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APPROXIMATION OF EVOLUTION EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH HURST PARAMETER $0 < H < 1/2$ †

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ABSTRACT

We consider the problem for approximate solution of linear stochastic evolution equations driven by infinite-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$. The error of the approximate solution for the explicit Euler scheme is investigated.

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1. INTRODUCTION

In various interesting applications of probability, several types of stochastic evolution equations are often used to model many random phenomena of scientific objects. In reality we cannot expect the solution of an equation to be observed at all space and time. It is usually assumed that we have only a finite dimension projection of the solution and sampling instants over a specified time. There have been the studies of the approximation of evolution equations driven by Brownian motion (see *e.g.*, Greksch and Kloeden, 1996; Gyöngy, 1998, 1999; Hausenblas, 2003). We refer, in particular, to the work of Hausenblas (2003) in which the accuracy of approximation of quasi-linear evolution equation was investigated through space and time discretization. Kim and Rhee (2004) have studied the approximation of evolution equation driven by fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1/2)$. Note that fBm has the difference of the behavior between the cases $H \in (0, 1/2)$ and $H \in (1/2, 1)$.

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For stochastic differential equation with respect to fBm in infinite dimensional case, very recently there have been the few works (see, *e.g.*, Grecksch and Ahn, 1999; Duncan *et al.*, 2000, 2002; Maslowski and Nualart, 2003; Tindel *et al.*, 2003). Among these works, Maslowski and Nualart (2003) have studied a linear equation with multiplicative noise but they treat only the case $H \in (1/2, 1)$. On the other hand, Tindel *et al.* (2003) have considered a linear stochastic evolution equation in a Hilbert space driven by cylindrical fBm with $H \in (0, 1)$.

In this paper we investigate the rate of convergence for approximation of linear evolution equation considered by Tindel *et al.* (2003) in the case of $H \in (0, 1/2)$. For this we take the method of moments as the spatial approximation and Euler scheme as time discretization respectively.

2. Stochastic Evolution Equation and Approximation

Let V be a separable Hilbert space. Assume that $A : Dom(A) \supset V \to V$ generates the strongly continuous semigroup $T(t) := e^{tA}$, $t \geq 0$, and $\Phi \in \mathcal{L}(V, V)$. We study an approximation of Stochastic Evolution Equation driven by V-valued fBm $(B^H(t))$, $t \geq 0$, with $H \in (0, 1)$:

$$\begin{cases} dX(t) &= AX(t)dt + \Phi dB^H(t), \quad t \in [0, T], \\ X(0) &= x_0 \in V. \end{cases} \tag{2.1}$$

Since the operator A can be shifted by $A - \lambda I$, we may assume that A generates an semigroup of strictly negative type. We denote by V_δ, $\delta \geq 0$, the domain of the fractional power $(-A)^\delta$ equipped with the norm $\|x\|_\delta = \|(-A)^\delta x\|$ for $x \in Dom((-A)^\delta)$. The solution of the initial value problem (2.1) will be given as the mild solution, i.e., for $t \in [0, T]$,

$$X(t) = T(t)x + \int_0^t e^{(t-s)A} \Phi dB^H(s). \tag{2.2}$$

Here for the definition of Wiener integral with respect to fBm, see Tindel *et al.*(2003).

ASSUMPTION 2.1. *It will be assumed that*

(1) $(-A)$ *is a nonnegative self-adjoint operator on* V.

(2) *There exist* $\alpha \in (0, H)$, $\gamma > 0$ *and* $\rho > 0$ *such that* $\gamma > \alpha + \rho$ *and the operator* $\Phi^*(-A)^{2(\delta+\alpha-H)}\Phi$ *for all* $\delta \in [\rho, \gamma]$ *is trace class, where* Φ^* *is the adjoint operator of* Φ.

By a standard argument (see *e.g.*, Theorem 5.9, DaPrato and Zabczyk, 1992) and Theorem 4 of Tindel *et al.* (2003), we have

THEOREM 2.1. *Under Assumption 2, there exists a unique mild solution* $(X(t))$, $t \in [0,T]$, *belonging to*

$$\mathcal{L}^2([0,T] \times \Omega; Dom((-A)^{\gamma+\alpha})) \cap C([0,T]; Dom((-A)^{\gamma}))$$

for $X(0) = x_0 \in Dom((-A)^{\gamma})$.

