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SCALAR EXTENSION OF SCHUR ALGEBRAS

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ABSTRACT. Let K be an algebraic number field. If k is the maximal cyclotomic subextension in K then the Schur K-group $S(K)$ is obtained from the Schur k-group $S(k)$ by scalar extension. In the paper we study projective Schur group $PS(K)$ which is a generalization of Schur group, and prove that a projective Schur K-algebra is obtained by scalar extension of a projective Schur k-algebra where k is the maximal radical extension in K with mild condition.

1. Introduction

Let K be a field. A finite dimensional central simple K-algebra is a Brauer algebra. A Brauer K-algebra A is a projective Schur algebra if there is a finite group G and a 2-cocycle $\alpha \in Z^2(G, K^*)$ such that A is a homomorphic image of the twisted group algebra KG^α, where $K^* = K - \{0\}$ is regarded as a G-module with respect to the trivial G-action. The similarity class containing a Brauer K-algebra A is denoted by $[A]$, and they form a Brauer group $B(K)$. The projective Schur group $PS(K)$ is a subgroup of $B(K)$ consisting of similarity classes which are represented by projective Schur K-algebras (refer to [1, 4]). When $\alpha = 1$, the projective Schur K-algebra A is called a Schur K-algebra, and the set of $[A]$'s forms the Schur group $S(K)$. If characteristic of K is positive then $S(K)$ is trivial.

Assume that K is an algebraic number field (i.e., a finite extension of the rational field $\mathbb{Q}$) with algebraic closure E. Let $\mathbb{Q}(\mu)$ denote the maximal cyclotomic extension of $\mathbb{Q}$ contained in E, where μ is the group of all roots of unity in E. Let $k = \mathbb{Q}(\mu) \cap K$ and $K \otimes_k S(k)$ be the subgroup of $B(K)$ obtained from $S(k)$ by extension of scalars. Then

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$K \otimes_k S(k)$ is contained in $S(K)$, and it was proved in [9, (4.6)] that

$$S(K) = K \otimes_k S(k) = \{[K \otimes_k A] \mid [A] \in S(k)\}.$$

We may refer to [5, Theorem 3.4], and [7] over ring K of algebraic integers.

The purpose of this paper is to study situations that a projective Schur K-algebra is obtained from a projective Schur k-algebra by scalar extensions where k is a subfield of K. Upon using projective characters, we recast the definition of projective Schur algebra in Theorem 7, and prove that if K is an algebraic number field and k is a maximal radical extension field of $\mathbb{Q}$ contained in K with mild conditions then a projective Schur K-algebra A can be written as $A \cong K \otimes_k A'$ for a projective Schur k-algebra A' in Theorem 8. Moreover we construct a subgroup $PS_F(K)$ of $PS(K)$ with $F \subset K$ such that $PS_F(K) = K \otimes_k PS_F(k)$ for some subfield k of K in Theorem 12. As an application to a special class of algebras, radical algebra was discussed in Theorem 13.

In what follows, we mean that K^* is the multiplicative subgroup of a field K, ε_n is a primitive n-th root of unity for $n > 0$ and μ is the set of roots of unity. For a finite group G, $|G|$ is the order of G and $o(g)$ is the order of $g \in G$. We denote by $\mathbb{Q}$ the rational number field, and by $Z(A)$ the center algebra of an algebra A.

2. Projective group character

We always assume that K is a field of characteristic 0 with an algebraic closure E, and G is a finite group of exponent n. Let ρ be an irreducible E-representation of G, and χ be the E-character of G afforded by ρ. Then

$$e(\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g \tag{1}$$

is a block idempotent of the group algebra EG (see [9, (1.1)] or [3, Vol.1(19.2.7)]). Moreover $e(\chi)$ is a block idempotent of $K(\varepsilon_n)G$, because all values of χ are contained in $K(\varepsilon_n)$. And the Galois group $\mathrm{Gal}(K(\varepsilon_n)/K)$ acts on the set of idempotents

$$\{e(\chi) \mid \chi \text{ irreducible } E\text{-characters of } G\}$$

by $\tau \cdot e(\chi) = e(\chi^\tau)$, where $\chi^\tau(g) = \tau(\chi(g))$ for all $\tau \in \mathrm{Gal}(K(\varepsilon_n)/K)$ and $g \in G$.

Let $K(\chi)$ denote the subfield of E generated by K and the character values $\chi(g)$ for all $g \in G$. Since $\chi(g)$ is a sum of $o(g)$-th roots of unity over K, $K(\chi) = K(\varepsilon_d)$ where $d|n$. And the block idempotent $v(\chi)$ of KG such that $e(\chi)$ is a summand of $v(\chi)$ forms (see [3, Vol.3(14.1.14)])

$$(2) \qquad v(\chi) = \sum_{\tau \in \mathrm{Gal}(K(\chi)/K)} e(\chi^\tau) \quad \text{(the } e(\chi^\tau) \text{ are all distinct).}$$

We shall denote the simple component A of KG corresponding to χ by $KGv(\chi)$. Then A is central over K if and only if $K = K(\chi)$ ([9, (1.4)]). Hence in this case, $\mathrm{Gal}(K(\chi)/K) = 1$, $e(\chi) = v(\chi)$ and A is isomorphic to $KGe(\chi)$. Thus the definition of Schur algebra can be stated as follow.

DEFINITION 1. [3, Vol.3, p.819] Let K be a field of characteristic 0 and E be an algebraic closure of K. Then a central simple algebra A is a Schur K-algebra if and only if there is a finite group G and an irreducible E-character χ of G such that $K = K(\chi)$ and $A \cong KGe(\chi)$ where $e(\chi)$ is as in (1).

