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T-upper approximation spaces

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Abstract

We define extensional spaces. Moreover, we investigate the relations among T-upper-approximation spaces, T-quasi-equivalence relations and extensional spaces.

Key Words : T-upper-approximation spaces, T-quasi-equivalence relations, extensional spaces. E-maps, c-maps, A-maps.

1. Introduction

Zadeh[13] introduced the concept of fuzzy equivalence relations. It has a significant concern in various fields.

The rough set concept proposed by Pawlak [10] is a new mathematical approach to imprecision, vagueness and uncertainty. Yao [12,13] investigated algebraic structures of rough sets as upper approximation operators. Dubois and Prade [4,5] introduced fuzzy rough sets as a fuzzy generalization of rough sets.

In this paper, we define extensional spaces. Moreover, we investigate the relations among T-upper-approximation spaces, T-quasi-equivalence relations, extensional spaces.

2. Preliminaries

Definition 2.1 A binary operation $T: [0,1]\times[0,1]\to[0,1]$ is called a t-norm if it satisfies the following conditions: for each $x, y, z\in[0,1]$,

(T1) $T(x,y) = T(y,z)$,

(T2) $T(x, T(y,z)) = T(T(x,y), z)$

(T3) $T(x,1) = x$,

(T4) if $y\le z$, then $T(x,y)\le T(x,z)$.

We denote $T(x,y) = x\odot y$.

Definition 2.2 Let T be a t-norm. A binary operation $\to: [0,1]\times[0,1]\to[0,1]$ is called a residual implication on X defined by

$$x\to y = \bigvee\{z\in[0,1] \mid T(x,z)\le y\}$$

Theorem 2.3 [2] Let $\odot$ be a t-norm. Then the following statements are equivalent:

(1) $\odot$ is left-continuous;

(2) $x\odot(x\to y)\le y$ for all $x,y\in[0,1]$;

(3) $x\le(y\to z)$ iff $x\odot y\le z$ for all $x,y,z\in[0,1]$;

(4) $(x\to y)\odot(y\to z)\le(x\to z)$ for all $x,y,z\in[0,1]$.

In this paper, we assume that $\odot$ is left continuous.

Definition 2.4.[14] A map $E: X\times X\to[0,1]$ is called a T-quasi-equivalence relation on X if the following properties hold:

(E1) $E(x,x) = 1$, for each $x\in X$,

(E2) $T(E(x,y), E(y,z))\le E(x,z)$, for each $x,y,z\in X$.

A T-fuzzy quasi-equivalence relation is called a T-equivalence relation on X if it satisfies:

(E3) $E(x,y) = E(y,x)$, for each $x,y\in X$.

A T-fuzzy equivalence relation is called a T-equality on X if it satisfies:

(E) if $E(x,y)=1$ for each $x,y\in X$, then $x=y$.

Let (X, E_1) and (Y, E_2) be $\odot$-fuzzy quasi-equivalence relations. A function $\phi: X\to Y$ is called E-map if $E_1(x,y)\le E_2(\phi(x), \phi(y))$ for each $(x,y)\in X\times X$.

Remark 2.5 (1) If a t-norm T_1 is weaker than a t-norm T_2, then a T_2-fuzzy (quasi-)equivalence E on X is a T_1-fuzzy (quasi-)equivalence E on X. Thus, $\wedge$-fuzzy (quasi-)equivalence E on X is a T-fuzzy (quasi-)equivalence E on X because $T(x,y)\le x\wedge y$ for every t-norm T.

(2) Let E be a T-fuzzy quasi-equivalence relation on X. Define $E^{-1}(x,y) = E(y,x)$ for all $x,y\in X$. Then E^{-1} is a T-fuzzy quasi-equivalence relation on X.

3. T-upper approximation operators

Definition 3.1 [9] An operator $c: [0,1]^X\to[0,1]^X$ is called an $\odot$-upper quasi-approximation operator on X if it satisfies the following conditions:

(C1) $1_x\le c(1_x)$,

(C2) $\bigvee_{z\in X}(c(1_x)(z)\odot c(1_z)(y))\le c(1_x)(y)$,

(C3) $c(\bigvee_{j\in J}\mu_j) = \bigvee_{j\in J} c(\mu_j)$,

(C4) $c(a\odot\mu) = a\odot c(\mu)$ where $a(x) = a$.