Let us denote (A_n, V_n) the approximation of (A, V) by the method of moments where V_n is the d_n-dimensional subspace of V (see *e.g.*, Harrington, 1993). Throughout this paper we assume the following conditions:

ASSUMPTION 2.2. *For every* $n \geq 1$, $P_n : V_n \to V$ *and* $D_n : V_n \to V$ *are bounded operators such that*

(1) $\|P_n\| \leq C_1$, $\|D_n\| \leq C_2$, *where* C_1 *and* C_2 *are independent of* n.

(2) $P_n D_n = I_n$ *where* I_n *is the identity operator on* V_n.

(3) *There exists a function* $\varphi_\delta : \mathbb{N} \to [0,1]$ *such that* $\lim_{n\to\infty} \varphi_\delta(n) = 0$ *and*

  (i) $\|(I - D_n P_n)x\| \leq \varphi_\delta(n)\|x\|_\delta$

  (ii) $\|A(I - D_n P_n)x\| \leq \varphi_\delta(n)\|A_n\|\|x\|_\delta$ *for all* $x \in Dom((-A)^\delta)$ *and* $\delta \in (0, \gamma]$, *where* $A_n = P_n A D_n$

(4) *For a sequence* $\tau_n > 0$ *such that* $\lim_{n\to\infty} \tau_n = 0$, *the following stability condition is satisfied:* $\tau_n \|\tilde{A}_n\| \leq 1$ *for all* $n = 1, 2, \cdots$, *where* $\tilde{A}_n = D_n A_n P_n$

Our approximation is given by

$$\begin{cases} dX_n(t) = A_n X_n(t)dt + \Phi_n dP_n B^H(t), \quad t \in [0,T], \\ X_n(0) = P_n x_0 \in V, \end{cases} \tag{2.3}$$

where Φ_n is a bounded linear operator defined on V_n such that $\Phi_n = P_n \Phi D_n$. Note that $P_n B^H(t)$ is a d_n-dimensional fBm, *i.e.*, $P_n B^H(t) := B_n^H(t) = (\beta_1^H(t), \cdots, \beta_n^H(t))$ where (β_i^H), $i = 1, \cdots, n$ are independent, real valued fBm each with the same Hurst parameter H. Let $\{\tau_n, n \geq 1\}$ be the sequence of time step sizes corresponding to the space V_n for time discretization. Let us denote by $Y_n(k)$ the approximation of $X_n(k\tau_n)$. Then

$$\begin{cases} Y_n(k+1) - Y_n(k) = \tau_n A_n Y_n(k) + \Phi_n \Delta_k B_n^H \\ Y_n(0) = P_n x_0 \in V, \end{cases} \tag{2.4}$$

where $\Delta_k B_n^H = B_n^H((k+1)\tau_n) - B_n^H(k\tau_n)$. For every integer $n \geq 1$, we construct the approximation:

$$Y_n(k) = (I+\tau_n A_n)^k P_n x_0 + \sum_{l=0}^{k-1} (I+\tau_n A_n)^{k-l-1} \Phi_n \Delta_l B_n^H. \tag{2.5}$$

The rate of convergence for the approximation (2.5) of the solution given in (2.1) is given by the following Theorem.

THEOREM 2.2. *For $H \in (0, 1/2)$, we have the following error bound*

$$\begin{aligned} &\mathbb{E}\|X(k\tau_n) - D_n Y_n(k)\|_\rho^2 \\ &\leq C_1(\varphi_\gamma^2(n) + \tau_n^2 \varphi_\gamma^2(n)\|A_n\|^{2\rho} + \tau_n^2)\|x_0\|_\gamma^2 + C_2 \kappa(n)\varphi_\alpha(n) \\ &\quad + C_3 \tau_n \|A_n\|^{2\max(0,1+\rho-\gamma-\alpha)} + C_4 \tau_n^{2H} \|A_n\|^{2(H-\alpha)} \\ &\quad + C_5 \tau_n^2 \|A_n\|^{2\max(0,2+\rho-\gamma-\alpha)} + C_6 \tau_n \|A_n\|^{2(\max(0,1+\rho-\gamma-\alpha)+H)}, \end{aligned} \tag{2.6}$$

where $\kappa(n) = \|A_n\|\|A_n^{-1}\|$ and C_i, $i = 1, \cdots 6$, is a generic notation for positive constant which does not depend on n, but Hurst parameter H.

