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ASYMPTOTIC LIMITS FOR THE SELF-DUAL CHERN-SIMONS $CP(1)$ MODEL

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ABSTRACT. In this paper we study the asymptotics for the energy density in the self-dual Chern-Simons $CP(1)$ model. When the sequence of corresponding multivortex solutions converges to the topological limit, we show that the field configurations saturating the energy bound converges to the limit function. Also, we show that the energy density tends to be concentrated at the vortices and antivortices as the Chern-Simons coupling constant κ goes to zero.

1. the Chern-Simons $CP(1)$ model

In this paper, we consider the Chern-Simons $CP(1)$ model ([8]) where the gauge field dynamics is solely governed by the Chern-Simons term. Gauging classical $CP(1)$ model, a useful toy model for the instantons in non-abelian Yang-Mills theories, is useful to obtain the finite energy solitons and yield to a Bogomol'nyi limit or self-dual equations which are easy to analyze mathematically compared to full second order Euler-Lagrange equation. We note that the Bogomol'nyi limit in superconductivity plays an important role as it permits to distinguish between the type of superconductors ([2]). Under the doubly periodic boundary condition, multiple existence of multivortex solutions ([3]) and its asymptotics ([9]) are studied. In this paper, we show that when the sequence of corresponding multivortex solutions converges to the topological limit, the field configurations saturating the energy bound converges to the limit function and the energy density tends to be concentrated at the vortices and antivortices as the Chern-Simons coupling constant

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κ goes to zero. We remark that the studies on the asymptotic behaviors of solutions of the Chern-Simon-Higgs model in a doubly periodic domain ([4, 5]) motivate our work here.

The CP(1) model consists of two complex scalar fields z_1, z_2 in $\mathbb{R}^2$. Denoting $\mathbf{z} = (z_1, z_2)$, the model requires that $|\mathbf{z}|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ and $\mathbf{z}$ is equivalent to the overall phase rotations. Thus if we can find $\phi = z_2/z_1$, then we have that $\mathbf{z} \sim (1, \phi)/\sqrt{1+|\phi|^2}$.

The Lagrangian for the self-dual Chern-Simons CP(1) model is

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + |\nabla_\mu \mathbf{z}|^2 - V(\mathbf{z}),$$

where $\epsilon^{\mu\nu\rho}$ is the totally skew-symmetric tensor with $\epsilon^{012} = 1$, $R = \mathrm{diag}(1/2, -1/2)$, $A_\mu R$ is the matrix valued gauge field, $V(\mathbf{z})$ is a potential term which will be fixed later, and the "covariant derivatives" ∇_μ and D_μ are defined as follows:

$$\nabla_\mu \mathbf{z} = D_\mu \mathbf{z} - (\bar{\mathbf{z}} D_\mu \mathbf{z})\mathbf{z}, \quad D_\mu \mathbf{z} = \partial_\mu \mathbf{z} - iA_\mu R\mathbf{z}.$$

The Gauss law constraint obtained from the variation of A_0 is given by

$$\kappa F_{12} = i\{\nabla_0 \bar{\mathbf{z}}[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}] - \text{h.c.}\},$$

where $F_{12} = \partial_1 A_2 - \partial_2 A_1$. The theory possesses the following conserved topological current

$$K^\mu = -i\epsilon^{\mu\nu\rho}\partial_\nu(\bar{\mathbf{z}}D_\rho \mathbf{z})$$

and a conserved global $U(1)$ current for the generator R,

$$J^\mu = i\{\nabla^\mu \bar{\mathbf{z}}[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}] - \text{h.c.}\}.$$

If we choose the potential as given by

$$V(\mathbf{z}) = \frac{1}{\kappa^2}\Big|\big[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}\big](\bar{\mathbf{z}}R\mathbf{z} - s)\Big|^2,$$

for a free real parameter s, then we can rewrite the energy density as

$$\begin{aligned}\mathcal{E} &= |\nabla_0 \mathbf{z}|^2 + |\nabla_1 \mathbf{z}|^2 + |\nabla_2 \mathbf{z}|^2 + V(\mathbf{z}) \\ &= |(\nabla_1 \pm i\nabla_2)\mathbf{z}|^2 + \Big|\nabla_0 \mathbf{z} \mp \frac{i}{\kappa}\{\big[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}\big](\bar{\mathbf{z}}R\mathbf{z} - s)\}\Big|^2 \\ &\quad \pm (K_0 + \frac{s}{\kappa}J_0).\end{aligned}$$