The pair (X, c) is an $\odot$-upper quasi-approximation space.

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An operator c is called an $\odot$-upper approximation operator on X if it satisfies:

(C) $c(1_x)(y)=c(1_y)(x)$,

Let (X,c_1) and (Y,c_2) be $\odot$-upper quasi approximation spaces. A function $\phi:X\to Y$ is called a c-map if $\phi(c_1(\mu))\leq c_2(\phi(\mu))$ for each $\mu\in[0,1]^X$.

Theorem 3.2 Let (X,c) be an $\odot$-upper quasi-approximation space. Then, for all $\mu\in[0,1]^X$.

(1) $1_x\leq c(1_x)$, for all $x\in X$ iff $\mu\leq c(\mu)$,

(2) it satisfies (C2) iff $c(c(\mu))=c(\mu)$.

Proof.(1) Since $\mu(x)=\bigvee_{z\in X}(\mu(z)\odot 1_z)(x))$, then

$$\begin{aligned} c(\mu)(x) &= c(\bigvee_{z\in X}(\mu(z)\odot 1_z)(x))) \\ &= \bigvee_{z\in X}(\mu(z)\odot c(1_z)(x)). \\ &\geq \bigvee_{z\in X}(\mu(z)\odot \bigvee_{y\in X}(c(1_z)(y)\odot c(1_y)(x)). \\ &= \bigvee_{z\in X}\bigvee_{y\in X}(\mu(z)\odot(c(1_z)(y)\odot c(1_y)(x))). \\ &= \bigvee_{y\in X}\bigvee_{z\in X}(\mu(z)\odot c(1_z)(y))\odot c(1_y)(x))). \\ &= \bigvee_{y\in X}(\bigvee_{z\in X}(\mu(z)\odot c(1_z)(y))\odot c(1_y)(x))). \\ &= \bigvee_{y\in X}(c(\mu)(y)\odot c(1_y)(x))). \\ &= c(c(\mu))(x). \end{aligned}$$

Theorem 3.3 Let (X,E)be an $\odot$-quasi-equivalence relation space. Define $c_E:[0,1]^X\to[0,1]^X$ as follows:

$$c_E(\mu)(x)=\bigvee_{z\in X}(\mu(z)\odot E(z,x))$$

Then c_E is an $\odot$-upper quasi-approximation operator on X.

Proof (C2)

$$\begin{aligned} &\bigvee_{z\in X}(c(1_x)(z)\odot c(1_z)(y)) \\ &= \bigvee_{z\in X}\{(\bigvee_{w\in X}(1_x(w)\odot E(w,z)))\odot(\bigvee_{p\in X}(1_z(p)\odot E(p,y)))\} \\ &= \bigvee_{z\in X}(E(x,z)\odot E(z,y)) \\ &\leq E(x,y)=\bigvee_{z\in X}(1_x(z)\odot E(z,y) \\ &= c(1_x)(z). \end{aligned}$$

Other cases are easily proved.

Corollary 3.4. Let (X,E)be an $\odot$-quasi equivalence relation space. Define $c_{E^{-1}}:[0,1]^X\to[0,1]^X$ as

$$c_{E^{-1}}(\mu)(x)=\bigvee_{z\in X}(\mu(z)\odot E(x.z))$$

Then $c_{E^{-1}}$ is an $\odot$-upper quasi-approximation operator on X.

Definition 3.5 [3] Let E be a quasi-equivalence relation on X. A fuzzy set $\mu\in I^X$ is called:

(1) left-extensional with respect to E on X if $\mu(x)\odot E(x,y)\leq\mu(y)$ for all $x,y\in[0,1]$. A fuzzy set $\overline{\mu}\in I^X$ is called the left-extensional hull defined as

$\overline{\mu}=\bigwedge\{\rho\mid\mu\leq\rho,\ \rho$ is left-extensional w,r.t $E\}$

(2) right-extensional with respect to E on X if $\mu(y)\odot E(x,y)\leq\mu(x)$ for all $x,y\in[0,1]$. A fuzzy set $\overline{\mu^{-1}}\in I^X$ is called the right-extensional hull defined as

$\overline{\mu^{-1}}=\bigwedge\{\rho\mid\mu\leq\rho,\ \rho$ right-extensional w,r.t $E\}$

