22}, \quad \rho_1' I^{-1}{}_2(\theta)\nabla_t(\theta)=I^{11}-I^{12},$$

$$\rho_2' I^{-1}{}_2(\theta)\nabla_t(\theta)=I^{21}-I^{22},$$

$$\rho_3' I^{-1}{}_2(\theta)\nabla_t(\theta)=\rho_4' I^{-1}{}_2(\theta)\nabla_t(\theta)=\rho_2' I^{-1}(\theta)\nabla_t(\theta)=0.$$

Thus, the above equation (3.3) becomes

$$\frac{\partial}{\partial\theta_1}\left\{\frac{I^{11}-I^{12}}{\sqrt{I^{11}-2I^{12}+I^{22}}}\pi(\theta)\right\}+\frac{\partial}{\partial\theta_2}\left\{\frac{I^{21}-I^{22}}{\sqrt{I^{11}-2I^{12}+I^{22}}}\pi(\theta)\right\}=0, \tag{3.6}$$

where

$$I^{11}=\frac{I_{22}}{I_{11}I_{22}-I_{12}^2}=\frac{\dfrac{k_2p_2(p_2-1)[k+k_2p_2(p_2-1)\theta_2^2]}{2k(1-\theta_2)^2[1+(p_2-1)\theta_2]^2}}{\dfrac{k_1p_1k_2p_2(p_1-1)(p_2-1)\{k_1p_1[1+(p_2-1)\theta_2^2]+k_2p_2[1+(p_1-1)\theta_1^2]\}}{4k^2[1+(p_1-1)\theta_1]^2(1-\theta_1)^2[1+(p_2-1)\theta_2]^2(1-\theta_2)^2}},$$

$$I^{12}=I^{21}=\frac{I_{12}}{I_{11}I_{22}-I^2_{12}}=\frac{-\dfrac{p_1k_1p_2k_2(p_1-1)(p_2-1)\theta_1\theta_2}{2k(1-\theta_1)(1-\theta_2)[1+(p_1-1)\theta_1][1+(p_2-1)\theta_2]}}{\dfrac{k_1p_1k_2p_2(p_1-1)(p_2-1)\{k_1p_1[1+(p_2-1)\theta^2_2]+k_2p_2[1+(p_1-1)\theta^2_1]\}}{4k^2[1+(p_1-1)\theta_1]^2(1-\theta_1)^2[1+(p_2-1)\theta_2]^2(1-\theta_2)^2}},$$

$$I^{22}=\frac{I_{22}}{I_{11}I_{22}-I^2_{12}}=\frac{\dfrac{k_1p_1(p_1-1)[k+k_1p_1(p_1-1)\theta^2_1]}{2k(1-\theta_1)^2[1+(p_1-1)\theta_1]^2}}{\dfrac{k_1p_1k_2p_2(p_1-1)(p_2-1)\{k_1p_1[1+(p_2-1)\theta^2_2]+k_2p_2[1+(p_1-1)\theta^2_1]\}}{4k^2[1+(p_1-1)\theta_1]^2(1-\theta_1)^2[1+(p_2-1)\theta_2]^2(1-\theta_2)^2}},$$

and

$I^{11}-2I^{12}+I^{22}=(I_{11}I_{22}-I^2_{12})^{-1}[D(\theta_1,\theta_2)]^2$,

where

$$[D(\theta_1,\theta_2)]^2=\left\{\frac{k_2p_2(p_2-1)}{2[1+(p_2-1)\theta_2]^2(1-\theta_2)^2}+\frac{k_1p_1(p_1-1)}{2[1+(p_1-1)\theta_1]^2(1-\theta_1)^2}\right.$$
$$\left.+\frac{1}{2k}\left[\frac{k_2p_2(p_2-1)\theta_2}{[1+(p_2-1)\theta_2](1-\theta_2)}-\frac{k_1p_1(p_1-1)\theta_1}{[1+(p_1-1)\theta_1](1-\theta_1)}\right]^2\right\}.$$

Let

$\pi_M(\theta)=(\sqrt{I^{11}-2I^{12}+I^{22}})(I_{11}I_{22}-I^2_{12})(1-\theta_1)(1-\theta_2)[1+(p_1-1)\theta_1][1+(p_2-1)\theta_2]g(\theta_3,\theta_4,\theta_5)$.

Then $\pi_M(\theta)$ is matching prior which is solution to the above equation (3.6).

Thus,

$$\pi_M(\theta)=\frac{D(\theta_1,\theta_2)}{\sqrt{I_{11}I_{22}-I^2_{12}}}(I_{11}I_{22}-I^2_{12})(1-\theta_1)(1-\theta_2)[1+(p_1-1)\theta_1][1+(p_2-1)\theta_2]g(\theta_1,\theta_2,\theta_3)$$
$$=D(\theta_1,\theta_2)(I_{11}I_{22}-I^2_{12})^{\frac{1}{2}}(1-\theta_1)(1-\theta_2)[1+(p_1-1)\theta_1][1+(p_2-1)\theta_2]g(\theta_1,\theta_2,\theta_3).$$