Thus the field configurations saturating the energy bound $E = \pm\int(K_0 + (s/\kappa)J_0) \geq 0$ satisfy the Gauss law constraint and the following self-dual equations,

$$(\nabla_1 \pm i\nabla_2)\mathbf{z} = 0$$

$$\nabla_0 \mathbf{z} \mp \frac{i}{\kappa}\Big\{\big[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}\big](\bar{\mathbf{z}}R\mathbf{z} - s)\Big\} = 0.$$

Without loss of generality, we can choose the upper signs. Introducing complex differentiation $\bar{\partial} = (\partial_1 + i\partial_2)/2$ and $\bar{\alpha} = (A_1 + iA_2)/2$, we obtain that

$$z_2\bar{\partial}z_1 - z_1\bar{\partial}z_2 = 2iz_1z_2\bar{\alpha}.$$

Thus, away from the zeroes of z_1 and z_2, $\phi = z_2/z_1$ satisfies

$$\bar{\partial}\ln\phi = -2i\bar{\alpha}. \tag{1}$$

This equation plays a key role in the next section.

Noting that the Lagrangian $\mathcal{L}$ is invariant under the following gauge transformations

$$\mathbf{z}(x) \mapsto e^{i\theta(x)}\mathbf{z}(x), \quad A_0(x) \mapsto A_0(x), \quad A_j(x) \mapsto A_j(x) + \partial_j\theta(x),$$

where θ is a real-valued function and $j = 1, 2$, we impose a doubly periodic boundary conditions due to 't Hooft in the doubly periodic region Ω. See [3], [9] for details.

Following the argument of Taubes[7], we can set

$$z_1 = \psi_1 \prod_l (z - p_l)^{n_l}, \ z_2 = \psi_2 \prod_j (z - q_j)^{m_j}, \ \sum_l n_l = n, \ \sum_j m_j = m,$$

where p_l's and q_j's are distinct and ψ_1, ψ_2 are nonvanishing smooth functions. Using gauge invariance, we introduce a real valued function u defined by

$$\phi = \frac{z_2}{z_1} = \exp\Big(\frac{u}{2} + i\big(\sum_j \arg(z - q_j) - \sum_l \arg(z - p_l)\big)\Big), \tag{2}$$

then the self-dual equations reduces to

$$\Delta u = \frac{4(1+2s)}{\kappa^2}\frac{e^u}{(1+e^u)^3}\Big(e^u - \frac{1-2s}{1+2s}\Big) + 4\pi\sum_j m_j\delta_{q_j} - 4\pi\sum_l n_l\delta_{p_l} \text{ in } \Omega.$$

If u is once found, then we can recover $(\mathbf{z}, A)$ by (1), (2) and the equivalence relation

$$\mathbf{z} \sim (1, \phi)/\sqrt{1 + |\phi|^2}. \tag{3}$$

We recall some preliminary results [9].

THEOREM 1. (topological limit) *Let $-1/2 < s < 1/2$. Then there exists a sequence of solutions $\{u_\kappa\}$ which satisfy the followings.*

(1) $F_{12}^{\kappa} = \frac{1+2s}{\kappa^2}\frac{e^{u_\kappa}}{(1+e^{u_\kappa})^3}(e^{u_\kappa} - \frac{1-2s}{1+2s})$
$\to -\pi\sum_j m_j\delta_{q_j} + \pi\sum_l n_l\delta_{p_l}$ *in the sense of measure as* $\kappa \to 0^+$.

(2) $\|u_\kappa - \ln\frac{1-2s}{1+2s}\|_{L^\infty(K)} \le C(K)\kappa^2$,
for $K \subset\subset \Omega' = \Omega \setminus \bigcup_l\{p_l\}\bigcup_j\{q_j\}$.

In this paper, we will prove the following theorem :

THEOREM 2. *Let* $\{u_\kappa\}$ *be a sequence converging to topological limit as in Theorem 1 and* $K \subset\subset \Omega'$. *Then the corresponding field configurations* $\{(\mathbf{z}^\kappa, A^\kappa)\}$ *given by the formulas* (1), (2), (3), *and the energy density* $\{\mathcal{E}^\kappa\}$ *satisfy the followings.*

(1) $(\mathbf{z}^\kappa, A^\kappa) \to (\mathbf{z}^*, A^*)$ *smoothly in* K *where*

$$\mathbf{z}^* \sim (1,\phi^*)/\sqrt{1+|\phi^*|^2}$$
$$\phi^* = \sqrt{\frac{1-2s}{1+2s}}\exp i(\sum_j \arg(z-q_j) - \sum_l \arg(z-p_l))$$
$$A_1^* + iA_2^* = \partial(\sum_l \arg(z-p_l) - \sum_j \arg(z-q_j)).$$

(2) $\mathcal{E}^\kappa \to (1+2s)\pi\sum_l n_l\delta_{p_l} + (1-2s)\pi\sum_j m_j\delta_{q_j}$ *as* $\kappa \to 0^+$.