Example 3.6 Let $X=\{x,y,z\}$ be a set and

$x\odot y=(x+y-1)\vee 0$ and $x\to y=(1-x+y)\wedge 1$

for all $x,y\in[0,1]$. Define an $\odot$-fuzzy quasi-equivalence relation E on X as follows:

$E(x,x)=E(y,y)=E(z,z)=1, E(x,y)=0.8, E(y,x)=0.7,$
$E(x,z)=0.6, E(y,z)=0.7, E(z,y)=0.9, E(z,x)=0.7$

For $\mu(x)=0.7, \mu(y)=0.1, \mu(z)=0.3,$

$0.5=\mu(x)\odot E(x,y)>\mu(y)=0.1$

So, μ is not left-extensional with respect to E

Definition 3.7 A subset Ω of $[0,1]^X$ is called an it extensional system on X if it satisfies: for each $\{\mu_i\}_{i\in I}\subset\Omega, \mu\in\Omega$

(A1) $\bigvee\mu_i\in\Omega$.

(A2) $\bigwedge\mu_i\in\Omega$,

(A3) $a\odot\mu\in\Omega$,

(A4) $(a\to\mu)\in\Omega$.

The pair (X,Ω) is called an extensional space. Let Ω_1 and Ω_2 be extensional systems on X.

The triple (X,Ω_1,Ω_2) is a called bi-extensional space.

Let (X,Ω_1) and (Y,Ω_2) be extensional spaces.

A function $\phi:X\to Y$ is called an A-map if $\phi^{-1}(\mu)\in\Omega_1$ for each $\mu\in\Omega_2$.

Theorem 3.8 Let E be an $\odot$-quasi-equivalence relation on X and let $\mu\in[0,1]^X$. Then

(1) $\overline{\mu}(x)=c_E(\mu)(x)=\bigvee_{z\in X}(\mu(z)\odot E(z,x))$

(2) $\overline{\mu}$ is left-extensional w.r.t E

(3) $\overline{\overline{\mu}}=\overline{\mu}$.

(4) If $\overline{\mu}=\mu$, then $\overline{a\to\mu}=a\to\mu$, $\overline{a\odot\mu}=a\odot\mu$.

(5) μ is left-extensional w.r.t E iff $E(x,y)\leq\mu(x)\to\mu(y)$.

Proof (1) $c_E(\mu)$ is extensional w.r.t. E because

$$\begin{aligned} c_E(\mu)(x)\odot E(x,y) &= (\bigvee_{z\in X}(\mu(z)\odot E(z,x)))\odot E(x,y) \\ &= \bigvee_{z\in X}(\mu(z)\odot(E(z,x)\odot E(x,y)) \\ &\leq \bigvee_{z\in X}(\mu(z)\odot(E(z,y)) \\ &= c_E(\mu)(y). \end{aligned}$$

$$\mu(x)=\mu(x)\odot E(x,x)\leq\bigvee_{z\in X}(\mu(z)\odot E(z,x))=c_E(\mu)(x)$$

If $\mu\leq\rho$ and ρ is extensional w.r.t E, then $c_E(\mu)\leq\rho$ because

$$c_E(\mu)(x)=\bigvee_{z\in X}(\mu(z)\odot E(z,x))\le\bigvee_{z\in X}(\rho(z)\odot E(z,x))\le\rho(x)$$

(4) Since $(p\to q)\odot r\le p\to(q\odot r)$,

$$\begin{aligned}\overline{a\to\mu}(x)&=(\bigvee_{z\in X}((a\to\mu)(z)\odot E(z,x)))\\&\le(\bigvee_{z\in X}a\to(\mu(z)\odot E(z,x)))\\&=a\to(\bigvee_{z\in X}(\mu(z)\odot E(z,x)))\\&=a\to\overline{\mu}(x)\\&=a\to\mu(x).\end{aligned}$$

Other cases are easy.

Corollary 3.9 Let E be an $\odot$-quasi-equivalence relation on X and let $\mu\in[0,1]^X$. Then

(1) $\overline{\mu^{-1}}(x)=c_E(\mu)(x)=\bigvee_{z\in X}(\mu(z)\odot E(x,z))$

(2) $\overline{\mu^{-1}}$ is right-extensional w.r.t E

(3) $\overline{\overline{\mu^{-1}}^{-1}}=\overline{\mu^{-1}}$.