Therefore,

$$\pi_M(\theta)=\pi_M(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$$
$$\propto\left[\frac{k_1p_1k_2p_2(p_1-1)(p_2-1)\{k_1p_1[1+(p_2-1)\theta^2_2]+k_2p_2[1+(p_1-1)\theta^2_1]\}}{4k[1+(p_1-1)\theta_1]^2(1-\theta_1)^2[1+(p_2-1)\theta_2]^2(1-\theta_2)^2}\right]^{\frac{1}{2}}$$
$$\times\left\{\frac{k_2p_2(p_2-1)}{2[1+(p_2-1)\theta_2]^2(1-\theta_2)^2}+\frac{k_1p_1(p_1-1)}{2[1+(p_1-1)\theta_1]^2(1-\theta_1)^2}\right.$$
$$\left.+\frac{1}{2k}\left[\frac{k_2p_2(p_2-1)\theta_2}{[1+(p_2-1)\theta_2](1-\theta_2)}-\frac{k_1p_1(p_1-1)\theta_1}{[1+(p_1-1)\theta_1](1-\theta_1)}\right]^2\right\}^{\frac{1}{2}}$$
$$\times(1-\theta_1)(1-\theta_2)[1+(p_1-1)\theta_1][1+(p_2-1)\theta_2]g(\theta_3,\theta_4,\theta_5)$$

$$\propto [k_1p_1[1+(p_2-1)\,\theta^2_2]+k_2p_2[1+(p_1-1)\,\theta^2_1]^{1/2}g(\theta_3,\theta_4,\theta_5)$$
$$\times\left\{\frac{k_2p_2(p_2-1)}{2[1+(p_2-1)\theta_2]^2(1-\theta_2)^2}+\frac{k_1p_1(p_1-1)}{2[1+(p_1-1)\theta_1]^2(1-\ \theta_1^2}\right.$$
$$\left.+\frac{1}{2k}\left[\frac{k_2p_2(p_2-1)\theta_2}{[1+(p_2-1)\theta_2](1-\theta_2)}-\frac{k_1p_1(p_1-1)\theta_1}{[1+(p_1-1)\theta_1](1-\theta_1)}\right]^2\right\}^{\frac{1}{2}},$$

where $k=k_1p_1+k_2p_2$ and g is any positive differentiable function.

By using invariance property of probability matching priors under one-to-one transformation of the parameter vector, we obtain $\pi_M(\eta)$ in (3.4).

4. Posterior Distributions

In this chapter, we provide posterior distributions under the noninformative priors (3.1), (3.2), and (3.4).

Theorem 4.1

The posterior distributions π_J, π_R and π_M are given, respectively, by

$\pi_J(\eta|\,x,y)$

$$\propto(2\pi)^{-\frac{k}{2}}(1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}-\frac{1}{2}}$$
$$\times(1-\eta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\eta_2]^{\frac{k_2}{k}-\frac{1}{2}}\eta_3^{-\frac{k}{2}-2}B(\eta_1,\eta_2)$$
$$\times\exp\left\{-\frac{1}{2}(1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}}(1-\eta_2)^{\frac{(p_2-1)k_2}{k}}\right.$$
$$\times[1+(p_2-1)\eta_2]^{\frac{k_2}{k}}\eta^{-1}_3\left[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\eta_4)^2}{1+(p_1-1)(\eta_1+\eta_2)}+\frac{1}{1-\eta_1-\eta_2}\sum_{m=2}^{p_1}x^2_{im}\right)\right.$$
$$\left.\left.+\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\eta_5)^2}{1+(p_2-1)\eta_2}+\frac{1}{1-\eta_2}\sum_{n=2}^{p_2}y^2_{jn}\right)\right]\right\},$$

$$\propto(2\pi)^{-\frac{k}{2}}\eta^{-\frac{k}{2}-2}_3B(\eta_1,\eta_2)$$
$$\times\exp\left\{-\frac{1}{2}(1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}}(1-\eta_2)^{\frac{(p_2-1)k_2}{k}}\right.$$
$$\times[1+(p_2-1)\eta_2]^{\frac{k_2}{k}}\eta^{-1}_3\left[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\eta_4)^2}{1+(p_1-1)(\eta_1+\eta_2)}+\frac{1}{1-\eta_1-\eta_2}\sum_{m=2}^{p_1}x^2_{im}\right)\right.$$

$$+\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\eta_5)^2}{1+(p_2-1)\eta_2}+\frac{1}{1-\eta_2}\sum_{n=2}^{p_2}y^2_{jn}\right)\Bigg]\Bigg\}\text{, and}$$

$\pi_M(\eta|x,y)$

$$\propto(2\pi)^{-\frac{k}{2}}\eta_3^{-\frac{k}{2}}B(\eta_1,\eta_2)D(\eta_1,\eta_2)$$

$$\times(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)][1+(p_2-1)\eta_2]$$

$$\times\exp\Bigg\{-\frac{1}{2}(1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}}(1-\eta_2)^{\frac{(p_2-1)k_2}{k}}$$

$$\times[1+(p_2-1)\eta_2]^{\frac{k_2}{k}}\eta_3^{-1}\Bigg[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\eta_4)^2}{1+(p_1-1)(\eta_1+\eta_2)}+\frac{1}{1-\eta_1-\eta_2}\sum_{m=2}^{p_1}x^2_{im}\right)$$

$$+\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\eta_5)^2}{1+(p_2-1)\eta_2}+\frac{1}{1-\eta_2}\sum_{n=2}^{p_2}y^2_{jn}\right)\Bigg]\Bigg\}g(\eta_3,\eta_4,\eta_5),$$

where

$D(\eta_1,\eta_2)=$

$$\Bigg\{\frac{k_2p_2(p_2-1)}{2[1+(p_2-1)\eta_2]^2(1-\eta_2)^2}+\frac{k_1p_1(p_1-1)}{2[1+(p_1-1)(\eta_1+\eta_2)]^2(1-\eta_1-\eta_2)^2}$$

$$+\frac{1}{2k}\left[\frac{k_2p_2(p_2-1)\eta_2}{[1+(p_2-1)\eta_2](1-\eta_2)}-\frac{k_1p_1(p_1-1)(\eta_1+\eta_2)}{[1+(p_1-1)(\eta_1+\eta_2)](1-\eta_1-\eta_2)}\right]^2\Bigg\}^{\frac{1}{2}}, \quad (4.1)$$

$k=k_1p_1+k_2p_2$, and g is any positive differentiable function.