2. Proof of Theorem 2

The proof of the theorem consists of several lemmas and we begin with establishing C^k-norm estimates. For simplicity, we will omit the script κ if there is no confusion.

LEMMA 1. *Let* $\{u_\kappa\}$ *be as in Theorem 1. Then for any nonnegative integer* k *and* $K \subset\subset \Omega'$, *we have that*

$$\|u_\kappa\|_{C^{k+1}(K)} \le C, \quad \|e^{u_\kappa} - \frac{1-2s}{1+2s}\|_{C^k(K)} \le C\kappa^2.$$

PROOF. If $k=0$, it follows from Theorem 1. Since the argument is similar for $k \ge 1$ by induction, we will only consider the case $k=1$.

For any given point $x_0 \in K$, we choose $r > 0$ sufficiently small such that $B_{2r}(x_0) \subset K$ and set $w = (e^u - (1-2s)/(1+2s))/\kappa^2$. We note that

$$\begin{aligned}\Delta w &= \frac{e^u}{\kappa^2}|\nabla u|^2 + \frac{e^u}{\kappa^2}\Delta u \\ &= \frac{e^u}{\kappa^2}|\nabla u|^2 + \frac{4(1+2s)}{\kappa^2}\frac{e^{2u}}{(1+e^u)^3}w \quad \text{in } B_{2r}(x_0) \subset K.\end{aligned}$$

Since $\Delta u \in L^\infty(K)$, we have that $\|\nabla u\|_{L^\infty(K)} \leq C$ and

$$\|\nabla w\|^2_{L^\infty(B_{2r})} \leq C\|w\|_{L^\infty(K)}(\|w\|_{L^\infty(K)} + \|\Delta w\|_{L^\infty(K)}) \leq \frac{C}{\kappa^2}. \tag{4}$$

For this type of estimates, see [1]. Similarly, differentiating Δu and Δw, we are led to

$$\|D^2 u\|_{L^\infty(B_{2r})} \leq \frac{C}{\sqrt{\kappa}}, \quad \|D^2 w\|_{L^\infty(B_{2r})} \leq \frac{C}{\kappa^2}. \tag{5}$$

On the other hand, we rewrite Δw as

$$\Delta w = \nabla w \cdot \nabla u + \frac{e^u}{\kappa^2}\Delta u = \nabla w \cdot \nabla u + \frac{4(1+2s)}{\kappa^2}\frac{e^{2u}}{(1+e^u)^3}w$$

to obtain

$$\begin{aligned}&- \kappa^2\Delta(\partial_j w) + \frac{4(1+2s)e^{2u}}{(1+e^u)^3}(\partial_j w) \\ =& - \kappa^2(\partial_j(\nabla w)\cdot\nabla u + \nabla w \cdot (\partial_j \nabla u)) - 4(1+2s)\partial_j(\frac{e^{2u}}{(1+e^u)^3})w \\ \equiv& \ h.\end{aligned}$$

Now we choose a constant c_0 such that

$$\frac{4(1+2s)e^{2u}}{(1+e^u)^3} \geq c_0 \quad \text{in } K.$$

It follows from (4) and (5) that

$$\|h\|_{L^\infty(B_{2r})} \leq C, \ \|\nabla w\|_{L^\infty(B_{2r})} \leq C/\kappa$$

and

$$\begin{aligned}&-\kappa^2\Delta\left(\partial_j w - \frac{\|h\|_{L^\infty(B_{2r})}}{c_0}\right) \\ &\qquad + \frac{4(1+2s)e^{2u}}{(1+e^u)^3}\left(\partial_j w - \frac{\|h\|_{L^\infty(B_{2r})}}{c_0}\right) \leq 0 \quad \text{in} \quad B_{2r}, \\ &\qquad \partial_j w - \frac{\|h\|_{L^\infty(B_{2r})}}{c_0} \leq \frac{\tilde{C}}{\kappa} \quad \text{on} \quad \partial B_{2r}.\end{aligned}$$

If we can find a positive function W which satisfies the following conditions,

(1) $-\kappa^2\Delta W + c_0 W \geq 0$ in B_{2r} and $W = \tilde{C}/\kappa$ on ∂B_{2r},
(2) W is bounded in B_r as $\kappa \to 0^+$,

then, by the maximum principle,

$$\partial_j w - \frac{\|h\|_{L^\infty(B_{2r})}}{c_0} \leq W \quad \text{in } B_r.$$

It is easily verified that

$$W(x) = W(|x - x_0|) = \frac{\tilde{C}}{\kappa}\exp\left(\frac{-\kappa + \sqrt{\kappa^2 + 4r^2 c_0}}{16r^2\kappa}(|x - x_0|^2 - 4r^2)\right)$$

is a desired function.