(4) If $\overline{\mu^{-1}}=\mu$, then
$\overline{a\to\mu^{-1}}=a\to\mu$, $\overline{a\odot\mu^{-1}}=a\odot\mu$.

(5) μ is right-extensional w.r.t E iff
$E(x,y)\le\mu(y)\to\mu(x)$.

Theorem 3.10 Let E be an $\odot$- quasi-equivalence relation on X and Ω_E denote the collection of fuzzy sets that are left-extensional w.r.t. E. Then

(1) (X,Ω_E) is an extensional space.

(2) If E is a equivalence relation on X, then $(\mu\to a)\in\Omega_E$ for $\mu\in\Omega_E$ and $a\in[0,1]$.

Proof (1)(A1) For all $\mu_i\in\Omega_E$,

$(\bigvee_{i\in\Gamma}\mu_i(x))\odot E(x,y)=\bigvee_{i\in\Gamma}(\mu_i(x)\odot E(x,y))\le\bigvee_{i\in\Gamma}\mu_i(y)$.

(A2) and (A3) are easy.

(A4) $(a\odot(a\to\mu(x)))\odot E(x,y)\le\mu(x)\odot E(x,y)\le\mu(y)$

(2) $(\mu(y)\odot(\mu(x)\to a))\odot E(x,y)\le\mu(x)\odot(\mu(x)\to a)\le a$

Corollary 3.11 Let E be an $\odot$- quasi-equivalence relation on X and $\Omega_{E^{-1}}$ denote the collection of fuzzy sets that are right-extensional w.r.t. E. Then

(1) $(X,\Omega_{E^{-1}})$ is an extensional space.

(2) $(\mu\to a)\in\Omega_E$ for $\mu\in\Omega_{E^{-1}}$ and $a\in[0,1]$.

Remark 3.12 Let E be an $\odot$- quasi-equivalence relation on X. The triple $(X,\Omega_E,\Omega_{E^{-1}})$ is a bi-extensional space.

Definition 3.13 Let $\odot,\otimes$ be t-norms. $\otimes$ dominates $\odot$ if for each $x_1,x_2,y_1,y_2\in[0,1]$
$(x_1\odot,y_1)\otimes(x_2\odot y_2))\ge(x_1\otimes x_2)\odot(y_1\otimes y_2)$.

Theorem 3.14 Let $\odot,\otimes$ be t-norms. $\otimes$ dominates $\odot$. Let $\Omega=\{h_j\mid j\in J\}$ be an extensional system. Then

(1) There exists an unique quasi-equivalence relation E_Ω on X such that $\Omega=\Omega_{E_\Omega}$ defined as

$$E_\Omega(x,y)=\bigwedge_{j\in J}(h_j(x)\to h_j(y))$$

where Ω_{E_Ω} is the collection of fuzzy sets that are left-extensional w.r.t. E_Ω.

(2) There exists an unique quasi-equivalence relation E_Ω^{-1} on X such that $\Omega=\Omega_{E_\Omega^{-1}}$ defined as

$$E_\Omega^{-1}(x,y)=\bigwedge_{j\in J}(h_j(y)\to h_j(x))$$

where $\Omega_{E_\Omega^{-1}}$ is the collection of fuzzy sets that are right-extensional w.r.t. E_Ω^{-1}.

(3) If $(h\to a)\in\Omega$ for each $h\in\Omega$, $a\in[0,1]$. there exists a unique equivalence relation $E_\wedge$ on Xsuch that $\Omega=\Omega_{E_\wedge}$defined as

$$E_\wedge(x,y)=\bigwedge_{j\in J}((h_j(x)\to h_j(y))\wedge(h_j(y)\to h_j(x))).$$

where $\Omega_{E_\wedge}$ is the collection of fuzzy sets that are extensional w.r.t. $E_\wedge$.

