The *BMAP/G/*1 Queue with Correlated Flows of Customers and Disasters

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Abstract A single-server queueing model with the Batch Markovian Arrival Process and disaster ow correlated with the arrival process is analyzed. The numerically stable algorithm for calculating the steady state distribution of the system is presented.

Key Words : Batch Markovian Arrival Process, Correlated Flow, Disasters

1. Introduction

Notion of a negative customer, i.e., the customer, which removes one or a group of usual ("positive") customer from a system upon the arrival, was introduced by E. Gelenbe in [10]. During the last decade considerable attention has been paid to study queueing systems with negative arrivals. For a comprehensive analysis of queueing systems and networks with negative customers, readers may refer to [1, 2, 4, 10, 11, 13].

The term 'disaster' was introduced by Jain and Sigman [12]. Disaster is a kind of negative customer which removes all the customers from the system, including one in service, upon its arrival.

Chen and Renshaw [6] considered the *M/M/*1 queue with disasters. Jain and Sigman [12] derived a Pollaczek-Khinchine formula for the Laplace transform of the steady state work in the system *M/G/*1 with Poisson arrivals of disasters.

Dudin and Nishimura [7] investigated in detail the *BMAP/SM/*1 queue with the *MAP* input of disasters and instantaneous recovering of the server. Dudin and Karolik [8] investigated analogous system in case when the recovery of the server takes some random time during which customers are accumulated or lost. Numerical analysis of the models is given in [8]. Shin [16] considered the *BMAP/G/*1 queue with correlated arrivals of customers and disasters and non-instantaneous server recovery after disaster arrival. Semenova investigated controlled *BMAP/SM/*1 type queues with the input flow, service process and disaster flow depending on the current operation mode which dynamically depends on the number of customers in the system, see, e.g.,[15].

In this work, we examine stationary queue length distribution in *BMAP/G/*1 queue with correlated arrival of customers and disasters with instantaneous server recovery. In [7, 8, 16] the so called transform approach is exploited. This approach accounts the analyticity of the vector generating function of the stationary distribution in a unit disk of a complex plane.

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It includes calculation of roots of the polynomial matrix determinant which is known to be a serious issue. In opposite to these papers and to Dudin and Semenova [9] where the matrix-analytic approach was used in combination with the method of embedded Markov chains and the theory of Markov renewal processes (see [5]), here the matrix-analytic approach is used in combination with the method of supplementary variables. The stable procedure for computing the steady state distribution of the system at arbitrary epoch is presented.

2. Model Descriptions

The input flow into the system is the following modification of the well-known (see, e.g., [3],[14]) *BMAP*. In this input flow, the inter-arrival times of customers are directed by an irreducible continuous time process $v_t, t \geq 0$ (directing process) with the state space $\{0,1,\cdots,W\}$. This process behaves as follows. Sojourn time of process v_t in state v is exponentially distributed with parameter, λ_v, $\lambda_v \geq 0$, $v = \overline{0,W}$. After this time expires, the process v_t jumps to the state r and the group consisting of k customers arrives with probability $d_k(v,r), k \geq 0$ or the disaster happens with probability $d_{-\infty}(v,r), v,r = \overline{0,W}$. It is supposed that $d_0(v,v) = 0$, $v = \overline{0,W}$, and the following normalization condition fulfills:

$$\sum_{r=0}^{W}\left(\sum_{k=0}^{\infty} d_k(v,r) + d_{-\infty}(v,r)\right) = 1,\ v = \overline{0,W}$$

Introduce into consideration matrices $D_k, k \geq 0$ and $D_{-\infty}$ having the entries defined as follows:

$$(D_0)_{v,v} = -\lambda_v, v = \overline{0,W},$$
$$(D_0)_{v,r} = \lambda_v d_0(v,r),\ v = \overline{0,W}, v \neq r$$
$$(D_k)_{v,r} = \lambda_v d_k(v,r),\ k \geq 1,$$
$$(D_{-\infty})_{v,r} = \lambda_v d_{-\infty}(v,r),\ v,r = \overline{0,W}$$

So, the input flow is completely defined by the set of matrices $D_{-\infty}$, $D_k, k \geq 0$.

Denote $D(z) = \sum_{k=0}^{\infty} D_k z^k,\ |z| \leq 1$

An arriving group of customers is placed into the buffer of infinite size if the server is busy at the moment of its arrival; otherwise, the service of a first customer in a group begins while the rest of a group enters the buffer. Service time is characterized by the distribution function $B(t)$ with the Laplace-Stieltjes Transform $B^*(s) = \int_0^{\infty} e^{-st} dB(t),\ s \geq 0$. When disaster happens, all the customers leave the system immediately. It is assumed that the service device also recovers immediately.