In [9, (4.6)], the equality $S(K) = K \otimes_k S(k)$ was proved by employing Brauer-Witt theorem which states that every Schur K-algebra is similar to a cyclotomic algebra. Though there is a radical algebra which is an analog of cyclotomic algebra in projective Schur algebra theory, only projective Schur *division* algebra is a radical algebra ([1]). Hence in next theorem we will prove the identity $S(K) = K \otimes_k S(k)$ by making use of group characters, so that the similar method will be extended to projective Schur algebra case.

THEOREM 2. *Let K be an algebraic number field and k be the maximal cyclotomic extension field contained in K. Then $S(K) = K \otimes_k S(k)$.*

Proof. Let $[S] \in S(K)$. Then there is a Schur algebra $A \in [S]$ such that

$$A \cong KGe(\chi) \ \ \text{(as } K\text{-algebras)} \ \ \text{and } K = K(\chi),$$

for a finite group G, an irreducible E-character χ of G, and the block idempotent $e(\chi)$ of EG as in (1).

Let A' be the simple component of kG corresponding to χ, and $k(\chi)$ be the extension field adjoining all values of χ to k. Then $k \subseteq k(\chi) \subseteq E$. Since $e(\chi)$ belongs to $k(\chi)G$, $v'(\chi) = \sum_{\tau \in \mathrm{Gal}(k(\chi)/k)} e(\chi^\tau)$ (the $e(\chi^\tau)$ are all distinct) is a block idempotent of kG (see (2)). Thus $A' \cong kGv'(\chi)$.

All $\chi(g)$ ($g \in G$) are contained in $K(\chi) \cap \mathbb{Q}(\mu) = K \cap \mathbb{Q}(\mu)$, where μ is the set of primitive roots of unity in E. But since $K \cap \mathbb{Q}(\mu)$ is a cyclotomic extension of $\mathbb{Q}$ in K, we have $K \cap \mathbb{Q}(\mu) \subseteq k$ and $\chi(g) \in k$ for all $g \in G$. Thus $k(\chi) = k$, $v'(\chi) = e(\chi)$ and A' is a central simple k-algebra such that $A' \cong kGe(\chi)$. Hence we have $[S] = [A] = [KGe(\chi)] = [K \otimes_k kGe(\chi)] = K \otimes_k [A'] \in K \otimes_k S(k)$. □

Throughout the paper we always assume that K is a field of characteristic 0 and E is an algebraic closure of K. Let α be a 2-cocycle in $Z^2(G, K^*)$ with $\alpha(x, 1) = \alpha(1, x) = 1$ for all $x, y \in G$, and $\{a_x | x \in G\}$ be a basis of the twisted group algebra KG^α satisfying $a_x a_y = \alpha(x, y) a_{xy}$. We denote by ρ an irreducible projective α-representation of G over E and by χ_α the α-character afforded by ρ.

THEOREM 3. *Let K, E and χ_α be defined as above. Then there is a finite Galois radical extension F over K in E containing $K(\chi_\alpha)$. That is, $F = K(\Omega)$, where Ω is a Gal(F/K)-invariant subgroup of F^* such that $\Omega K^*/K^*$ is finite, and $K(\chi_\alpha) \subseteq F$.*

Proof. For any $g \in G$, let

$$\lambda_g = \prod_{i=1}^{o(g)} \alpha(g^i, g) \in K \tag{3}$$

and let δ_g in E be any $o(g)$-th root of λ_g. Let Ω_α be the subset

$$\Omega_\alpha = \langle \mu, \{\delta_g |\ g \in G\} \rangle \subseteq E^*,$$

where μ is the set of $|G|$-th root of unity in E. Then $K \subseteq K(\Omega_\alpha) \subseteq E$, and $\Omega_\alpha K^*$ is torsion over K^*. And since δ_g is a root of the polynomial $X^{o(g)} - \lambda_g \in K[X]$, any automorphism on $K(\Omega_\alpha)$ maps δ_g to another root of $X^{o(g)} - \lambda_g$ that belongs to Ω_α. Thus Ω_α is Gal$(K(\Omega_\alpha)/K)$-invariant, and $K(\Omega_\alpha)$ is a finite Galois radical extension field of K. Moreover since $\chi_\alpha(g)$ is a sum of δ_g ([3, Vol.3(1.2.6)]), it follows that $\chi_\alpha(g)$ belongs to $K(\Omega_\alpha)$, hence $K(\chi_\alpha)$ is a subfield of $K(\Omega_\alpha)$. □

Maintaining the above notations, we get next corollary.

COROLLARY 4. *Let $\alpha \in Z^2(G, K^*)$ be of finite order $o(\alpha)$. Then $K(\chi_\alpha)$ is a subfield of a cyclotomic extension field over K in E.*

Proof. For $g \in G$, we use the same notations $\lambda_g \in K$ and $\delta_g \in E$ as in Theorem 3. Let ρ be the irreducible α-representation of G over E affording χ_α.

Consider any positive multiple $n = o(g)s$ with some $s > 0$. Let $\lambda'_g = \prod_{i=1}^{n} \alpha(g^i, g)$ and let δ'_g be an n-th root of λ'_g in E. Then $\rho(g)^n =$

$\prod_{i=1}^{n} \alpha(g^i, g)\rho(g^n) = \lambda'_g I$, where I is the identity matrix, and

$$(\delta'_g)^n = \lambda'_g = \prod_{i=1}^{n} \alpha(g^i, g) = \left(\prod_{i=1}^{o(g)} \alpha(g^i, g)\right)^s = (\lambda_g)^s = \left(\delta_g^{o(g)}\right)^s = (\delta_g)^n.$$