3. PROOF OF THEOREM 2

We write $[t]^+(\tau_n) = (k+1)\tau_n$ and $[t]^-(\tau_n) = k\tau_n$ if $k\tau_n \leq t < (k+1)\tau_n$, and $m(t) = [t]^-(\tau_n)/\tau_n$. Then (2.5) can be written as

$$Y_n(m(t)) = (I+\tau_n A_n)^{m(t)} P_n x_0 + \int_0^{[t]^-(\tau_n)} (I+\tau_n A_n)^{m(t)-m(s)-1} \Phi_n dB_n^H(s). \tag{3.1}$$

Hence

$$X(k\tau_n) - D_n Y_n(k) := \sum_{i=1}^{5} I_n^i(t), \tag{3.2}$$

where

$$\begin{aligned} I_n^1(t) &= e^{[t]^-(\tau_n)A} x_0 - D_n (I+\tau_n A_n)^{m(t)} P_n x_0, \\ I_n^2(t) &= \int_0^{[t]^-(\tau_n)} e^{([t]^-(\tau_n)-s)A} [\Phi - D_n \Phi_n P_n] dB^H(s), \\ I_n^3(t) &= \int_0^{[t]^-(\tau_n)} [e^{([t]^-(\tau_n)-s)A} - D_n e^{([t]^-(\tau_n)-s)A_n} P_n] D_n \Phi_n P_n dB^H(s), \\ I_n^4(t) &= \int_0^{[t]^-(\tau_n)} D_n e^{([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)A_n} [e^{([s]^-(\tau_n)+\tau_n-s)A_n} - I] P_n D_n \Phi_n P_n dB^H(s), \\ I_n^5(t) &= \int_0^{[t]^-(\tau_n)} D_n [e^{([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)A_n} - (I+\tau_n A_n)^{m(t)-m(s)-1}] P_n D_n \Phi_n P_n dB^H(s). \end{aligned}$$

Let K^* be the operator in $L^2[0,T]$ given by

$$K_T^*(h)(s) = K(t,s)h(s) + \int_s^t (h(u) - h(s))\frac{\partial K}{\partial u}(u,s)du, \tag{3.3}$$

where $K(t,s)$ is the kernel in the Wiener integral representation of fBm and given by

$$K^H(t,s) = c_H\left(\frac{t}{s}\right)^{H-(1/2)}(t-s)^{H-(1/2)} + s^{(1/2)-H}F\left(\frac{t}{s}\right) \tag{3.4}$$

c_H being constant and

$$F(x) = c_H\left(\frac{1}{2} - H\right)\int_0^{x-1} u^{H-(1/2)}\left(1-(1+u)^{H-(1/2)}\right)du. \tag{3.5}$$

• $I_n^1(t)$ term: First we write

$$\begin{aligned}
&\|I_n^1(t)\|_\rho^2 \\
&\le \|T([t]^-(\tau_n))x_0 - e^{[t]^-(\tau_n)\tilde{A}_n}x_0\|_\rho^2 + \|e^{[t]^-(\tau_n)\tilde{A}_n}x_0 - D_n(I+\tau_nA_n)^{[t]^-(\tau_n)}P_nx_0\|_\rho^2 \\
&:= J_n^1(t) + J_n^2(t).
\end{aligned}$$

Let $E(\lambda)$, $\lambda \ge 0$, be the resolution of the identity of a non-negative self-adjoint operator $(-A)$. Then by using the spectral representation of self-adjont operator and (1) of Assumption 2,

$$J_n^1(t) = \int_0^\infty \lambda^{2\rho}e^{-2[t]^-(\tau_n)\lambda}d\|E(\lambda)(D_nP_n - I)x_0\|^2 \le C\varphi_\gamma^2(n)\|x_0\|_\gamma^2. \tag{3.6}$$

Since $\tilde{A}_n$ is a bounded self-adjoint operator on V, there exist real numbers a_n, b_n and the resolution of the identity $\tilde{E}_n(\lambda)$ such that $\tilde{E}_n(\lambda) = 0$ for $\lambda < a_n$ and I for $\lambda \ge b_n$, where $a_n = \inf\{< \tilde{A}_nx, x >: \|x\| = 1\}$ and $b_n = \sup\{< \tilde{A}_nx, x >: \|x\| = 1\}$. By using $(-A)^\rho D_n = D_n(-A_n)^\rho$, $e^{[t]^-(\tau_n)\tilde{A}_n} = D_ne^{[t]^-(\tau_n)A_n}P_n$ and $(I+\tau_n\tilde{A}_n)^{m(t)} = D_n(I+\tau_nA_n)^{m(t)}P_n$, we get

$$\begin{aligned}
J_n^2(t) &= \int_0^{\|\tilde{A}_n\|} \lambda^{2\rho}(e^{-[t]^-(\tau_n)\lambda} - (1-\tau_n\lambda)^{m(t)})d\|\tilde{E}_n(\lambda)x_0\|^2 \\
&\quad + \|\tilde{A}_n\|^{2\rho}(e^{-[t]^-(\tau_n)\|\tilde{A}_n\|} - (1-\tau_n\|\tilde{A}_n\|)^{m(t)})^2\|(I - \tilde{E}_n(\|\tilde{A}_n\|-)x_0\|^2 \\
&:= J_n^{21}(t) + J_n^{22}(t).
\end{aligned}$$