(4) If (a) $(h\to a)\in\Omega$ for each $h\in\Omega$, $a\in[0,1]$. (b) $(h_1\otimes h_2)\in\Omega$ for each $h_1,h_2\in\Omega$ where $\otimes$ dominates $\odot$, then there exists a unique equivalence relation $E_\otimes$ on X such that $\Omega=\Omega_{E_\otimes}$ defined as

$$E_\otimes(x,y)=\bigwedge_{j\in J}((h_j(x)\to h_j(y))\otimes(h_j(y)\to h_j(x)))$$

where $\Omega_{E_\otimes}$ is the collection of fuzzy sets that are extensional w.r.t. $E_\otimes$.

Proof. (1) For all $\mu\in\Omega$, since μ is extensional w.r.t. E_Ω, $\mu\in\Omega_{E_\Omega}$. Hence $\Omega\subset\Omega_{E_\Omega}$.

Let $\rho\in\Omega_{E_\Omega}$. Define

$$\rho_z(x)=\rho(z)\odot E_\Omega(z,x)=\rho(z)\odot\bigwedge_{\nu\in\Omega}(\nu(z)\to\nu(x))$$

By (A2)-(A4), $\rho_z\in\Omega$. Since ρ is extensional w.r.t. E_Ω, then $\rho_z(x)=\rho(z)\odot E_\Omega(z,x)\le\rho(x)$.

Since $\rho_z(z)=\rho(z)\odot E_\Omega(z,z)=\rho(z)$, by (A1)
$\rho=\bigvee_{z\in X}\rho_z\in\Omega$.

Let an quasi-equivalence relation F on X with $\Omega_F=\Omega$. Then $F\le E_\Omega$. Define $\rho_y(z)=F(y,z)$. Since
$\rho_x(y)\odot F(y,z)=F(x,y)\odot F(y,z)\le F(x,z)=\rho_x(z)$
ρ_y is left-extensional w.r.t. F. By $\Omega_F=\Omega_E=\Omega$,
$\rho_y\in\Omega$ for all $y\in X$. Thus ρ_x is extensional w.r.t. E_Ω; i.e. $E_\Omega(x,y)=\rho_x(x)\odot E_\Omega(x,y)\le\rho_x(y)=F(x,y)$.

(4) We easily show that $E_\otimes$ is an $\odot$-equivalence relation from:

$$\begin{aligned}&E_\otimes(x,y)\odot E_\otimes(y,z)\\&=\bigwedge_{j\in J}((h_j(x)\to h_j(y))\otimes(h_j(y)\to h_j(x)))\\&\quad\odot\bigwedge_{j\in J}((h_j(y)\to h_j(z))\otimes(h_j(z)\to h_j(y)))\end{aligned}$$

$\leq \bigwedge_{j\in J}\{((h_j(x)\to h_j(y))\odot(h_j(y)\to h_j(z)))$

$\otimes((h_j(y)\to h_j(x))\odot(h_j(z)\to h_j(y)))\}$

$\leq \bigwedge_{j\in J}((h_j(x)\to h_j(z))\otimes(h_j(z)\to h_j(x)))$

$= E_{\otimes}(x,z).$

For all $\mu\in\Omega$, since μ is extensional w.r.t $E_{\otimes}$. $\mu\in\Omega_{E_{\otimes}}$. Hence $\Omega\subset\Omega_{E_{\otimes}}$.

Let $\rho\in\Omega_{E_{\otimes}}$. Define

$\rho_z(x)=\rho(z)\odot E_{\otimes}(z,x)$

$=\rho(z)\odot\bigwedge_{\nu\in\Omega}(\nu(z)\to\nu(x))\otimes(\nu(z)\to\nu(x)))$

By (A1)-(A4) and conditions, we have $\rho_z\in\Omega$.

Since ρ is left-extensional w.r.t. $E_{\otimes}$,

$\rho_z(x)=\rho(z)\odot E_{\otimes}(z,x)\leq\rho(x).$

Since $\rho_z(z)=\rho(z)\odot E_{\otimes}(z,z)=\rho(z)$, by (A1),

$\rho=\bigvee_{z\in X}\rho_z\in\Omega.$

Other case are similarly proved.

Theorem 3.15 Let $\odot$ be a continuous t-norm. Let $\Omega=\{h_j\mid j\in J\}$ be an extensional system. Then $c_{\Omega}:[0,1]^X\to[0,1]^X$ defined by

$c_{\Omega}(\lambda)=\bigwedge\{\rho\mid\lambda\leq\rho,\ \ \rho\in\Omega\}$

is a quasi-approximation operator.