Similarly, we can find a negative function W' which is bounded in B_r as $\kappa \to 0^+$ such that

$$\partial_j w - \|h\|_{L^\infty(B_{2r})} \geq W' \quad \text{in } B_r.$$

Since x_0 is arbitrary, we conclude that $\|\nabla w\|_{L^\infty(K)} \leq C$ and hence $\|D^2 u\|_{L^\infty(K)} \leq C$. □

LEMMA 2. *$(\phi^\kappa, A^\kappa) \to (\phi^*, A^*)$ smoothly in $K \subset\subset \Omega'$ where*

$$\phi^* = \sqrt{\frac{1-2s}{1+2s}}\exp\left(i\left(\sum_j \arg(z - q_j) - \sum_l \arg(z - p_l)\right)\right)$$

$$A_1^* + iA_2^* = \partial\left(\sum_l \arg(z - p_l) - \sum_j \arg(z - q_j)\right).$$

PROOF. By Lemma 1, $|\phi|^2 \to (1-2s)/(1+2s)$ smoothly in K and we obtain that

$$\begin{aligned}\bar\partial|\phi|^2 = \bar\phi\bar\partial\phi + \phi\bar\partial\bar\phi &= -2i\bar\alpha|\phi|^2 + |\phi|^2\bar\partial\ln\bar\phi \\ &= |\phi|^2(-2i\bar\alpha + \bar\partial\ln\bar\phi) \to 0 \quad \text{as} \quad \kappa \to 0^+.\end{aligned}$$

Thus the limit function ϕ^* satisfies

$$|\phi^*|^2 = \frac{1-2s}{1+2s}, \quad \bar\partial\ln\bar\phi^* = 2i\bar\alpha^*.$$

By (2), we obtain the desired results. □

LEMMA 3. *For $K \subset\subset \Omega'$, $\int_K \mathcal{E}^\kappa \to 0$ as $\kappa \to 0^+$.*

PROOF. We divide the integral into two parts as

$$\begin{aligned}\int_K \mathcal{E} &= \int_K K_0 + (s/\kappa)J_0 \\ &= \int_K i(\partial_2(\bar{\mathbf{z}}\partial_1\mathbf{z}) - \partial_1(\bar{\mathbf{z}}\partial_2\mathbf{z}) \\ &\quad + \int_K \partial_2(A_1(\bar{\mathbf{z}}R\mathbf{z} - s)) - \partial_1(A_2(\bar{\mathbf{z}}R\mathbf{z} - s)) \\ &\equiv (I) + (II).\end{aligned}$$

Since $|z_1|^2 = 1/(1+e^u) \to (1+2s)/2$ in $C^k(K)$ by Lemma 1, we deduce that

$$\bar{z}_1\partial_j z_1 + z_1\partial_j\bar{z}_1 = o(1), \quad j = 1,2$$

and this implies

$$\begin{aligned}&z_1\bar{z}_1(\partial_2\bar{z}_1\partial_1 z_1 - \partial_1\bar{z}_1\partial_2 z_1) \\ &= (z_1\partial_2\bar{z}_1)(\bar{z}_1\partial_1 z_1) - (z_1\partial_1\bar{z}_1)(\bar{z}_1\partial_2 z_1) \\ &= (o(1) - \bar{z}_1\partial_2 z_1)(\bar{z}_1\partial_1 z_1) - (o(1) - \bar{z}_1\partial_1 z_1)(\bar{z}_1\partial_2 z_1) \\ &= o(1).\end{aligned}$$

Thus

$$\partial_2\bar{z}_1\partial_1 z_1 - \partial_1\bar{z}_1\partial_2 z_1 = o(1).$$

Similarly, we obtain that

$$\partial_2\bar{z}_2\partial_1 z_2 - \partial_1\bar{z}_2\partial_2 z_2 = o(1).$$

Hence $(I) = o(1)$.