Since $\sup_{0\le x\le 1}|e^{-nx} - (1-x)^n| \le \frac{1}{n}$ for $n = 1, 2, \cdots$, we find that from $(-\tilde{A}_n)^\rho = D_n(-A_n)^\rho P_n$ and Remark 4.2 in Hausenblas (2003)

$$J_n^{21}(t) \le C\tau_n^2\int_0^\infty (\lambda)^{2\rho}d\|\tilde{E}_n(\lambda)x_0\|^2 \le C\tau_n^2[\varphi_\gamma^2(n)\|A_n\|^{2\rho}\|x_0\|_\gamma^2 + \|x_0\|_\rho^2]. \tag{3.7}$$

Let $E_n(\|A_n\|-) = s{*}\lim_{\epsilon\to 0^+} E_n(\|A_n\|-\epsilon)$ where $s{*}lim$ means strong convergence of operators. By the same estimate as for $J_n^{21}(t)$, we have

$$\begin{aligned}
J_n^{22}(t) &\le C\tau_n^2\|\tilde{A}_n\|^{2\rho} s * \lim_{\epsilon\to 0^+} \|[I-\tilde{E}_n(\|A_n\|-\epsilon)]x_0\|^2 \\
&\le C\tau_n^2\|\tilde{A}_n\|^{2\rho} \lim_{\epsilon\to 0^+}(\|\tilde{A}_n\|-\epsilon)^{-2\rho}\int_0^\infty (-\lambda)^{2\rho} d\|\tilde{E}_n(\lambda)x_0\|^2 \\
&\le C\tau_n^2[\varphi_\gamma^2(n)\|A_n\|^{2\rho}\|x_0\|_\gamma^2 + \|x_0\|_\rho^2]. \qquad (3.8)
\end{aligned}$$

From (3.6), (3.7) and (3.8), we obtain

$$I_n^1(t) \le C(\varphi_\gamma^2(n) + \tau_n^2\varphi_\gamma^2(n)\|A_n\|^{2\rho} + \tau_n^2)\|x_0\|_\gamma^2. \qquad (3.9)$$

• $I_n^2(t)$ term: Using the relationship between Wiener integral with respect to fBm and Wiener integral with respect to Wiener process (see e.g., Tindel et al., 2003),

$$\begin{aligned}
\mathbb{E}\|I_n^2(t)\|_\rho^2 \le\ & 2\sum_{l=1}^\infty \int_0^{[t]^-(\tau_n)} \left|(-A)^\rho e^{([t]^-(\tau_n)-s)A}\Phi[I-D_nP_n]e_l\right| K^2(t,s)ds. \\
& +2\sum_{l=1}^\infty \int_0^{[t]^-(\tau_n)} \left| \int_s^{[t]^-(\tau_n)} \left((-A)^\rho e^{([t]^-(\tau_n)-u)A} - (-A)^\rho e^{([t]^-(\tau_n)-s)A}\right)\right. \\
& \left. \Phi[I-D_nP_n]e_l \frac{\partial K}{\partial u}(u,s)du\right|^2 ds \\
:=\ & L_n^{21}(t) + L_n^{22}(t).
\end{aligned}$$