Proof (C1) it is easy.

(C2) Since $c_{\Omega}(\mu)\in\Omega$, then $c_{\Omega}(c_{\Omega}(\mu))=c_{\Omega}(\mu)$

(C3) $\bigvee_{i\in\Gamma}c_{\Omega}(\mu_i)\leq c_{\Omega}(\bigvee_{i\in\Gamma}\mu_i)$.

Since $\bigvee_{i\in\Gamma}c_{\Omega}(\mu_i)\in\Omega$ and $\bigvee_{i\in\Gamma}\mu_i\leq\bigvee_{i\in\Gamma}c_{\Omega}(\mu_i)$,

$\bigvee_{i\in\Gamma}c_{\Omega}(\mu_i)\geq c_{\Omega}(\bigvee_{i\in\Gamma}\mu_i)$

(C4) Since all $a\in[0,1]$ and $\rho\in\Omega$, then $(a\odot\rho)\in\Omega$. So, $\bigwedge\{a\odot\rho\mid\lambda\leq\rho,\ \rho\in\Omega\}\geq\bigwedge\{\mu\mid a\odot\lambda\leq\mu,\ \mu\in\Omega\}$.

For $\mu\in\Omega$ with $a\odot\lambda\leq\mu$, we have $\lambda\leq(a\to\mu)\in\Omega$. It implies $a\odot\lambda\leq(a\odot(a\to\mu))\in\Omega$. Since $(a\odot(a\to\mu))\leq\mu$, $\bigwedge\{a\odot\rho\mid a\odot\lambda\leq a\odot\rho,\ \rho\in\Omega\}\leq\bigwedge\{\mu\mid a\odot\lambda\leq\mu,\ \mu\in\Omega\}$

Since $\odot$ is continuous,

$$\begin{aligned}a\odot c_{\Omega}(\lambda)&=a\odot\bigwedge\{\rho\mid\lambda\leq\rho,\ \rho\in\Omega\}\\&=\bigwedge\{a\odot\rho\mid\lambda\leq\rho,\ \rho\in\Omega\}\\&=\bigwedge\{\mu\mid a\odot\lambda\leq\mu,\ \mu\in\Omega\}\\&=c_{\Omega}(a\odot\lambda).\end{aligned}$$

Theorem 3.16 Let (X,c) be an $\odot$-upper quasi-approximation space. Define an operator $E_c:X\times X\to[0,1]$ as follows:

$$E_c(x,y)=c(1_x)(y)$$

Then

(1) E_c is an $\odot$-quasi-equivalence relation on X.

(2) $E_{c_E}=E,\ \ c_{E_c}=c,\ \ E_{\Omega_E}=E.$

Proof (1) Since $\mu(x)=\bigvee_{z\in X}(\mu(z)\odot 1_z(x))$, we have $c(\mu)(x)=c(\bigvee_{z\in X}(\mu(z)\odot 1_z(x)))=\bigvee_{z\in X}(\mu(z)\odot c(1_z)(x))$.

(E3)

$$\begin{aligned}E_c(x,y)&=c(1_x)(y)=c(c(1_x))(y)\\&=\bigvee_{z\in X}((c(1_x)(z)\odot c(1_z)(y)))\\&=\bigvee_{z\in X}(E_c(x,z)\odot E_c(z,x)).\end{aligned}$$

(2)

$E_{c_E}(x,y)=c_E(1_x)(y)=\bigvee_{z\in X}(1_x(z)\odot E(z,y))=E(x,y).$

$$\begin{aligned}c_{E_c}(\mu)(x)&=\bigvee_{z\in X}(\mu(z)\odot E_c(z,x))\\&=\bigvee_{z\in X}(\mu(z)\odot c(1_z)(x))=c(\mu).\end{aligned}$$

Since $\mu\in\Omega_E$, then $E(x,y)\leq\mu(x)\to\mu(y)$.

Hence $E_{\Omega_E}(x,y)\geq E(x,y)$

Since $\overline{1_x}\in\Omega_E$, then

$E_{\Omega_E}(x,y)\leq(\overline{1_x}(x)\to\overline{1_x}(y))=\overline{1_x}(y)=E(x,y).$

Corollary 3.17 Let (X,c) be an $\odot$-upper quasi-approximation space. Define an operator $E_c:X\times X\to[0,1]$ as $E_c^{-1}(x,y)=c(1_y)(x)$. Then

(1) E_c^{-1} is an $\odot$-quasi-equivalence relation on X.