Since $o(\alpha)$ is finite, if we consider $n = o(g)o(\alpha)$ then

$$(\delta_g)^n = \left(\prod_{i=1}^{o(g)} \alpha(g^i, g)\right)^{o(\alpha)} = \prod_{i=1}^{o(g)} \alpha^{o(\alpha)}(g^i, g) = 1,$$

and we may choose δ_g as an n-root of unity. Thus $K(\chi_\alpha)$ is contained in a cyclotomic subfield of E. $\square$

Since the algebraic closure E is a splitting field for EG^α,

$$e(\chi_\alpha) = \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1})\chi_\alpha(g^{-1})a_g \tag{4}$$

is a block idempotent of EG^α associated with χ_α ([3, Vol.3(1.11.1)]). If Γ is a finite dimensional K-algebra and V is a Γ-module then $\Gamma^E = E \otimes_k \Gamma$ and $V^E = E \otimes_k V$ are E-algebra and Γ^E-module respectively. And any block idempotent v of Γ is a central idempotent of Γ^E. Thus v can be written uniquely as a sum of distinct block idempotents e of Γ^E ([3, Vol.3(7.1.1)]).

THEOREM 5. *For $\alpha \in Z^2(G, K^*)$, let χ_α be the irreducible α-character of G over E afforded by an irreducible α-representation ρ of G. Let $e(\chi_\alpha)$ be the block idempotent of EG^α as in (4), and $v(\chi_\alpha)$ be the block idempotent of KG^α such that $e(\chi_\alpha)$ is a summand of $v(\chi_\alpha)$. Then, as K-algebras,*

$$KG^\alpha v(\chi_\alpha) \cong \rho(KG^\alpha) \text{ and } K(\chi_\alpha) \cong Z(\rho(KG^\alpha)).$$

Proof. When $e(\chi_\alpha)$ is a summand of $v(\chi_\alpha)$, we shall write $e(\chi_\alpha) \subset v(\chi_\alpha)$. Let U be a simple EG^α-module corresponding to χ_α and V be a simple KG^α-module such that U is an irreducible constituent of V^E. Let Ω_α be the subset of E^* consisting of the set μ of $|G|$-th roots of unity in E and the set $\{\delta_g | g \in G\}$, where δ_g is an $o(g)$-th root of $\prod_{i=1}^{o(g)} \alpha(g^i, g)$ (see (3)). Then $K(\Omega_\alpha)$ is a finite Galois radical extension field of K containing $K(\chi_\alpha)$ (Theorem 3).

Clearly the block idempotent

$$e(\chi_\alpha) = \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1}) \chi_\alpha(g^{-1}) a_g$$

of EG^α belongs to $K(\chi_\alpha)G^\alpha$. And the Galois group $\mathrm{Gal}(K(\Omega_\alpha)/K) = \mathcal{G}$ acts on the twisted group algebra $K(\Omega_\alpha)G^\alpha$ by $\tau \cdot \sum_{g \in G} x_g a_g = \sum_{g \in G} \tau(x_g) a_g$ for $\tau \in \mathcal{G}$ and $x_g \in K(\Omega_\alpha)$. Thus if we consider $\chi_\alpha^\tau = \tau\chi_\alpha$ then χ_α^τ is an α-character of G over E corresponding to the EG^α-module U^τ. Since $\tau e(\chi_\alpha) = \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1}) \chi_\alpha^\tau(g^{-1}) a_g = e(\chi_\alpha^\tau)$, we have

$$\sigma\left(\sum_{\tau \in \mathcal{G}} e(\chi_\alpha^\tau)\right) = \sum_{\tau \in \mathcal{G}} \sigma e(\chi_\alpha^\tau) = \sum_{\tau \in \mathcal{G}} e(\chi_\alpha^{\sigma\tau}) = \sum_{\tau \in \mathcal{G}} e(\chi_\alpha^\tau)$$

for all $\sigma \in \mathcal{G}$, thus $\sum_{\tau \in \mathcal{G}} e(\chi_\alpha^\tau)$ is contained in KG^α.

We notice however that $\sum_{\tau \in \mathcal{G}} e(\chi_\alpha^\tau)$ may not be a block idempotent in KG^α, because some of idempotents $e(\chi_\alpha^\tau)$ might appear more than once in the summation. We now let

$$v(\chi_\alpha) = \sum e(\chi_\alpha^\tau), \tag{5}$$

where the sum runs over $\tau \in \mathcal{G}$ such that $e(\chi_\alpha^\tau)$ are all distinct. And we may generously assume that $e(\chi_\alpha) \subset v(\chi_\alpha)$. Let σ be any element in $\mathcal{G}$. Since τ runs over $\mathcal{G}$ where $e(\chi_\alpha^\tau)$ are all distinct in the summation $v(\chi_\alpha)$, so does $\sigma\tau$ and $e(\chi_\alpha^{\sigma\tau})$ are all distinct. Hence $\sigma(v(\chi_\alpha)) = \sum e(\chi_\alpha^{\sigma\tau})$ is the sum of all distinct idempotents of EG^α, so is equal to $v(\chi_\alpha)$ for all $\sigma \in \mathcal{G}$. Thus $v(\chi_\alpha)$ is a block idempotent in KG^α associated with χ_α. We note that $e(\chi_\alpha) \subset v(\chi_\alpha) \subset e(\chi_\alpha) + \sum_{\tau \in \mathcal{G} - \mathcal{G}_0} e(\chi_\alpha^\tau)$ where $\mathcal{G}_0 = \mathrm{Gal}(K(\Omega_\alpha)/K(\chi_\alpha))$.