We define the measure $\mu_n^l(B)$, $B \in \mathcal{B}(\mathbb{R})$, by $\mu_n^l(B) = \|E(B)\Phi[I-D_nP_n]e_l\|^2$. By using the properties of spectral representation and the same method as the proof of main Theorem of Tindel *et al.* (2003), we have

$$\begin{aligned}
L_n^{21}(t) &\le C(H)\sum_{l=1}^\infty \int_0^{[t]^-(\tau_n)}\int_0^\infty \lambda^{2\rho} e^{-2([t]^-(\tau_n)-s)\lambda} d\mu_n^l(\lambda)[([t]^-(\tau_n)-s)s]^{2H-1}ds \\
&\le Ctr[(I-D_nP_n)\Phi(-A)^{2(\rho-H)}\Phi^*(I-D_nP_n)] \\
&\le C\kappa(n)\varphi_\alpha(n), \qquad (3.10)
\end{aligned}$$

and also

$$\begin{aligned}
L_n^{22}(t) \le\ & C(H)\sum_{l=1}^\infty \int_0^\infty \lambda^{2\rho}\int_0^t\int_0^s\int_0^s (s-u)^{H-(3/2)}(s-v)^{H-(3/2)} \\
& \times\left(e^{-(u+v)\lambda} - e^{(s+u)\lambda} - e^{-(s+v)\lambda} + e^{-2s\lambda}\right) d\mu_n^l(\lambda) \\
\le\ & C\kappa(n)\varphi_\alpha(n). \qquad (3.11)
\end{aligned}$$

From (3.10) and (3.11), it follows that

$$\mathbb{E}\|I_n^2(t)\|_\rho^2 \leq C\kappa(n)\varphi_\alpha(n). \tag{3.12}$$

- $I_n^3(t)$ term: It is obvious that $\mathbb{E}\|I_n^3(t)\|_\rho^2 = 0$.
- $I_n^4(t)$ term: By the definition of stochastic integrals with respect to fBm and $(-A)^\rho D_n = D_n(-A_n)^\rho$, we have

$$\begin{aligned}
&\mathbb{E}\|I_n^4(t)\|_\rho^2 \\
&\leq 2\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)} \left\|(-\tilde{A}_n)^\rho e^{([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)\tilde{A}_n}[e^{([s]^-(\tau_n)+\tau_n-s)\tilde{A}_n} - I]D_n\Phi_n P_n e_l\right\|^2 \\
&\qquad \times K^2(t,s)ds. \\
&\quad +2\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)} \left\|\int_s^{[t]^-(\tau_n)} (-\tilde{A}_n)^\rho \Big(e^{([t]^-(\tau_n)-[u]^-(\tau_n)-\tau_n)\tilde{A}_n}[e^{([u]^-(\tau_n)+\tau_n-u)\tilde{A}_n} - I]\right. \\
&\quad \left. -e^{([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)\tilde{A}_n}[e^{([s]^-(\tau_n)+\tau_n-s)\tilde{A}_n} - I]\Big)D_n\Phi_n P_n e_l \frac{\partial K}{\partial u}(u,s)du\right\|^2 ds \\
&:= L_n^{41}(t) + L_n^{42}(t).
\end{aligned}$$

Using the inequalities $K(t,s) \leq c(H)(t-s)^{H-(1/2)}s^{H-(1/2)}$, $0 \leq s - [s]^-(\tau_n) \leq \tau_n$ and $1 - e^{-x} \leq x$ for $x \geq 0$, we have

$$\begin{aligned}
L_n^{41}(t) &\leq C\sum_{l=1}^{\infty}\int_0^\infty \lambda^{2\rho}\int_0^{[t]^-(\tau_n)} e^{-2([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)\lambda}\left(e^{-([s]^-(\tau_n)+\tau_n-s)\lambda} - 1\right)^2 \\
&\qquad \times[([t]^-(\tau_n)-s)s]^{2H-1}ds\,d\|\tilde{E}_n(\lambda)D_n\Phi_n P_n e_l\|^2. \\
&\leq C\tau_n^2\sum_{l=1}^{\infty}\int_0^\infty \lambda^{2(1+\rho-H)}e^{2\tau_n\lambda}\int_0^{2\lambda[t]^-(\tau_n)} e^{-s}s^{2H-1}\left(t-\frac{s}{2\lambda}\right)^{2H-1}ds \\
&\qquad \times d\|\tilde{E}_n(\lambda)D_n\Phi_n P_n e_l\|^2 \\
&\leq C\tau_n^2\sum_{l=1}^{\infty}\int_0^\infty \lambda^{2(1+\rho-H)}e^{4\tau_n\lambda}d\|\tilde{E}_n(\lambda)D_n\Phi_n P_n e_l\|^2 \\
&\leq C\tau_n^2\|A_n\|^{2\max(0,1+\rho-\gamma-\alpha)}.
\end{aligned} \tag{3.13}$$