3. Stationary Distribution

Consider the following probabilities:

$$f_t(0,v) = P\{i_t = 0, v_t = v\}$$
$$f_t(i,v,x) = P\{i_t = 0, v_t = v, x_t \langle x\},$$
$$i \geq 1, v = \overline{0,W}, x \geq 0$$

where i_t is the number of customers in the system, v_t is the state of the directing process of *BMAP*, x_t is the amount of time left to finish the service of the customer being served at the epoch $t, t \geq 0$ (residual service time). For every t the following normalization condition holds:

$$\sum_{v=0}^{W}\left(f_t(0,v) + \sum_{i=1}^{\infty} f_t(i,v,+\infty) \right) = 1,\ t \geq 0$$

Denote the following probability vectors:

$$\vec{f}_t(0) = (f_t(0,0), \cdots, f_t(0,W))$$
$$\vec{f}_t(i,x) = (f_t(i,0,x), \cdots, f_t(i,W,x))$$
$$i \geq 1, x \geq 0, t \geq 0$$

As it is shown in [16], the presence of the input flow of disasters with non-zero intensity, $D_{-\infty} \neq 0$ is the sufficient condition for the stationary queue length distribution existence for the considered system.

Denote the following stationary probability vectors:

$$\vec{p}_0 = \lim_{t\to\infty} \vec{f}_t(0),$$
$$\vec{p}_i(x) = \lim_{t\to\infty} \vec{f}_t(i,x)$$

and their Laplace-Stieltjes transforms, derivatives and generating functions by

$$\vec{p}_i^{\,*}(s) = \int_0^{\infty} e^{-st} d\vec{p}_i(x),\ s > 0,$$
$$\vec{p}_i'(x) = d\vec{p}_i(x)/dx$$
$$\vec{p}^{\,*}(z,s) = \sum_{i=1}^{\infty} \vec{p}_i^{\,*}(s) z^i,$$
$$\vec{p}(z,x) = \sum_{i=1}^{\infty} \vec{p}_i(x) z^i$$
$$\vec{p}\,'(z,x) = \sum_{i=1}^{\infty} p_i'(x) z^i,\ x \geq 0, |z| \leq 1$$

Lemma 1. *Vectors* $\vec{p}^{\,*}(z,s)$, $\vec{p}_0$ *and* $\vec{p}\,'(z,0)$ *satisfy to the following equation:*

$$\vec{p}^{\,*}(z,s)(sI + D(z)) + \vec{p}_0 D(z) B^*(s) +$$
$$\vec{x} D_{-\infty} B^*(s) + z^{-1}\vec{p}(z,0)\left(B^*(s) - z\right) = \vec{0}$$

where x is the stationary probability vector of the process $v_t, t \geq 0$; which is calculated as the unique solution of the following system of linear equations:

$$\vec{x}\left(D(1) + D_{-\infty}\right) = \vec{0},\ \vec{x}\vec{e} = \vec{1}$$

where $\vec{e}$ is the column-vector of proper dimension consisting of all 1's, $\vec{0}$ is the row vector consisting of all 0's, I is the identity matrix.

Proof. It can be easily seen that the process $(i_t, v_t, x_t), t \geq 0$, is a Markov process. Thus, we can compose Kolmogorov-Chapman equations for the probabilities $f_t(0,v)$ and $f_t(i,v,x)$, $i \geq 1, v = \overline{0,W}, x \geq 0$. In the matrix form they look as follows:

$$\vec{f}_{t+\Delta}(0) = \vec{f}_t(0)(I + D_0\Delta) + \vec{f}_t(1,\Delta) +$$
$$\left(\vec{f}_t(0) + \sum_{i=1}^{\infty} \vec{f}_t(i,+\infty) \right) D_{-\infty}\Delta + o(\Delta)$$
$$\vec{f}_{t+\Delta}(i,x) = \vec{f}_t(0) D_i \Delta B(x) +$$
$$+ \{\vec{f}_t(i,x+\Delta) - \vec{f}_t(i,\Delta)\}(I + D_0\Delta)$$
$$+ \sum_{j=1}^{i-1} \vec{f}_t(j,x+\Delta) D_{i-j}\Delta$$
$$+ \vec{f}_t(i+1,\Delta) B(x) + o(\Delta),\ i \geq 1$$

From these difference equations, following differential equations for stationary probability vectors are easy derived:

$$\vec{p}_0 D_0 + \vec{p}\,'(1,0) + \left(\vec{p}_0 + \sum_{i=0}^{\infty} \vec{p}(i,+\infty)\right) D_{-\infty} = \vec{0}$$

$$\vec{p}\,'(i,x) + \vec{p}_0 D_i B(x) + \sum_{j=1}^{i-1} \vec{p}(j,x) D_{i-j} + \\ + \sum_{j=1}^{i-1} \vec{p}(j,x) D_{i-j} + \vec{p}\,'(i+1,0) B(x) \\ - \vec{p}(i,0) = \vec{0},\ i \geq 1$$

Summing all these equations up and putting $x \to +\infty$ we get the following system of linear equations:

$$\left(\vec{p}_0 + \sum_{i=0}^{\infty} \vec{p}_i(+\infty)\right)(D(1) + D_{-\infty}) = \vec{0}$$

Taking into account normalization condition (1) when $x \to +\infty$ and equation (3) for the vector x, we conclude that $\vec{p}_0 + \sum_{i=1}^{\infty} \vec{p}_i(+\infty) = \vec{x}$.