For the α-representation ρ on G, the mapping on EG^α defined by $\sum x_g a_g \mapsto \sum x_g \rho(g)$ $(x_g \in E)$ is a homomorphism of E-algebras. We shall use the same notation ρ for the homomorphism on EG^α. Since U is a simple EG^α-module corresponding to χ_α, $e(\chi_\alpha)$ acts as identity and the other block idempotents must annihilate U. Thus

$$\rho(v(\chi_\alpha)) = 1 \quad \text{and} \quad \rho(KG^\alpha v(\chi_\alpha)) = \rho(KG^\alpha).$$

Hence ρ induces a surjective homomorphism of $KG^\alpha v(\chi_\alpha)$ onto $\rho(KG^\alpha)$. But since $KG^\alpha v(\chi_\alpha)$ is simple, ρ is one to one and $KG^\alpha v(\chi_\alpha) \cong \rho(KG^\alpha)$.

And the second statement $K(\chi_\alpha) \cong Z(\rho(KG^\alpha))$ follows immediately from Theorem 7.3.8 (iii) in [3, Vol.3]. □

COROLLARY 6. *Let the context be the same as in Theorem 5. Let A be a simple component of KG^α corresponding to χ_α. Then A is central over K if and only if $K = K(\chi_\alpha)$.*

Proof. The simple component A of KG^α is isomorphic to $KG^\alpha v(\chi_\alpha)$ with a block idempotent $v(\chi_\alpha)$ of KG^α. Thus A is central over K if and only if $K = Z(A) \cong Z(KG^\alpha v(\chi_\alpha)) \cong K(\chi_\alpha)$ by Theorem 5. □

We are now able to recast the definition of projective Schur algebra in the following form.

THEOREM 7. *An algebra A is a projective Schur K-algebra if and only if there exists a finite group G, a 2-cocycle $\alpha \in Z^2(G, K^*)$ and an irreducible α-character χ_α of G over E such that $K = K(\chi_\alpha)$ and $A \cong KG^\alpha e(\chi_\alpha)$, where $e(\chi_\alpha)$ is as in (4).*

Proof. Let A be a projective Schur K-algebra. Then A is a central simple K-algebra that is a homomorphic image of KG^α for a finite group G and a 2-cocycle $\alpha \in Z^2(G, K^*)$.

Let v be a block idempotent of KG^α such that $A = KG^\alpha v$. Since v is a sum of distinct block idempotents of EG^α, we may let e be a block idempotent of EG^α which is a summand of v. Let χ_α be the irreducible α-character of G over E associated with e. Then we can write $e = e(\chi_\alpha) = (\chi_\alpha(1)/|G|)\sum_{g\in G}\alpha^{-1}(g, g^{-1})\chi_\alpha(g^{-1})a_g$. By considering the fields $K \subset K(\chi_\alpha) \subset K(\Omega_\alpha) \subset E$ as in Theorem 3 and by letting $\mathcal{G} = \mathrm{Gal}(K(\Omega_\alpha)/K)$, without loss of generality we may write $v = v(\chi_\alpha) = \sum e(\chi_\alpha^\tau)$ which is the sum of distinct $e(\chi_\alpha^\tau)$ for $\tau \in \mathcal{G}$ as in (5).

Since $A = KG^\alpha v(\chi_\alpha)$ is central, we have $K = K(\chi_\alpha)$ due to Corollary 6. And since $e(\chi_\alpha) \subset v(\chi_\alpha) \subset e(\chi_\alpha) + \sum e(\chi_\alpha^\tau)$ where the sum ranges over $\tau \in \mathrm{Gal}(K(\Omega_\alpha)/K) - \mathrm{Gal}(K(\Omega_\alpha)/K(\chi_\alpha))$ by the proof of Theorem 5, it follows that $e(\chi_\alpha) = v(\chi_\alpha)$, so $A = KG^\alpha v(\chi_\alpha)$ is isomorphic to $KG^\alpha e(\chi_\alpha)$. The other direction is easy to see. □

3. Projective Schur algebra over a field

A K-algebra Γ is said to be definable over a subfield L of K if Γ is isomorphic to $K \otimes_L \Gamma' = \Gamma'^K$ for some L-algebra Γ'. It is known that KG is definable over $\mathbb{Q}$ if $\mathrm{char}K = 0$, and KG^α is definable over a subfield L of K if L contains the values of $\alpha \in Z^2(G, K^*)$ [3, Vol.3(7.1.1)]. For a simple Γ-module V, V^E need not be a semisimple Γ^E-module. However if Γ is definable over a perfect subfield of K (or if K itself is perfect) then V^E is semisimple ([3, Vol.3(7.1.3)]).

THEOREM 8. *Let K be an algebraic number field and A be a projective Schur K-algebra which is an image of KG^{α} for a finite group G and $\alpha \in Z^2(G, K^*)$. Let M_α be a subfield of K containing the values of α. Then for the maximal radical extension field k of M_α in K, there is a projective Schur k-algebra A' such that $A \cong K \otimes_k A'$.*

Proof. Let χ_α be the α-character of G over an algebraic closure E that corresponds to the simple KG^{α}-algebra A. Let $\Omega_\alpha = \langle \mu, \{\delta_g | g \in G\} \rangle$ be the subset of E^*, where μ is the set of $|G|$-th root of unity in E and $\delta_g^{o(g)} = \prod_{i=1}^{o(g)} \alpha(g^i, g)$ (see Theorem 3). Then there is a tower of fields $K \subseteq K(\chi_\alpha) \subseteq K(\Omega_\alpha) \subseteq E$, and by Theorem 7 we are able to write

$$A \cong KG^{\alpha} e(\chi_\alpha) \quad \text{and} \quad K(\chi_\alpha) = K,$$

where $e(\chi_\alpha) = (\chi_\alpha(1)/|G|) \sum_{g \in G} \alpha^{-1}(g, g^{-1}) \chi_\alpha(g^{-1}) a_g$ is the block idempotent of EG^{α} (see (4)).