On the other hand,

$$
\begin{aligned}
&L_n^{42}(t)\\
&= 2\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)}\int_s^{[t]^-(\tau_n)}\int_s^{[t]^-(\tau_n)}\Bigg\langle(-\tilde{A}_n)^{\rho}\Bigg(e^{([t]^-(\tau_n)-[u]^-(\tau_n)-\tau_n)\tilde{A}_n}\\
&\qquad\times[e^{([u]^-(\tau_n)+\tau_n-u)\tilde{A}_n}-I]-e^{([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)\tilde{A}_n}[e^{([s]^-(\tau_n)+\tau_n-s)\tilde{A}_n}-I]\Bigg)\\
&\quad D_n\Phi_nP_ne_l,(-\tilde{A}_n)^{\rho}\Bigg(e^{([t]^-(\tau_n)-[v]^-(\tau_n)-\tau_n)\tilde{A}_n}[e^{([v]^-(\tau_n)+\tau_n-v)\tilde{A}_n}-I]\\
&\qquad-e^{([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)\tilde{A}_n}[e^{([s]^-(\tau_n)+\tau_n-s)\tilde{A}_n}-I]\Bigg)D_n\Phi_nP_ne_l\Bigg\rangle\\
&\qquad\times\frac{\partial K}{\partial u}(u,s)\frac{\partial K}{\partial v}(v,s)dudvds\\
&= 2\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)}\int_s^{[t]^-(\tau_n)}\int_s^{[t]^-(\tau_n)}\int_0^{\infty}\lambda^{2\rho}h(u,v,s,\lambda)d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2\\
&\qquad\times\frac{\partial K}{\partial u}(u,s)\frac{\partial K}{\partial v}(v,s)dudvds, \qquad (3.14)
\end{aligned}
$$

where $h(u,v,s,\lambda)$ is given by $h(u,v,s,\lambda)=\tilde{h}(u,s,\lambda)\tilde{h}(v,s,\lambda)$,

$$
\begin{aligned}
\tilde{h}(x,s,\lambda) &= e^{-([t]^-(\tau_n)-[x]^+(\tau_n))\lambda}[1-e^{-([x]^+(\tau_n)-x)\lambda}]\\
&\quad -e^{-([t]^-(\tau_n)-[s]^+(\tau_n))\lambda}[1-e^{-([s]^+(\tau_n)-s)\lambda}].
\end{aligned}
$$

From $|(\partial K/\partial u)(u,s)|\le C(H)(u-s)^{H-(3/2)}$, the above (3.14) becomes

$$
\begin{aligned}
&L_n^{42}(t)\\
&\le C_1\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)}\left(\int_s^{[s]^+(\tau_n)}\tilde{h}(u,s,\lambda)(u-s)^{H-(3/2)}du\right)^2ds\\
&\qquad\times d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2\\
&\quad +C_2\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)}\left(\int_{[s]^+(\tau_n)}^{[t]^-(\tau_n)}\tilde{h}(u,s,\lambda)(u-s)^{H-(3/2)}du\right)^2ds\\
&\qquad\times d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2\\
&:= L_n^{421}(t)+L_n^{422}(t).
\end{aligned}
$$

As for $L_n^{421}(t)$,

$$L_n^{421}(t) = C\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)}\left(\int_s^{[s]^+(\tau_n)}\tilde{h}(u,s,\lambda)(u-s)^{H-(3/2)}du\right)^2 ds \times d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2,$$

where $\tilde{h}(u,s,\lambda)$ is given by

$$\begin{aligned}&\tilde{h}(u,s,\lambda)\\ &= e^{-([t]^-(\tau_n)-[s]^+(\tau_n))\lambda}[1-e^{-([s]^+(\tau_n)-u)\lambda}] - e^{-([t]^-(\tau_n)-[s]^+(\tau_n))\lambda}[1-e^{-([s]^+(\tau_n)-s)\lambda}].\end{aligned}$$

By the integration by parts, we can write $L_n^{421}(t)$ as follows:

$$\begin{aligned}L_n^{421}(t) &= C\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)} e^{-2([t]^-(\tau_n)-[s]^+(\tau_n))\lambda}e^{-2([s]^+(\tau_n)-s)\lambda}\\ &\quad\times(e^{([s]^+(\tau_n)-s)\lambda}-1)^2([s]^+(\tau_n)-s)^{2H-1}dsd\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2\\ &\quad+C\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2(\rho+1)}\int_0^{[t]^-(\tau_n)} e^{-2([t]^-(\tau_n)-[s]^+(\tau_n))\lambda}e^{-2([s]^+(\tau_n)-s)\lambda}\\ &\quad\times\left(\int_s^{[s]^+(\tau_n)} e^{(u-s)\lambda}(u-s)^{H-(1/2)}du\right)^2 dsd\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2\\ &:= L_n^{4211}+L_n^{4212}.\end{aligned}$$