Proof

The likelihood function of $(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)$ is

$$l(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5|x,y)=(2\pi)^{-\frac{k}{2}}\eta_3^{-\frac{k}{2}}$$

$$\times\exp\Bigg\{-\frac{1}{2}(1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}}(1-\eta_2)^{\frac{(p_2-1)k_2}{k}}$$

$$\times[1+(p_2-1)\eta_2]^{\frac{k_2}{k}}\eta_3^{-1}\Bigg[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\eta_4)^2}{1+(p_1-1)(\eta_1+\eta_2)}+\frac{1}{1-\eta_1-\eta_2}\sum_{m=2}^{p_1}x^2_{im}\right)$$

$$+\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\eta_5)^2}{1+(p_2-1)\eta_2}+\frac{1}{1-\eta_2}\sum_{n=2}^{p_2}y^2_{jn}\right)\Bigg]\Bigg\},$$

where $k=k_1p_1+k_2p_2$.

Then we can easily obtain above results.

Theorem 4.2

The posterior distribution $\pi_J(\boldsymbol{\eta}|\boldsymbol{x},\boldsymbol{y})$ and $\pi_R(\boldsymbol{\eta}|\boldsymbol{x},\boldsymbol{y})$ are proper, and the posterior distribution $\pi_M(\boldsymbol{\eta}|\boldsymbol{x},\boldsymbol{y})$ is proper for $a < r$, $a \in R^1$.

Proof

We only prove the result for Matching prior π_M. It suffices to show that

$$\int_{-1-\frac{1}{p_1-1}}^{1+\frac{1}{p_2-1}} \int_{-\frac{1}{p_2-1}}^{1} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi_M(\boldsymbol{\eta}|\boldsymbol{x},\boldsymbol{y})\, d\eta_5\, d\eta_4\, d\eta_3\, d\eta_2\, d\eta_1 < \infty .$$

Now,

$$\pi_M(\eta_1,\eta_2|\boldsymbol{x},\boldsymbol{y}) = \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (2\pi)^{-\frac{k}{2}} \eta_3^{-\frac{k}{2}} B(\eta_1,\eta_2)\, D(\eta_1,\eta_2)$$

$$\times(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)][1+(p_2-1)\eta_2]$$

$$\times\exp\Bigg\{-\frac{1}{2}(1-\eta_1-\eta_2)^{-\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{-\frac{k_1}{k}}(1-\eta_2)^{-\frac{(p_2-1)k_2}{k}}$$

$$\times[1+(p_2-1)\eta_2]^{-\frac{k_2}{k}}\eta_3^{-1}\Bigg[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\eta_4)^2}{1+(p_1-1)(\eta_1+\eta_2)}+\frac{1}{1-\eta_1-\eta_2}\sum_{m=2}^{p_1}x_{im}^2\right)$$

$$+\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\eta_5)^2}{1+(p_2-1)\eta_2}+\frac{1}{1-\eta_2}\sum_{n=2}^{p_2}y_{jn}^2\right)\Bigg]\Bigg\}\, g(\eta_3,\eta_4,\eta_5)\, d\eta_5\, d\eta_4\, d\eta_3.$$

Here,

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\eta_3^{-1}Z(\eta_1,\eta_2)\left[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\eta_4)^2}{1+(p_1-1)(\eta_1+\eta_2)}+\frac{1}{1-\eta_1-\eta_2}\sum_{m=2}^{p_1}x_{im}^2\right)\right]\right\} g(\eta_3,\eta_4,\eta_5)\, d\eta_4$$

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{Z(\eta_1,\eta_2)\eta_3^{-1}}{2[1+(p_1-1)(\eta_1+\eta_2)]}\left[\sum_{i=1}^{k_1}x_{i1}^2-2\sqrt{p_1}\eta_4\sum_{i=1}^{k_1}x_{i1}+k_1p_1\eta_4^2\right]d\eta_4\right.$$

$$=\exp\left\{-\frac{Z(\eta_1,\eta_2)\eta_3^{-1}}{2}\left[\frac{\sum_{i=1}^{k_1}\sum_{m=2}^{p_1}x_{im}^2}{1-\eta_1-\eta_2}-\frac{\left(\sum_{i=1}^{k_1}x_{i1}\right)^2}{k_1[1+(p_1-1)(\eta_1+\eta_2)]}+\frac{\sum_{i=1}^{k_1}x_{i1}^2}{1+(p_1-1)(\eta_1+\eta_2)}\right]\right\}$$

$$\times\int_{-\infty}^{\infty}\exp\left\{\frac{-\left(\eta_4-\frac{\sum x_{i1}^2}{k_1\sqrt{p_1}}\right)^2}{\frac{2Z^{-1}(\eta_1,\eta_2)\,\eta_3^{-1}[1+(p_1-1)(\eta_1+\eta_2)]}{k_1p_1}}\right\}$$