(2) $E^{-1}_{c_E^{-1}}=E^{-1},\ \ c_{E_c^{-1}}=c,\ \ E^{-1}_{\Omega_E^{-1}}=E^{-1}.$

Theorem 3.18 Let (X,c) be an $\odot$-quasi-approximation space. Then

(1) $\Omega_c=\{\mu\in[0,1]^X\mid c(\mu)=\mu\}$ is an extensional space on X such that $c=c_{\Omega_c}$ and $\Omega_{c_\Omega}=\Omega$

(2) $E_c=E_{\Omega_c}$

Proof. (1) Since $c_{\Omega_c}(\lambda)=\bigwedge\{\rho\mid\lambda\leq\rho,\ \rho\in\Omega_c\}$ and $\rho=c(\rho)$, then $c_{\Omega_c}(\lambda)\geq c(\lambda)$. Since $\lambda\leq c(\lambda)$, $c_{\Omega_c}(\lambda)\leq c(\lambda)$. Hence $c=c_{\Omega_c}$.

Let $\mu\in\Omega_{c_\Omega}$. Then $c_{\Omega}(\mu)=\mu\in\Omega$. Let $\rho\in\Omega$. Then $c_{\Omega}(\rho)=\rho$. $\rho\in\Omega_{c_\Omega}$.

(2) For $c(1_x)\in\Omega_c$ for each $x\in X$,

$E_c(x,y)=c(1_x)(y)=(c(1_x)(x)\to c(1_x)(y))\geq E_{\Omega_c}(x,y).$

Since $\mu(y)=\bigvee_{z\in X}(1_z(y)\odot\mu(z))$,

$c(\mu)(y)=\bigvee_{z\in X}c(1_z)(y)\odot\mu(z)$

It implies $c(1_x)(y)\leq(\mu(x)\to c(\mu)(y))$.

Thus

$E_c(x,y)=c(1_x)(y)\leq\bigwedge_{\mu\in\Omega_c}(\mu(x)\to c(\mu)(y)).$

Theorem 3.19. (1) Let (X,c_1) and (Y,c_2) be $\odot$-upper quasi approximation spaces. The following statements are equivalent

(a) $\phi:X\to Y$ is a c-map,

(b) $\phi:(X,E_{c_1})\to(Y,E_{c_2})$ is an E-map.

(c) $\phi:(X,\Omega_{c_1})\to(Y,\Omega_{c_2})$ is an A-map.

(2) Let (X,E_1) and (Y,E_2) be $\odot$-quasi equiv-

alence spaces.

(d) $\phi: X \to Y$ is an E-map.

(e) $\phi:(X, c_{E_1}) \to (Y, c_{E_2})$ is a c-map

(f) $\phi:(X, \Omega_{E_1}) \to (Y, \Omega_{E_2})$ is an A-map.

(3) Let (X, Ω_1) and (Y, Ω_2)be extensional spaces. Let $\odot$ be a continuous t-norm.

(g) $\phi: X \to Y$ is an A-map.

(h) $\phi:(X, E_{\Omega_1}) \to (Y, E_{\Omega_2})$ is an E-map.

(i) $\phi:(X, c_{\Omega_1}) \to (Y, c_{\Omega_2})$ is a c-map.

(4) Let (X, c_1) and (Y, c_2) be $\odot$-upper quasi-approximation spaces. Then $\phi:(X, c_1) \to (Y, c_2)$ is a c-map iff $\phi(c_1(1_x)) \le c_2(1_{\phi(x)})$ for each $x \in X$.

Poof (1) (a) $\Rightarrow$(b)

$$E_{c_1}(x,y) = c(1_x)(y) \le \phi(c_1(1_x))(\phi(y))$$
$$\le c_2(1_{\phi(x)})(\phi(y)) = E_{c_2}(\phi(x), \phi(y)).$$

Conversely, it follows (2-e) and $c_{E_{c_i}} = c_i$for $i = 1, 2$.