Using the fact that $\tilde{E}_n(\lambda)$ is the resolution of the identity of bounded operator $\tilde{A}_n$, we obtain

$$\begin{aligned}L_n^{4211}(t) &\le C\tau_n^{2H}\|A_n\|^{2(H-\alpha)}\int_0^{\infty}\lambda^{2(\rho+\alpha-H)}d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2\\ &\le C\tau_n^{2H}\|A_n\|^{2(H-\alpha)},\end{aligned}\tag{3.15}$$

and also

$$L_n^{4212}(t) \le C\tau_n\|A_n\|^{2\max(0,1+\rho-\gamma-\alpha)}.\tag{3.16}$$

Now we consider the term $L_n^{422}(t)$. First we write $\tilde{h}(u,s,\lambda) = \tilde{h}_1(u,\lambda)+\tilde{h}_2(s,\lambda)$, where

$$\begin{aligned}\tilde{h}_1(u,\lambda) &= e^{-([t]^-(\tau_n)-[u]^+(\tau_n))\lambda}[1-e^{-([u]^+(\tau_n)-u)\lambda}]\\ \tilde{h}_2(s,\lambda) &= -e^{-([t]^-(\tau_n)-[s]^+(\tau_n))\lambda}[1-e^{-([s]^+(\tau_n)-s)\lambda}].\end{aligned}$$

Using $\tilde{h}_1$ and $\tilde{h}_2$, we have

$$\begin{aligned} L_n^{422}(t) \le\ & C\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)}\left(\int_{[s]^+(\tau_n)}^{[t]^-(\tau_n)}\tilde{h}_1(u,s,\lambda)(u-s)^{H-(3/2)}du\right)^2 ds \\ & \times d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2 \\ & +C\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)}\left(\int_{[s]^+(\tau_n)}^{[t]^-(\tau_n)}\tilde{h}_2(s,\lambda)(u-s)^{H-(3/2)}du\right)^2 ds \\ & \times d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2 \\ := & L_n^{4221}(t)+J_n^{4222}(t). \end{aligned}$$

By the inequality $1-e^{-x}\le x$ for $0\le x\le 1$, the term $L_n^{4221}(t)$ can be bounded by

$$\begin{aligned} L_n^{4221}(t) &\le C\tau_n^2\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2(\rho+1)}[([t]^-(\tau_n))^{2H}+\tau_n^{2H}]d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2 \\ &\le C[([t]^-(\tau_n))^{2H}\tau_n^2\|\tilde{A}_n\|^{2H}+\tau_n^{2(1+H)}\|\tilde{A}_n\|^{2H}] \\ &\quad\times\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2(\rho+1-H)}d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2 \\ &\le C\tau_n^2\|A_n\|^{2(\max(0,1+\rho-\gamma-\alpha)+H)}. \end{aligned} \tag{3.17}$$

By a similar estimate as for $L_n^{4221}(t)$, we have

$$L_n^{4222}(t)\le C\tau_n^2\|A_n\|^{2(\max(0,1+\rho-\gamma-\alpha)+H)}. \tag{3.18}$$

Combining (3.13), (3.15), (3.16), (3.17) and (3.18), we obtain

$$\begin{aligned} \mathbb{E}\|I_n^4(t)\|_\rho^2 \le\ & C_1\tau_n^2\|A_n\|^{2\max(0,1+\rho-\gamma-\alpha)}+C_2\tau_n^{2H}\|A_n\|^{2(H-\alpha)} \\ & +C_3\tau_n\|A_n\|^{2\max(0,1+\rho-\gamma-\alpha)}+C_4\tau_n^2\|A_n\|^{2(\max(0,1+\rho-\gamma-\alpha)+H)}. \end{aligned} \tag{3.19}$$