$$=\exp\left\{-\frac{Z(\eta_1,\eta_2)\,\eta_3^{-1}}{2}\left[\frac{\sum_{i=1}^{k_1}\sum_{m=2}^{p_1}x_{im}^2}{1-\eta_1-\eta_2}-\frac{\left(\sum_{i=1}^{k_1}x_{i1}\right)^2}{k_1[1+(p_1-1)(\eta_1+\eta_2)]}+\frac{\sum_{i=1}^{k_1}x_{i1}^2}{1+(p_1-1)(\eta_1+\eta_2)}\right]\right\}$$

$$\times\left[\frac{2\pi[1+(p_1-1)(\eta_1+\eta_2)]}{k_1p_1}\right]^{\frac{1}{2}}Z^{-\frac{1}{2}}(\eta_1,\eta_2)\,\eta_3^{\frac{1}{2}}\, g(\eta_3,\eta_4,\eta_5).$$

Similarly,

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\eta^{-1}{}_3 Z(\eta_1,\eta_2)\left[\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\eta_5)^2}{1+(p_2-1)\eta_2}+\frac{1}{1-\eta_2}\sum_{n=2}^{p_2}y^2{}_{jn}\right)\right]\right\}g(\eta_3,\eta_4,\eta_5)d\eta_5$$

$$=\exp\left\{-\frac{Z(\eta_1,\eta_2)\,\eta^{-1}{}_3}{2}\left[\frac{\sum_{j=1}^{k_2}\sum_{n=2}^{p_2}y^2{}_{jn}}{1-\eta_2}-\frac{\left(\sum_{j=1}^{k_2}y_{j1}\right)^2}{k_2[1+(p_2-1)\eta_2]}+\frac{\sum_{j=1}^{k_2}y^2{}_{j1}}{1+(p_2-1)\eta_2}\right]\right\}$$

$$\times\left[\frac{2\pi[1+(p_2-1)\eta_2]}{k_2p_2}\right]^{\frac{1}{2}}Z^{-\frac{1}{2}}(\eta_1,\eta_2)\ \eta^{\frac{1}{2}}{}_3\,g(\eta_3,\eta_4,\eta_5),$$

where

$$Z(\eta_1,\eta_2)=(1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}}(1-\eta_2)^{\frac{(p_2-1)k_2}{k}}[1+(p_2-1)\eta_2]^{\frac{k_2}{k}}.$$

Then

$$\pi_M(\eta_1,\eta_2|\ \boldsymbol{x},\boldsymbol{y})\propto\int_0^{\infty}\eta_3^{-\frac{k}{2}+1}(1-\eta_1-\eta_2)(1-\eta_2)$$

$$\times[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{3}{2}}[1+(p_1-1)\eta_2]^{\frac{3}{2}}B(\eta_1,\eta_2)D(\eta_1,\eta_2)$$

$$\times\exp\left\{-\frac{Z(\eta_1,\eta_2)\eta_3^{-1}}{2}\left[\sum_{i=1}^{k_1}\left(\frac{(x_{i1}-\sqrt{p_1}\eta_4)^2}{1+(p_1-1)(\eta_1+\eta_2)}+\frac{1}{1-\eta_1-\eta_2}\sum_{m=2}^{p_1}x^2{}_{im}\right)\right.\right.$$

$$\left.\left.+\sum_{j=1}^{k_2}\left(\frac{(y_{j1}-\sqrt{p_2}\eta_5)^2}{1+(p_2-1)\eta_2}+\frac{1}{1-\eta_2}\sum_{n=2}^{p_2}y^2{}_{jn}\right)\right]\right\}Z^{-1}(\eta_1,\eta_2)g(\eta_3,\eta_4,\eta_5)d\eta_3.$$

Let $\eta^{-1}{}_3=t$, then

$$\int_0^{\infty}\eta_3^{-\frac{k}{2}+1}e^{-\frac{\eta_3^{-1}Z_1(\eta_1,\eta_2)C_1(\eta_1,\eta_2)}{2}}\eta_3^a d\eta_3=\int_0^{\infty}t^{\frac{k}{2}-3}e^{-\frac{tZ_1(\eta_1,\eta_2)C_1(\eta_1,\eta_2)}{2}}t^{-a}dt\ ,$$

where

$$C_1(\eta_1,\eta_2)=\Big[(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)][1+(p_2-1)\eta_2]\sum_{i=1}^{k_1}\sum_{m=2}^{p_1}x^2{}_{im}$$

$$-\frac{(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_2-1)\eta_2]}{k_1}\left(\sum_{i=1}^{k_1}x_{i1}\right)^2+(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_2-1)\eta_2]\sum_{i=1}^{k_1}x^2{}_{i1}$$

$$+(1-\eta_1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)][1+(p_2-1)\eta_2]\sum_{j=1}^{k_2}\sum_{n=2}^{p_2}y^2{}_{jn}$$

$$-\frac{(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)]}{k_2}\left(\sum_{j=1}^{k_2}y_{j1}\right)^2$$

$$+(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)]\sum_{j=1}^{k_2} y^2{}_{j1}\Big], \tag{4.2}$$

$$Z_1(\eta_1,\eta_2) = (1-\eta_1-\eta_2)^{\frac{(p_1-1)k_1}{k}-1}[1+(p_1-1)(\eta_1+\eta_2)]^{\frac{k_1}{k}-1}$$

$$\times(1-\eta_2)^{\frac{(p_2-1)k_2}{k}-1}[1+(p_2-1)\eta_2]^{\frac{k_2}{k}-1}, \tag{4.3}$$

and $g(\eta_3,\eta_4,\eta_5)=\eta_3^a,\ a\in R^1$. Since,

$$\frac{\Gamma(r-a)}{\lambda^{r-a}}\int_0^\infty \frac{\lambda^{r-a}}{\Gamma(r-a)}t^{r-a-1}e^{-\lambda t}dt = \lambda^{a-r}\Gamma(r-a),\ r>a,$$

where $r=\frac{k-4}{2}$ and $\lambda=\frac{Z_1C_1}{2}$.