Finally, applying Laplace-Stieltjes transform and z-transform to the equations (4) we easy get (2).

Theorem 1. *The vectors* $\vec{p}_0$ *and* $\vec{p}(i,+\infty),\ i \geq 1$ *satisfy the following system of linear equations:*

$$\vec{p}_0 = \vec{p}_0 Y_0 + \vec{x} D_{-\infty} W_0 + \vec{p}(1,+\infty) Y_0$$

$$\vec{p}(i,+\infty) = \vec{p}_0 Y_i + \vec{x} D_{-\infty} W_i + \sum_{j=1}^{i+1} \vec{p}(j,+\infty) Y_{i+1-j},\ i \geq 1$$

where Y_i *and* $W_i,\ i \geq 0$ *are the coefficients of the following matrix expansions:*

$$\sum_{i=0}^{\infty} Y_i z^i = \int_0^{\infty} e^{D(z)t} dB(t),$$

$$\sum_{i=0}^{\infty} W_i z^i = \int_0^{\infty} e^{D(z)t} (1 - B(t))\, dt,\ |z| \leq 1$$

Proof. Substituting matrix $(-D(z))$ instead of scalar value s into formula (2), we get the following expression:

$$\vec{p}(z,0)(Y(z) - zI) + \vec{p}_0 D(z) Y(z) + \vec{x} D_{-\infty} Y(z) = \vec{0}$$

Multiply relation (2) by $(Y(z) - zI)$ and relation (7) by $(z - B^*(s))$. Sum these expressions up, put $s = 0$, multiply the result by $(D(z))^{-1}$ and take into account evident relations $W(z) = (D(z))^{-1}(Y(z) - I) = (Y(z) - I)(D(z))^{-1}$. As the result we get the following relation:

$$\vec{p}(z,+\infty)(Y(z) - zI) + \vec{p}_0 (Y(z) - I) + \vec{x} D_{-\infty} W(z) = \vec{0}$$

Now, equations (5) and (6) are the derived as the coefficients of the MacLaurent expansion of relation (8) at the point $z = 0$.

4. Stable Recursion for Computing the Stationary Queue Length Distribution

Theorem 2. *Probability vector* $\vec{p}_0$ *can be computed as follows:*

$$\vec{p}_0 = -\vec{x} D_{-\infty} \left(\sum_{k=0}^{\infty} D_k G^k\right)^{-1}$$

where G is the minimal nonnegative solution of *the matrix equation*

$$G = \sum_{i=0}^{\infty} Y_i G^i$$

Proof. Multiplying equations of the system (5) and (6) by and summing them up we get the following expression:

$$\vec{p}_0 (G - I) + \vec{x} D_{-\infty} \sum_{i=0}^{\infty} W_i G^i = 0$$

It can be shown that $(G - I)$ is invertible if $D_{-\infty} \neq 0$. Thus, multiplying relation (10) by $(G - I)^{-1}$ and taking into account $\sum_{i=0}^{\infty} W_i G^i (G - I)^{-1} = \left(\sum_{k=0}^{\infty} D_k G^k \right)^{-1}$, we get (9).

Theorem 3. *Probability vectors* $\vec{p}_i(+\infty), i \geq 1$, *can be computed recursively by*

$$\vec{p}_i(+\infty) = (\vec{p}_0 \overline{Y_i} + \vec{x} D_{-\infty} \overline{W_i} + \sum_{j=1}^{i-1} \vec{p}(j,+\infty) \overline{Y}_{i+1-j})(I - \overline{Y}_1)^{-1}, \; i \geq 1$$

where

$$\overline{Y_i} = \sum_{k=i}^{\infty} Y_k G^{k-i},$$

$$\overline{W_i} = \sum_{k=i}^{\infty} W_k G^{k-i}, \; i \geq 0$$

Proof. Proof is implemented by multiplying equations of the system (6) by $G^{k-i}, k \geq 1$ for $i \geq k$ and summing them up.

Numerical stability of the described procedures for calculating the vectors $\vec{p}_k(+\infty), k \geq 1$ stems from the fact that all matrices involved into recursion (11) are non-negative. Presented results can be easy extended to the case when the recovering of the server is not instantaneous.

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