Consider two extension fields $k(\chi_\alpha)$ and $k(\Omega_\alpha)$ of k adjoined by the values of χ_α and the set Ω_α to k respectively. Then $k \subseteq k(\chi_\alpha) \subseteq k(\Omega_\alpha) \subseteq E$, and $k(\Omega_\alpha)$ is a finite radical Galois extension of k because $\delta_g^{o(g)} = \prod_{i=1}^{o(g)} \alpha(g^i, g) \in M_\alpha \subseteq k$. We denote $\mathrm{Gal}(k(\Omega_\alpha)/k)$ by $\mathcal{G}'$.

Obviously KG^{α} is definable over k since all values of α are in $M_\alpha \subseteq k$. Thus $KG^{\alpha} = K \otimes_k kG^{\alpha}$, where the K-basis a_g of KG^{α} is also considered as a k-basis of kG^{α}, and the block idempotent $e(\chi_\alpha)$ of EG^{α} belongs to $k(\chi_\alpha)G^{\alpha} \subseteq k(\Omega_\alpha)G^{\alpha}$.

Let A' be the simple component of kG^{α} corresponding to χ_α. Due to Theorem 5, $v'(\chi_\alpha) = \sum_\tau e(\chi_\alpha^\tau)$ where τ runs over $\mathcal{G}'$ such that all $e(\chi_\alpha^\tau)$ are distinct is a block idempotent of kG^{α} associated with χ_α. Here we may assume $e(\chi_\alpha) \subset v'(\chi_\alpha)$. And the simple component A' is isomorphic to $kG^{\alpha} v'(\chi_\alpha)$. We are now enough to show that A' is a projective Schur k-algebra satisfying $A \cong K \otimes_k A'$.

Clearly $K \cap k(\Omega_\alpha)$ is a radical extension of k contained in K, thus $K \cap k(\Omega_\alpha)$ is also radical over M_α because k is radical over M_α (see [2, (3.10.1)]). But since k is a maximal radical extension of M_α in K, it follows that $k \subseteq K \cap k(\chi_\alpha) \subseteq K \cap k(\Omega_\alpha) \subseteq k$, and they are all same.

Every value of χ_α is contained in K, for $K = K(\chi_\alpha)$. And $\chi_\alpha(g) \in k(\chi_\alpha)$ for all $g \in G$. Thus $\chi_\alpha(g) \in K \cap k(\chi_\alpha) = k$, so $k = k(\chi_\alpha)$. Hence by making use of Corollary 6, A' is a central simple k-algebra.

Since $e(\chi_\alpha)$ belongs to $k(\chi_\alpha)G^{\alpha} = kG^{\alpha}$, every $\tau \in \mathcal{G}' = \mathrm{Gal}(k(\Omega_\alpha)/k)$ leaves $e(\chi_\alpha)$ fixed, so $v'(\chi_\alpha) = e(\chi_\alpha)$. Thus the simple algebra $A' \cong kG^{\alpha} v'(\chi_\alpha)$ is isomorphic to $kG^{\alpha} e(\chi_\alpha)$, hence A' is a projective Schur k-algebra due to Theorem 7. Therefore our required situation follows immediately that

$$A \cong KG^{\alpha}e(\chi_{\alpha}) \cong K \otimes_k kG^{\alpha}e(\chi_{\alpha}) \cong K \otimes_k A'. \qquad \square$$

Without loss of generality we may assume that M_{α} is the smallest subfield of K containing all values of α. We showed that a projective Schur K-algebra can be obtained by K-scalar extension of a projective Schur k-algebra where k is a certain subfield of K. This observation will be clear if we assume the following case.

COROLLARY 9. *Let A be a projective Schur K-algebra which is a homomorphic image of KG^{α}. With the same context in Theorem 8, let $M_{\alpha}(\Omega_{\alpha})$ be the extension field of M_{α} adjoining the set Ω_{α}. If $k = K \cap M_{\alpha}(\Omega_{\alpha})$, then A is a K-scalar extension of a projective Schur k-algebra.*

Proof. Due to Theorem 7, we may write $A \cong KG^{\alpha}e(\chi_{\alpha})$ and $K = K(\chi_{\alpha})$. Since values of α are contained in both K and M_{α}, KG^{α} is definable over k so that $KG^{\alpha} \cong K \otimes_k kG^{\alpha}$.

From $k(\chi_{\alpha}) = K(\chi_{\alpha}) \cap M_{\alpha}(\Omega_{\alpha}) = K \cap M_{\alpha}(\Omega_{\alpha}) = k$, $e(\chi_{\alpha}) \in EG^{\alpha}$ is contained in kG^{α} and is left fixed by all $\tau \in \mathrm{Gal}(k(\Omega_{\alpha})/k)$. Hence the block idempotent $v'(\chi_{\alpha})$ in kG^{α} which is a sum of distinct $e(\chi_{\alpha}^{\tau})$'s for $\tau \in \mathrm{Gal}(k(\Omega_{\alpha})/k)$ is equal to $e(\chi_{\alpha})$. Thus the central simple k-algebra $A' \cong kG^{\alpha}v'(\chi_{\alpha})$ associated with χ_{α} is isomorphic to $kG^{\alpha}e(\chi_{\alpha})$, and it follows that $A \cong KG^{\alpha}e(\chi_{\alpha}) = K \otimes_k kG^{\alpha}e(\chi_{\alpha}) \cong K \otimes_k A'$. $\square$

In Corollary 9, if $\alpha = 1$ then $k = K \cap M_{\alpha}(\Omega_{\alpha})$ equals $K \cap \mathbb{Q}(\mu)$, which is the same field chosen in Theorem 2 for Schur algebra. Theorem 8 provides a partial analog of Theorem 2 that $A \cong K \otimes_k A'$ for $[A] \in PS(K)$ and $[A'] \in PS(k)$. However it does not imply the equality $PS(K) = K \otimes_k PS(k)$, even it is not true. For instance, if K is an algebraic number field then $PS(K)$ is the whole Brauer group $B(K)$ due to [4], hence the equality would mean that every element in $B(K)$ comes from $B(\mathbb{Q})$, which is not correct.