(a) $\Rightarrow$(c) Since $c_1(\phi^{-1}(\mu)) \le \phi^{-1} c_2(\mu)$, for $\mu \in \Omega_{c_2}$, we have $\phi^{-1}(\mu) \in \Omega_{c_1}$

Conversely, it follows (3-i) and $c_{\Omega_{c_i}} = c_i$for $i = 1, 2$.

(2) (d) $\Rightarrow$(e)

$$\begin{aligned}\phi(c_{E_1}(\mu))(y) &= \bigvee_{x \in \phi^{-1}(\{y\})} c_{E_1}(\mu)(x) \\ &= \bigvee_{x \in \phi^{-1}(\{y\})} \bigvee_{z \in X} (E_1(z,x) \odot \mu(z)) \\ &\le \bigvee_{z \in X} (E_2(\phi(z), \phi(x)) \odot \phi(\mu)(\phi(z))) \\ &\le c_{E_1}(\phi(\mu))(y).\end{aligned}$$

Conversely, it follows (1-b) and $E_{c_{E_i}} = E_i$for $i = 1, 2$.

(d)$\Rightarrow$(f) For $\mu \in \Omega_{E_2}$,

$$\begin{aligned}\phi^{-1}(\mu)(x) \odot E_1(x,y) &\le \mu(\phi(x)) \odot E_2(\phi(x), \phi(y)) \\ &\le \mu(\phi(x)) = \phi^{-1}(\mu)(y).\end{aligned}$$

Hence $\phi^{-1}(\mu) \in \Omega_{E_1}$

Conversely, it follows (3-h) and $E_{\Omega_{E_i}} = E_i$for $i = 1, 2$.

(3) (g) $\Rightarrow$(h) Since $\mu \in \Omega_2$ implies $\phi^{-1}(\mu) \in \Omega_1$, we have

$$\begin{aligned}E_{\Omega_2}(\phi(x), \phi(y)) &= \bigwedge_{\mu \in \Omega_2} (\mu(\phi(x)) \to \mu(\phi(x))) \\ &= \bigwedge_{\mu \in \Omega_2} (\phi^{-1}(\mu)(x) \to \phi^{-1}(\mu)(y)) \\ &\ge \bigwedge_{\rho \in \Omega_1} (\rho(x) \to \rho(y)) \\ &= E_{\Omega_2}(x,y).\end{aligned}$$

Conversely, it follows (2-e) and $\Omega_{E_{\Omega_i}} = \Omega_i$for $i = 1, 2$.

(g) $\Rightarrow$ (i)

$$\begin{aligned}\phi^{-1}(c_{\Omega_2}(\phi(\lambda))) &= \phi^{-1}(\bigwedge\{\mu \mid \phi(\lambda) \le \mu,\ \mu \in \Omega_2\}) \\ &= \bigwedge\{\phi^{-1}(\mu) \mid \lambda \le \phi^{-1}(\mu),\ \mu \in \Omega_2\} \\ &\ge c_{\Omega_1}(\lambda).\end{aligned}$$

It implies $c_{\Omega_2}(\phi(\lambda)) \ge \phi(c_{\Omega_1}(\lambda))$

Conversely, it follows (1-c) and $\Omega_{c_{\Omega_i}} = \Omega_i$for $i = 1, 2$.

(4) Since $\lambda = \bigvee_{z \in X} \lambda(z) \odot 1_z$, we have

$$\begin{aligned}\phi(c_1(\lambda))(y) &= \bigvee_{x \in \phi^{-1}(\{y\})} c_1(\bigvee_{z \in X} \lambda(z) \odot 1_z)(x) \\ &= \bigvee_{x \in \phi^{-1}(\{y\})} \bigvee_{z \in X} \lambda(z) \odot c_1(1_z)(x) \\ &= \bigvee_{z \in X} \lambda(z) \odot (\bigvee_{x \in \phi^{-1}(\{y\})} c_1(1_z)(x)) \\ &= \bigvee_{z \in X} \lambda(z) \odot (\phi(c_1(1_z))(y)) \\ &\le \bigvee_{z \in X} \lambda(z) \odot (c_2(1_{\phi(z)}))(y)) \\ &\le \bigvee_{z \in X} \phi(\lambda)(\phi(z)) \odot (c_2(1_{\phi(z)}))(y)) \\ &\le c_2(\phi(\lambda))(y).\end{aligned}$$

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