Thus,

$$\pi_M(\eta_1,\eta_2\mid \boldsymbol{x},\ \boldsymbol{y}) \propto (1-\eta_1-\eta_2)^{-\frac{(p_1-1)k_1}{k}+1}(1-\eta_2)^{-\frac{(p_2-1)k_2}{k}+1}$$

$$\times[1+(p_1-1)(\eta_1+\eta_2)]^{-\frac{k_1}{k}+\frac{3}{2}}[1+(p_2-1)\eta_2]^{-\frac{k_2}{k}+\frac{3}{2}}$$

$$\times B(\eta_1,\eta_2)D(\eta_1,\eta_2)[C_1(\eta_1,\eta_2)Z_1(\eta_1,\eta_2)]^{a-r},\ r>a.$$

Let $\eta_1+\eta_2=a$, $\eta_2=b$, and then $1+(p_1-1)a=A,\ 1+(p_2-1)b=B$.

$$\pi_M(A,B\mid \boldsymbol{x},\ \boldsymbol{y}) \propto A^{\frac{k_1(a-r-1)-k(a-r)}{k}+\frac{3}{2}-1}B^{\frac{k_2(a-r-1)-k(a-r)}{k}+\frac{3}{2}-1}$$

$$\times\int_0^{p_2}\int_0^{p_1}\left(\frac{p_1-A}{p_1-1}\right)^{\frac{(p_1-1)k_1(a-r-1)-k(a-r)}{k}-1}\left(\frac{p_2-B}{p_2-1}\right)^{\frac{(p_2-1)k_2(a-r-1)-k(a-r)}{k}-1}$$

$$\times B_1(A,B)D_1(A,B)[C_1(A,B)]^{a-r}dAdB,\ r>a,$$

where

$$B_1(A,B) = [k_1p_1^2p_2^2+k_2p_1p_2^3-2k_1p_1^2p_2B+k_1p_1^2p_2B^2-k_2p_1p_2^2-2Ak_2p_2^3$$

$$+2Ak_2p_2^2+A^2k_2p_2^3-k_2p_2^2A^2-k_1p_1p_2^2+2Bp_2k_1p_1-B^2p_1p_2k_1]^{\frac{1}{2}},$$

$$D_1(A,B) = \Bigg[k_2p_2(p_2-1)A^2\left(\frac{p_1-A}{p_{1-1}}\right)^2+k_1p_1(p_1-1)A^2\left(\frac{p_2-B}{p_2-1}\right)^2$$

$$+\frac{1}{k}\left[k_1p_1A(B-1)\left(\frac{p_1-A}{p_1-1}\right)-k_2p_2B(A-1)\right]^2\left(\frac{p_2-B}{p_2-1}\right)\Bigg]^{-\frac{1}{2}}.$$

Since

$$[C_1(A,B)]^{a-r} = \Bigg[(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)][1+(p_2-1)\eta_2]\sum_{i=1}^{k_1}\sum_{m=2}^{p_1}x^2{}_{im}$$
$$-\frac{(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_2-1)\eta_2]}{k_1}\left(\sum_{i=1}^{k_1}x_{i1}\right)^2$$
$$+(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_2-1)\eta_2]\sum_{i=1}^{k_1}x^2{}_{i1}$$
$$+(1-\eta_1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)][1+(p_2-1)\eta_2]\sum_{j=1}^{k_2}\sum_{n=2}^{p_2}y^2{}_{jn}$$
$$-\frac{(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)]}{k_2}\left(\sum_{j=1}^{k_2}y_{j1}\right)^2$$
$$+(1-\eta_1-\eta_2)(1-\eta_2)[1+(p_1-1)(\eta_1+\eta_2)]\sum_{j=1}^{k_2}y^2{}_{j1}\Bigg]^{a-r}$$

has maximum value $F(p_1,p_2,k_1,k_2|\ \boldsymbol{x},\ \boldsymbol{y})$.
Thus, $B_1(A,B)[C_1(A,B)]^{a-r}D_1(A,B)$ has maximum value $M(p_1,p_2,k_1,k_2|\ \boldsymbol{x},\ \boldsymbol{y})$.

Let $\frac{p_1-A}{p_1}=u$ and $\frac{p_2-B}{p_2}=v$, then

$\pi_M(u,v|\ \boldsymbol{x},\boldsymbol{y}) \propto$

$$\int_0^1\int_0^1 u^{\frac{(p_1-1)k_1(a-r-1)-k(a-r)}{k}-1}\, v^{\frac{(p_2-1)k_2(a-r-1)-k(a-r)}{k}-1}$$
$$\times(1-u)^{\frac{k_1(a-r-1)-k(a-r)}{k}+\frac{3}{2}-1}(1-v)^{\frac{k_2(a-r-1)-k(a-r)}{k}+\frac{3}{2}-1}$$
$$\times M(p_1,p_2,k_1,k_2|\ \boldsymbol{x},\ \boldsymbol{y})du dv < \infty \text{ for } r > a .$$

The remaining cases can be similarly verified.