• $I_n^5(t)$ term: By the definition of stochastic integrals with respect to fBm with $H \in (0,1/2)$,

$$\begin{aligned}
&\mathbb{E}\|I_n^5(t)\|_\rho^2 \\
&\le 2\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)} \|(-\tilde{A}_n)^\rho[e^{([t]^-(\tau_n)-[u]^-(\tau_n)-\tau_n)\tilde{A}_n} - (I+\tau_n\tilde{A}_n)^{m(t)-m(u)-1}] \\
&\qquad D_n\Phi_nP_ne_l\|^2K^2([t]^-(\tau_n),s)ds \\
&\quad +2\sum_{l=1}^{\infty}\int_0^{[t]^-(\tau_n)}\left\|\int_s^{[t]^-(\tau_n)}(-\tilde{A}_n)^\rho\Big([e^{([t]^-(\tau_n)-[u]^-(\tau_n)-\tau_n)\tilde{A}_n}\right. \\
&\qquad -(I+\tau_n\tilde{A}_n)^{m(t)-m(u)-1}] - [e^{([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)\tilde{A}_n} \\
&\qquad \left. -(I+\tau_n\tilde{A}_n)^{m(t)-m(s)-1}]\Big)D_n\Phi_nP_ne_l\frac{\partial K}{\partial u}(u,s)du\right\|^2 ds \\
&:= L_n^{51}(t) + L_n^{52}(t).
\end{aligned}$$

By a similar estimate as for $L_n^{41}(t)$ and $e^{-nx} - (1-x)^n \le \sqrt{n}x$ for $0 \le x \le 1$ and $n = 1,2,\cdots$,

$$\begin{aligned}
&L_n^{51}(t) \\
&\le C\tau_n\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2(1+\rho-H)}e^{2\tau_n\lambda}\int_0^{2\lambda[t]^-(\tau_n)} e^{-s}s^{2H-1}\left(t-\frac{s}{2\lambda}\right)^{2H-1}ds \\
&\qquad d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2 \\
&\quad +C\tau_n^2\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2(2+\rho-H)}e^{2\tau_n\lambda}\int_0^{2\lambda[t]^-(\tau_n)} e^{-s}s^{2H-1}\left(t-\frac{s}{2\lambda}\right)^{2H-1}ds \\
&\qquad d\|\tilde{E}_n(\lambda)D_n\Phi_nP_ne_l\|^2 \\
&\le C\tau_n\|A_n\|^{2\max(0,1+\rho-\gamma-\alpha)} + C\tau_n^2\|A_n\|^{2\max(0,2+\rho-\gamma-\alpha)}. \qquad (3.20)
\end{aligned}$$

We define a function g by

$$\begin{aligned}
g(u,s,\lambda) = &[e^{-([t]^-(\tau_n)-[u]^-(\tau_n)-\tau_n)\lambda} - (1-\tau_n\lambda)^{m(t)-m(u)-1}] \\
&-[e^{-([t]^-(\tau_n)-[s]^-(\tau_n)-\tau_n)\lambda} - (1-\tau_n\lambda)^{m(t)-m(s)-1}].
\end{aligned}$$

Then by a similar estimate as for $L_n^{42}(t)$ and $g(u,s,\lambda)=0$ for $s \le u < [s]^+(\tau_n)$, we have

$$
\begin{aligned}
L_n^{52}(t) &\le C\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)}\left(\int_s^{[t]^-(\tau_n)} g(u,s,\lambda)(u-s)^{H-(3/2)}du\right)^2 ds \\
&\qquad \times d\|\tilde{E}_n(\lambda)D_n\Phi_n P_n e_l\|^2 \\
&\le C\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2\rho}\int_0^{[t]^-(\tau_n)}\left(\int_{[s]^+(\tau_n)}^{[t]^-(\tau_n)} g(u,s,\lambda)(u-s)^{H-(3/2)}du\right)^2 ds \\
&\qquad \times d\|\tilde{E}_n(\lambda)D_n\Phi_n P_n e_l\|^2 \\
&\le C\tau_n\sum_{l=1}^{\infty}\int_0^{\infty}\lambda^{2(1+\rho)}[(([t]^-(\tau_n))^{2H}+\tau_n^{2H}]d\|\tilde{E}_n(\lambda)D_n\Phi_n P_n e_l\|^2 \\
&\le C\tau_n\|A_n\|^{2(\max(0,1+\rho-\gamma-\alpha)+H)}. \qquad (3.21)
\end{aligned}
$$

From (3.20) and (3.21), it follows that

$$
\begin{aligned}
\mathbb{E}\|I_n^5(t)\|_\rho^2 &\le C_1\tau_n\|A_n\|^{2\max(0,1+\rho-\gamma-\alpha)} + C_2\tau_n^2\|A_n\|^{2\max(0,2+\rho-\gamma-\alpha)} \\
&\quad + C_3\tau_n\|A_n\|^{2(\max(0,1+\rho-\gamma-\alpha)+H)}. \qquad (3.22)
\end{aligned}
$$

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