On the other hand, from Theorem 1,

$$\begin{aligned}\bar{\mathbf{z}}R\mathbf{z} - s &= \frac{1}{2}(|z_1|^2 - |z_2|^2) - s \\ &= \frac{1}{2}(\frac{1}{1+e^u} - \frac{e^u}{1+e^u} - 2s) \\ &= -\frac{1+2s}{2(1+e^u)}(e^u - \frac{1-2s}{1+2s}) \\ &= o(1) \quad \text{in} \quad K.\end{aligned}$$

Then (II) becomes

$$\begin{aligned}(II) &= \int_K \nabla\cdot(-A_2(\bar{\mathbf{z}}R\mathbf{z}-s), A_1(\bar{\mathbf{z}}R\mathbf{z}-s))\\ &= \int_{\partial K}(\bar{\mathbf{z}}R\mathbf{z}-s)(-A_2,A_1)\cdot\mathbf{n}\\ &= o(1),\end{aligned}$$

where $\mathbf{n}$ is the unit outward normal vector. □

LEMMA 4. *For sufficiently small $r>0$, we have that*

$$\int_{B_r(p_l)}\mathcal{E}^\kappa \to (1+2s)n_l\pi,\qquad \int_{B_r(q_j)}\mathcal{E}^\kappa \to (1-2s)m_j\pi \quad \textit{as} \quad \kappa\to 0^+.$$

PROOF. Let r be sufficiently small such that $B_r(p_l)$ has no other points p_j's nor q_j's in it. For the unit outward normal vector $\mathbf{n}$ and unit tangent vector τ, we notice that

$$\begin{aligned}&\int_{B_r(p_l)}\partial_1(\bar{\mathbf{z}}\partial_2\mathbf{z})-\partial_2(\bar{\mathbf{z}}\partial_1\mathbf{z})\\ &= \int_{\partial B_r(p_l)}(\bar{\mathbf{z}}\partial_2\mathbf{z},-\bar{\mathbf{z}}\partial_1\mathbf{z})\cdot\mathbf{n}\\ &= \int_{\partial B_r(p_l)}\bar{\mathbf{z}}\nabla\mathbf{z}\cdot\tau\\ &= \int_{\partial B_r(p_l)}\bar{z}_1\frac{\partial z_1}{\partial\tau}+\bar{z}_2\frac{\partial z_2}{\partial\tau}\\ &\equiv (I)+(II).\end{aligned}$$

We recall that $\deg(z_1,p_l)$, the degree of $z_1/|z_1|$ considered as a map from $\partial B_r(p_l)$ into S^1, is well-defined as

$$\deg(z_1,p_l)=\frac{1}{2\pi i}\int_{\partial B_r(p_l)}\frac{\bar{z}_1}{|z_1|^2}\frac{\partial z_1}{\partial\tau},$$

where τ is the unit tangent vector field to $\partial B_r(p_l)$.

Using the facts $|\mathbf{z}|^2=1$, $\deg(z_1,p_l)=n_l$ and $\deg(z_2,q_j)=m_j$, we derive that

$$\begin{aligned}(I) &= \int_{\partial B_r(p_l)}\frac{1-(1-2s)/2}{|z_1|^2}\bar{z}_1\frac{\partial z_1}{\partial\tau}+\int_{\partial B_r(p_l)}\frac{(1-2s)/2-|z_2|^2}{|z_1|^2}\bar{z}_1\frac{\partial z_1}{\partial\tau}\\ &= (1+2s)\pi i\deg(z_1,p_l)+o(1),\end{aligned}$$

and similar arguments give

$$(II)=(1-2s)\pi i\deg(z_2,p_l)+o(1).$$

On the other hand,

$$\begin{aligned}&\int_{B_r(p_l)} \partial_2(A_1(\bar{\mathbf{z}}R\mathbf{z}-s))-\partial_1(A_2(\bar{\mathbf{z}}R\mathbf{z}-s))\\&=-\int_{\partial B_r(p_l)}(\bar{\mathbf{z}}R\mathbf{z}-s)(A_1,A_2)\cdot\tau\\&=o(1).\end{aligned}$$

Hence

$$\begin{aligned}\int_{B_r(p_l)}\mathcal{E}=i\int_{B_r(p_l)}&\partial_2(\bar{\mathbf{z}}\partial_1\mathbf{z})-\partial_1(\bar{\mathbf{z}}\partial_2\mathbf{z})\\&+\int_{B_r(p_l)}\partial_2(A_1(\bar{\mathbf{z}}R\mathbf{z}-s))-\partial_1(A_2(\bar{\mathbf{z}}R\mathbf{z}-s))\\&=(1+2s)n_l\pi+o(1).\end{aligned}$$

Similar arguments show that

$$\int_{B_r(q_j)}\mathcal{E}=(1-2s)m_j\pi+o(1),$$

and this completes the proof. □

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