Next, we provide the marginal posterior distribution of η_1 under priors π_J, π_R and π_M .

Theorem 4.3
Under the noninformative priors π_J, π_R and π_M the marginal posterior distribution of η_1 are, respectively,

$$\propto \int_{-\frac{1}{p_2-1}}^{1} B(\eta_1,\eta_2)[Z_1(\eta_1,\eta_2)C_1(\eta_1,\eta_2)]^{-\frac{k}{2}}d\eta_2\,,$$

$$\pi_R(\eta_1|\ \boldsymbol{x},\boldsymbol{y}) = \int_{-\frac{1}{p_2-1}}^{1}\pi_R(\eta_1,\eta_2|\ \boldsymbol{x},\ \boldsymbol{y})d\eta_2$$

$$\propto \int_{-\frac{1}{p_2-1}}^{1} [1+(p_1-1)(\eta_1+\eta_2)]^{-\frac{1}{2}}[1+(p_2-1)\eta_2]^{-\frac{1}{2}}$$

$$\times B(\eta_1,\eta_2)[Z_1(\eta_1,\eta_2)]^{-1}[Z_1(\eta_1,\eta_2)C_1(\eta_1,\eta_2)]^{-\frac{k}{2}} d\eta_2 ,$$

and

$$\pi_M(\eta_1 | \, \boldsymbol{x}, \boldsymbol{y}) = \int_{-\frac{1}{p_2-1}}^{1} \pi_M(\eta_1,\eta_2 | \, \boldsymbol{x}, \boldsymbol{y}) d\eta_2$$

$$\propto \int_{-\frac{1}{p_2-1}}^{1} [1+(p_1-1)(\eta_1+\eta_2)]^{-\frac{1}{2}}[1+(p_2-1)\eta_2]^{-\frac{1}{2}}$$

$$\times B(\eta_1,\eta_2)D(\eta_1,\eta_2)[Z_1(\eta_1,\eta_2)]^{-1}[Z_1(\eta_1,\eta_2)C_1(\eta_1,\eta_2)]^{a-r} d\eta_2 , \quad r > a ,$$

where

$B(\eta_1,\eta_2)$, $D(\eta_1,\eta_2)$, $Z_1(\eta_1,\eta_2)$, and $C_1(\eta_1,\eta_2)$ are from (3.5) , (4.1) , (4.2), and (4.3), respectively and $r = \frac{k-4}{2}$, $k = k_1p_1 + k_2p_2$.

5. A Simulation Study

It is assumed that

$$X_i \sim N_{p_1}(\mu_1, \Sigma_1) , \; i=1,2,\cdots,k_1 \text{ and } \; Y_j \sim N_{p_2}(\mu_2, \Sigma_2) , \; j=1,2,\cdots,k_2 .$$

We first generate random variables $(X_1,\cdots,X_{k_1})$, $(Y_1,\cdots,Y_{k_2})$ from p_1-variate normal distribution and p_2-variate normal distribution.

Let $\eta_1 = \rho_1 - \rho_2$ and $\eta_2 = \rho_2$ .Then the marginal posterior distribution of η_1 given $\boldsymbol{x,y}$ is

$$\pi_J(\eta_1 | \, \boldsymbol{x}, \boldsymbol{y}) = \int_{-\frac{1}{p_2-1}}^{1} \pi_J(\eta_1,\eta_2 | \, \boldsymbol{x}, \boldsymbol{y}) d\eta_2$$

$$\pi_R(\eta_1 | \, \boldsymbol{x}, \boldsymbol{y}) = \int_{-\frac{1}{p_2-1}}^{1} \pi_R(\eta_1,\eta_2 | \, \boldsymbol{x}, \boldsymbol{y}) d\eta_2$$

$$\propto \int_{-\frac{1}{p_2-1}}^{1} [1+(p_1-1)(\eta_1+\eta_2)]^{-\frac{1}{2}}[1+(p_2-1)\eta_2]^{-\frac{1}{2}}$$

$$\times B(\eta_1,\eta_2)[Z_1(\eta_1,\eta_2)]^{-1}[Z_1(\eta_1,\eta_2)C_1(\eta_1,\eta_2)]^{-\frac{k}{2}} d\eta_2 ,$$

and

$$\pi_M(\eta_1 | \, \boldsymbol{x}, \boldsymbol{y}) = \int_{-\frac{1}{p_2-1}}^{1} \pi_M(\eta_1,\eta_2 | \, \boldsymbol{x}, \boldsymbol{y}) d\eta_2$$

$$\propto \int_{-\frac{1}{p_2-1}}^{1} [1+(p_1-1)(\eta_1+\eta_2)]^{-\frac{1}{2}}[1+(p_2-1)\eta_2]^{-\frac{1}{2}}$$
$$\times B(\eta_1,\eta_2)D(\eta_1,\eta_2)[Z_1(\eta_1,\eta_2)]^{-1}[Z_1(\eta_1,\eta_2)C_1(\eta_1,\eta_2)]^{a-r}d\eta_2\,,\ r>a\,.$$

Table 5.1 provides frequentist coverage probabilities of $0.05(0.95)$ posterior quantile for η_1 under different value of k_1, p_1, k_2 and p_2. The computation of those numerical values is based on the following algorithm for any fixed true η_1. Here α is $0.05(0.95)$.