THEOREM 10. *Let K, E, $\alpha \in Z^2(G, K^*)$, χ_{α} and $v(\chi_{\alpha})$ be the same as in Theorem 8. Let $A \cong KG^{\alpha}v(\chi_{\alpha})$ be a simple component of KG^{α} corresponding to χ_{α}. If $\beta \in Z^2(G, K^*)$ is cohomologous to α* (denote it by $\alpha \sim \beta$) *then there is an irreducible β-character χ_{β} of G over E such that $K(\chi_{\alpha}) = K(\chi_{\beta})$ and $v(\chi_{\alpha}) = v(\chi_{\beta})$, so the simple component B of KG^{β} corresponding to χ_{β} is isomorphic to A, as K-algebras.*

Proof. Let ρ be an irreducible α-representation of G over E which affords χ_{α}. Let $\beta(g, x) = \alpha(g, x)t(g)t(x)t^{-1}(gx)$ with a map $t : G \to K^*$ $(t(1) = 1)$ for $g, x \in G$. Then it is easy to see that ρ' and χ_{β} defined by

$\rho'(g) = t(g)\rho(g)$ and $\chi_\beta(g) = t(g)\chi_\alpha(g)$ are irreducible β-representation and β-character of G respectively, and ρ' affords χ_β. Since $\chi_\beta(g) = t(g)\chi_\alpha(g) \in K(\chi_\alpha)$, $K(\chi_\beta) \subseteq K(\chi_\alpha)$ and they are equal. Moreover since

$$\prod_{i=1}^{o(g)} \beta(g^i, g) = \prod_{i=1}^{o(g)} \alpha(g^i, g)t(g^i)t(g)t^{-1}(g^{i+1}) = \lambda_g \cdot t(g)^{o(g)} \quad \text{for } g \in G,$$

where λ_g is in (3), we may take $o(g)$-th root δ'_g of $\prod_{i=1}^{o(g)} \beta(g^i, g)$ as $\delta_g \cdot t(g)$, where $\delta_g^{o(g)} = \lambda_g$. Hence

$$K(\Omega_\beta) = K\big(\langle \mu, \{\delta'_g | g \in G\}\rangle\big) = K\big(\langle \mu, \{\delta_g | g \in G\}\rangle\big) = K(\Omega_\alpha),$$

so we shall denote it by $K(\Omega) = K(\Omega_\alpha) = K(\Omega_\beta)$.

Let $\{a_g |\ g \in G\}$ be a K-basis of KG^α. Then $b_g = t(g)a_g$ forms a basis of KG^β, and $KG^\alpha \cong KG^\beta$ as K-algebras under $a_g \mapsto t^{-1}(g)b_g$ $(g \in G)$. Moreover the block idempotent $e(\chi_\beta)$ of EG^β is equal to $e(\chi_\alpha)$ of EG^α, because

$$\begin{aligned} & e(\chi_\beta) \\ = & \frac{\chi_\beta(1)}{|G|} \sum_{g \in G} \beta^{-1}(g, g^{-1})\chi_\beta(g^{-1})b_g \\ = & \frac{t(1)\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1})t^{-1}(g)t^{-1}(g^{-1})t(gg^{-1})t(g^{-1})\chi_\alpha(g^{-1})t(g)a_g \\ = & \frac{\chi_\alpha(1)}{|G|} \sum_{g \in G} \alpha^{-1}(g, g^{-1})\chi_\alpha(g^{-1})a_g = e(\chi_\alpha), \end{aligned}$$

thus $v(\chi_\beta)$ the sum of distinct $e(\chi_\beta^\tau)$ for $\tau \in \mathrm{Gal}(K(\Omega)/K)$ is equal to $v(\chi_\alpha)$. Hence it follows immediately that the simple component B of KG^β corresponding to χ_β is isomorphic to $KG^\beta v(\chi_\beta) \cong KG^\alpha v(\chi_\alpha) \cong A$. □

Let A be a projective Schur K-algebra. Then due to Theorem 7, $A \cong KG^\alpha e(\chi_\alpha)$ and $K(\chi_\alpha) = K$ with $\alpha \in Z^2(G, K^*)$ and an irreducible α-character χ_α of a finite group G. If we consider $\beta \in Z^2(G, K^*)$ such that $\alpha \sim \beta$ then A is isomorphic to a simple algebra $B \cong KG^\beta e(\chi_\beta)$ for some irreducible β-character χ_β due to Theorem 10. Furthermore since $K(\chi_\beta) = K(\chi_\alpha) = K$, B is a projective Schur K-algebra. Now applying Theorem 8 to both A and B, there exists a projective Schur k_α-algebra A' and a projective Schur k_β-algebra B' such that

$$K \otimes_{k_\alpha} A' \cong A \ \cong \ B \cong K \otimes_{k_\beta} B',$$

where M_α [resp. M_β] is the (smallest) subfield of K containing all values of α [resp. β], and k_α [resp. k_β] is the maximal radical extension of M_α [resp. M_β] contained in K. We observe that though $K(\chi_\alpha) = K(\chi_\beta)$ and $K(\Omega_\alpha) = K(\Omega_\beta)$, it is not necessarily M_α and M_β, and k_α and k_β are same respectively.