Let $\eta_1^{\alpha}(\alpha|\ X,\ Y)$ be the posterior α^{th} quantile of η_1 given X and Y Then the frequentist coverage probability of credible interval for η_1 is

$$P_{\eta}(\alpha;\eta_1) = P\left(-1-\frac{1}{p_1-1}<\eta_1\le\eta_1^{\pi}(\alpha|\ X,\ Y)\right).$$

The estimated $P(\alpha;\eta_1)$ for $\alpha=0.05(0.95)$ is shown in Table 5.1.

Note that under a prior π, for fixed X_i, Y_j, $\eta_1\le\eta_1^{\pi}(\alpha|\ X,\ Y)$ if and only if $F(\eta_1^{\pi}(\alpha|\ X,\ Y)|\ X,\ Y)\le\alpha$. Under a prior π, $P(\alpha;\eta_1)$ can be estimated by the relative frequency of $F(\eta_1^{\pi}(\alpha|\ X,\ Y))\le\alpha$.

Table 5.1 Frequentist coverage probability of $0.05(0.95)$ posterior quantiles.

$\mu=0$ $\sigma^2=1$ $p_1=p_2=3$ $\eta_1=-0.7$ $a=-4$					$\mu=0$ $\sigma^2=1$ $p_1=p_2=3$ $\eta_1=0.5$ $a=-3$				
k_1	k_2	Jeffreys	Referecne	Matching	k_1	k_2	Jeffreys	Referecne	Matching
4	4	0.0380 (0.9226)	0.0547 (0.9862)	0.0555 (0.9587)	4	4	0.0622 (0.9425)	0.0285 (0.9537)	0.0519 (0.9440)
4	5	0.0362 (0.9516)	0.0517 (0.9853)	0.0514 (0.9617)	4	5	0.0500 (0.9415)	0.0245 (0.9486)	0.0450 (0.9405)
5	4	0.0411 (0.9431)	0.0627 (0.9855)	0.0602 (0.9650)	5	4	0.0737 (0.9498)	0.0324 (0.9587)	0.0573 (0.9521)
5	5	0.0386 (0.9255)	0.0572 (0.9801)	0.0546 (0.9579)	5	5	0.0574 (0.9437)	0.0315 (0.9503)	0.0510 (0.9435)
5	7	0.0395 (0.9121)	0.0548 (0.9754)	0.0543 (0.9555)	5	7	0.0444 (0.9419)	0.0278 (0.9442)	0.0433 (0.9386)
6	3	0.0464 (0.9567)	0.0739 (0.9905)	0.0685 (0.9687)	6	3	0.1045 (0.9585)	0.0352 (0.9685)	0.0631 (0.9639)
6	4	0.0397 (0.9409)	0.0635 (0.9829)	0.0599 (0.9595)	6	4	0.0760 (0.9521)	0.0365 (0.9611)	0.0576 (0.9569)
6	5	0.0440 (0.9359)	0.0605 (0.9809)	0.0583 (0.9592)	6	5	0.0646 (0.9499)	0.0345 (0.9556)	0.0554 (0.9522)
6	6	0.0412 (0.9285)	0.0574 (0.9768)	0.0550 (0.9574)	6	6	0.0562 (0.9456)	0.0335 (0.9505)	0.0519 (0.9465)
7	5	0.0408 (0.9409)	0.0611 (0.9800)	0.0570 (0.9602)	7	5	0.0684 (0.9503)	0.0390 (0.9562)	0.0547 (0.9545)

$\mu=0 \quad \sigma^2=1 \quad p_1=p_2=4 \quad \eta_1=-0.6 \quad a=-5$

k_1	k_2	Jeffreys	Referecne	Matching
4	4	0.0334 (0.9151)	0.0470 (0.9794)	0.0531 (0.9518)
4	5	0.0337 (0.8970)	0.0491 (0.9768)	0.0545 (0.9476)
5	4	0.0382 (0.9253)	0.0536 (0.9784)	0.0567 (0.9524)
5	5	0.0326 (0.9172)	0.0495 (0.9751)	0.0533 (0.9496)
5	7	0.0371 (0.9101)	0.0483 (0.9719)	0.0540 (0.9533)
6	3	0.0390 (0.9427)	0.0600 (0.9836)	0.0613 (0.9572)
6	4	0.0390 (0.9338)	0.0569 (0.9774)	0.0594 (0.9544)
6	5	0.0389 (0.9294)	0.0539 (0.9761)	0.0562 (0.9565)
6	6	0.0406 (0.9230)	0.0552 (0.9716)	0.0578 (0.9526)
7	5	0.0397 (0.9324)	0.0589 (0.9753)	0.0606 (0.9567)

$\mu=0 \quad \sigma^2=1 \quad p_1=p_2=4 \quad \eta_1=0.5 \quad a=-4$

k_1	k_2	Jeffreys	Referecne	Matching
4	4	0.0876 (0.9622)	0.0294 (0.9576)	0.0522 (0.9535)
4	5	0.0757 (0.9617)	0.0286 (0.9548)	0.0499 (0.9535)
5	4	0.1022 (0.9642)	0.0359 (0.9598)	0.0549 (0.9554)
5	5	0.0846 (0.9657)	0.0315 (0.9591)	0.0500 (0.9567)
5	7	0.0661 (0.9591)	0.0296 (0.9518)	0.0427 (0.9495)
6	3	0.1297 (0.9663)	0.0340 (0.9637)	0.0528 (0.9602)
6	4	0.1297 (0.9663)	0.0353 (0.9605)	0.0493 (0.9587)
6	5	0.0899 (0.9629)	0.0412 (0.9582)	0.0539 (0.9541)
6	6	0.0791 (0.9630)	0.0359 (0.9562)	0.0500 (0.9541)
7	5	0.0935 (0.9645)	0.0387 (0.9613)	0.0493 (0.9587)