THEOREM 11. *Let K be an algebraic number field, and A and B be any projective Schur K-algebras. Then there exist a subfield k of K and projective Schur k-algebras A_0 and B_0 such that $A \cong K \otimes_k A_0$ and $B \cong K \otimes_k B_0$.*

Proof. Let A and B be homomorphic images of KG^α and KH^β respectively where G and H are finite groups, $\alpha \in Z^2(G, K^*)$ and $\beta \in Z^2(H, K^*)$. Due to Theorem 8 there are M_α [resp. M_β] which is the smallest subfield of K containing all values of α [resp. β], and k_α [resp. k_β] which is the maximal radical extension of M_α [resp. M_β] in K, satisfying

$$A \cong K \otimes_{k_\alpha} A' \quad \text{and} \quad B \cong K \otimes_{k_\beta} B',$$

where A' and B' are projective Schur k_α and k_β-algebras respectively.

Let F be a subfield of K containing the values of both α and β. And let k be the maximal radical extension of F in K. Obviously $M_\alpha \subseteq F$ and $M_\beta \subseteq F$.

It is easy to see that k_α and k_β are contained in k. In fact, since k_α is a radical extension of M_α, we may write $k_\alpha = M_\alpha(\Delta_\alpha)$ with a subset Δ_α of k_α^* such that $\Delta_\alpha M_\alpha^*/M_\alpha^*$ is torsion. Clearly $M_\alpha \subseteq F \subseteq k$. Moreover if $x \in \Delta_\alpha$ then $x^m \in M_\alpha \subseteq F$ for some $m > 0$, thus xF^* is of finite order in K^*/F^*. Due to the maximality of k in K, we have $x \in k$ and $\Delta_\alpha \subseteq k$, thus $k_\alpha \subseteq k$. Similarly we have $k_\beta \subseteq k$.

Now since both KG^α and KH^β are definable over k, we have $KG^\alpha = K \otimes_k kG^\alpha$ and $KH^\beta = K \otimes_k kH^\beta$. And by applying Theorem 8 to F and its maximal radical extension k in K, we can conclude that there exist projective Schur k-algebras A_0 and B_0 such that $A \cong K \otimes_k A_0$ and $B \cong K \otimes_k B_0$. □

Theorem 11 motivates to construct a subset $PS_F(K)$ of $PS(K)$ for a subfield F of K: let $F \subseteq K$ and let $PS_F(K)$ be the set of similar classes $[S]$ of K-algebras where $[S]$ contains a projective Schur K-algebra that is an image of KG^α definable over F for some finite group G and $\alpha \in Z^2(G, K^*)$.

Obviously, $PS_F(K)$ is a subgroup of $PS(K)$ for, let $[S_i] \in PS_F(K)$ be with $A_i \in [S_i]$ $(i = 1, 2)$ where A_i is an image of $KG_i^{\alpha_i}$ and $KG_i^{\alpha_i}$ is definable over F. Then $A_1 \otimes_K A_2$ is represented by $K(G_1 \times G_2)^{\alpha_1 \times \alpha_2}$, where

$\alpha_1 \times \alpha_2$ is defined by $\alpha_1 \times \alpha_2((g_1, g_2), (x_1, x_2)) = \alpha_1(g_1, x_1)\alpha_2(g_2, x_2)$ for $g_i, x_i \in G_i$ $(i = 1, 2)$. Moreover $K(G_1 \times G_2)^{\alpha_1 \times \alpha_2}$ is definable over F because $K(G_1 \times G_2)^{\alpha_1\times\alpha_2} = KG_1^{\alpha_1} \otimes_K KG_2^{\alpha_2} = (K \otimes_F FG_1^{\alpha_1}) \otimes_K (K \otimes_F FG_2^{\alpha_2}) = K \otimes_F (FG_1^{\alpha_1} \otimes_F FG_2^{\alpha_2}) = K \otimes_F F(G_1 \times G_2)^{\alpha_1\times\alpha_2}$. Thus $A_1 \otimes_K A_2 \in [S_1][S_2]$ and $[S_1][S_2] \in PS_F(K)$. In particular if α_i has values in F then so does $\alpha_1 \times \alpha_2$.

Let $[S]$ be any element in $PS_F(K)$ and $A \in [S]$ be an image of KG^α for some $\alpha \in Z^2(G, K^*)$. Then there is an irreducible α-character χ_α such that $K = K(\chi_\alpha)$ and $A \cong KG^\alpha e(\chi_\alpha)$ by Theorem 7, where $e(\chi_\alpha)$ is as in (4). We note that since KG^α is definable over F, it is also definable over the maximal radical extension field k of F in K. Moreover due to Theorem 8, $A \cong K \otimes_k kG^\alpha e(\chi_\alpha)$. But since $[kG^\alpha e(\chi_\alpha)] \in PS_F(k)$, $[A] = K \otimes_k [kG^\alpha e(\chi_\alpha)]$ belongs to $K \otimes_k PS_F(k)$.

Hence the following theorem is straightforward.

THEOREM 12. *Let K be an algebraic number field and $F \subset K$. Then $PS_F(K)$ is a subgroup of $PS(K)$ and $PS_F(K) = K \otimes_k PS_F(k)$ for the maximal radical extension field k of F in K.*

As an application to a special class of projective Schur algebras, we consider the radical (abelian) algebra ([1]) which is a crossed product algebra $(L/K, \alpha')$ where $L = K(\Omega)$ is a radical (abelian) G-Galois extension of K (that is, Ω is a $G = \mathrm{Gal}(L/K)$-invariant subgroup of L^* (i.e., $\sigma(\Omega) \subseteq \Omega$ for any $\sigma \in G$) such that $\Omega K^*/K^*$ is a torsion group), and $\alpha' \in Z^2(G, L^*)$ is the image of some $\alpha \in Z^2(G, \Omega)$ under the inclusion $\Omega \hookrightarrow L^*$. The radical algebra is an analogue of the cyclotomic algebra in the context of projective Schur algebra, and every projective Schur division algebra is itself a radical abelian algebra. The set of similarity classes of radical K-algebra forms a radical group $\mathrm{Rad}(K)$ which is a subgroup of $PS(K)$.