$\mu=0 \quad \sigma^2=1 \quad p_1=p_2=5 \quad \eta_1=-0.6 \quad a=-5$

k_1	k_2	Jeffreys	Referecne	Matching
2	3	0.0224 (0.8342)	0.0357 (0.9984)	0.0421 (0.9447)
3	3	0.0280 (0.8989)	0.0461 (0.9905)	0.0517 (0.9576)
3	4	0.0281 (0.8782)	0.0429 (0.9880)	0.0493 (0.9566)
3	5	0.0271 (0.8697)	0.0383 (0.9842)	0.0455 (0.9597)
4	4	0.0325 (0.9023)	0.0476 (0.9793)	0.0515 (0.9552)
4	5	0.0347 (0.8972)	0.0518 (0.9783)	0.0564 (0.9593)
5	4	0.0339 (0.9154)	0.0529 (0.9770)	0.0550 (0.9552)
5	5	0.0331 (0.9132)	0.0496 (0.9755)	0.0533 (0.9583)
6	3	0.0352 (0.9317)	0.0577 (0.9796)	0.0582 (0.9582)
6	4	0.0364 (0.9281)	0.0575 (0.9771)	0.0589 (0.9589)

$\mu=0 \quad \sigma^2=1 \quad p_1=p_2=5 \quad \eta_1=0.6 \quad a=-5$

k_1	k_2	Jeffreys	Referecne	Matching
3	3	0.1031 (0.9619)	0.0122 (0.9447)	0.0460 (0.9404)
4	2	0.2023 (0.9735)	0.0036 (0.9612)	0.0588 (0.9563)
4	4	0.0929 (0.9696)	0.0192 (0.9517)	0.0439 (0.9483)
4	5	0.0838 (0.9649)	0.0204 (0.9455)	0.0417 (0.9433)
5	2	0.2170 (0.9730)	0.0056 (0.9603)	0.0568 (0.9550)
5	4	0.1011 (0.9645)	0.0200 (0.9490)	0.0415 (0.9438)
5	5	0.0872 (0.9651)	0.0232 (0.9481)	0.0409 (0.9431)
6	3	0.1371 (0.9666)	0.0154 (0.9524)	0.0401 (0.9469)
6	4	0.1119 (0.9682)	0.0224 (0.9545)	0.0408 (0.9496)
6	5	0.1001 (0.9657)	0.0281 (0.9503)	0.0461 (0.9456)

For most of the cases presented in Table 5.1, we see that the matching prior π_M in Theorem 4.3 performs better than π_J, π_R in meeting the target coverage probabilities.

Reference

[1] Berger, J.O. and Bernardo, J.M.(1989). Estimating a product of means : Bayesian analysis with reference priors. *Journal of the American Statistical Association*, 84, 200-207.

[2] Berger, J.O. and Bernardo, J.M.(1992). On the Development of Reference Priors(with discussion). *Bayesian Statistics 4*(Bernardo, J.M. et al., eds.). Oxford University Press, Oxford, 35-60.

[3] Bernardo, J.M. (1979). Reference posterior distributions for Bayesian inference. *Journal of Royal Statistical Society* (Ser.B), 113-147.

[4] Datta, G.S. and Ghosh, J,K. (1995). On priors providing frequentist validity for Bayesian inference. *Biometrika*. 82, 37-45.

[5] Donner, A. and Bull ,S. (1983). Inferences concerning a common intraclass correlation coefficient. *Biometrics*, 39, 771-775.

[6] Donner, A. and Koval, J.J. (1980). The Estimation of Intraclass Correlation in the Analysis of Family Data. *Biometrics*, 19-25.

[7] Kim, D.H., Kang, S.G., and Lee, W.D. (2001). Noninformative Priors for Intraclass Correlation Coefficient in Familial Data. *Far East Journal of Theoretical Statistical*, 5, 51-65.

[8] Peers, H.W. (1965). On Confidence Sets and Bayesian Probability Points in the Case of Several Parameters. *Journal of Royal Statistical Society* (Ser B), 27, 9-16.

[9] Rao, C.R. (1973). *Linear Statistical Inference and Its Applications*. New York: Wiley.

[10] Rosner, B., Donner, A. and Honnekens, C.H. (1977). Estimation of intraclass correlations from familial data. *Applied Statistics*, 26, No.2, 179-187.

[11] Srivastava, M.S. (1984). Estimation of intraclass correlation in familial data. *Biometrica*, 71, 177-185.

[12] Srivastava, M.S. and Kaptapa, R.S. (1986). Comparison of estimators of interclass and intraclass correlations from familial data. *Canadian Journal of Statistics*, 14, 29-42.

[13] Young, D.J. and Bhandary, M. (1998). Test for equality of intraclass correlation coefficients under unequal family sizes. *Biometrics*, 54, 1363-1373.

[Received December 2004, Accepted March 2005]