THEOREM 13. *Let k be a maximal radical extension of $\mathbb{Q}$ contained in a field K. Then for any $[S] \in Rad(K)$, $[S]^h$ is a K-scalar extension of an element in $Rad(k)$ for some $h > 0$.*

Proof. Let $A = (K(\Omega)/K, \alpha')$ be a radical K-algebra contained in $[S]$. Then $[A] = [S]$, $K(\Omega)$ is a radical G-Galois extension of K with $G = \mathrm{Gal}(K(\Omega)/K)$, and $\alpha' \in Z^2(G, K(\Omega)^*)$ is the image of $\alpha \in Z^2(G, \Omega)$. And for any $\sigma \in G$ and $\omega \in \Omega$, $\omega^n \in K^*$ for some integer $n > 0$ and $\sigma(\omega)$ belongs to Ω. Now let

$$\Omega_0 = \{\omega \in \Omega |\ \omega^n \in k^* \text{ for some } n > 0\}.$$

Then $\Omega_0 < \Omega$, $\Omega_0 k^*/k^*$ is a torsion group and $k(\Omega_0)$ is a radical extension of k. Since k is the maximal radical extension contained in K, we have $K \cap k(\Omega_0) = k$.

Consider the field extensions $K \subseteq K(\Omega_0) \subseteq K(\Omega)$ and $k \subseteq k(\Omega_0) \subseteq k(\Omega)$. Let ω be any element in Ω_0. Then $\omega^n \in k^* \subset K$, and $\sigma(\omega)^n = \sigma(\omega^n) = \omega^n \in k^*$ (i.e., $\sigma(\Omega_0) \subset \Omega_0$) for any $\sigma \in \mathrm{Aut}_K K(\Omega_0)$. Let x be any element in $K(\Omega_0) - K$. Then $x \in K(\Omega) - K$ and there is $\tau \in \mathrm{Aut}_K K(\Omega)$ such that $\tau(x) \neq x$, for $K(\Omega)/K$ is Galois. Denote $\tau|_{K(\Omega_0)}$ by τ_0. If we write any element $y \in K(\Omega_0)$ by $y = \sum a_i\omega_i$ with $a_i \in K$, $\omega_i \in \Omega_0$ then $\tau_0(y) = \tau(y) = \sum a_i\tau(\omega_i) \in K(\Omega_0)$. This shows that τ_0 can be regarded as an element in $\mathrm{Aut}_K K(\Omega_0)$ satisfying $\tau_0(x) = \tau(x) \neq x$. Therefore $K(\Omega_0)$ is a radical G_0-Galois extension of K where $G_0 = \mathrm{Gal}(K(\Omega_0)/K)$.

Since $\mathrm{Gal}(K(\Omega_0)/K) \cong \mathrm{Gal}(k(\Omega_0)/(K \cap k(\Omega_0))) = \mathrm{Gal}(k(\Omega_0)/k)$, $k(\Omega_0)$ is also G_0-Galois radical over k; we shall denote $\mathrm{Gal}(k(\Omega_0)/k)$ by the same notation G_0. If we write $H = \mathrm{Gal}(K(\Omega)/K(\Omega_0))$ then G/H is isomorphic to G_0.

From $A = (K(\Omega)/K, \alpha')$, let Γ_α be the group extension of Ω by G

$$\alpha :\ 1 \to \Omega \to \Gamma_\alpha \xrightarrow{j} G \to 1,$$

which corresponds to $\alpha \in Z^2(G, \Omega)$. Then $A = K(\Gamma_\alpha)$ as a K-vector space.

Consider the homomorphism [8, (5.3.2)]

$$v_{G\to G/H} :\ H^2(G,\Omega) \to H^2(G/H, \Omega^H),$$

defined in the following manner. Let $j^{-1}(H) = W$ and let W_c be the commutator subgroup of W. Then there is a group extension

$$\alpha_c\ :\ 1 \to W/W_c \to \Gamma_\alpha/W_c \to G/H \to 1$$

having a factor set α_c in $Z^2(G/H, W/W_c)$. Denote by Λ the reduced group theoretical transfer map $W/W_c \to \Omega^H$.

The Λ is a G/H-homomorphism and induces a homomorphism of cohomology groups $\overline{\Lambda} : H^2(G/H, W/W_c) \to H^2(G/H, \Omega^H)$. Then $v_{G\to G/H}$ is defined by $v_{G\to G/H}(\bar{\alpha}) = \overline{\Lambda}(\bar{\alpha}_c)$, where $\bar{\alpha} \in H^2(G,\Omega)$ is the cohomology class of α. It can be seen that $v_{G\to G/H}$ is a homomorphism. And we denote $v_{G\to G/H}(\bar{\alpha})$ by $\bar{\beta} \in H^2(G/H, \Omega^H)$.

We observe $\Omega^H = \Omega_0$. In fact if $\omega \in \Omega^H$ then $\omega \in \Omega$ is fixed by all elements in $H = \mathrm{Gal}(K(\Omega)/K(\Omega_0))$, so $\omega \in \Omega_0$. Conversely if $\omega \in \Omega_0$ then $\omega \in \Omega \cap K(\Omega_0)$ is fixed by H. Hence we may regard β as an element in $Z^2(G_0, \Omega_0)$, and we have a group extension Γ_β of Ω_0 by G_0 :

$$\beta \; : \; 1 \to \Omega_0 \to \Gamma_\beta \to G_0 \to 1.$$

If let $\beta' \in Z^2(G_0, k(\Omega_0)^*)$ be an image of β under the inclusion $\Omega_0 \hookrightarrow k(\Omega_0)^*$ and let $B = (k(\Omega_0)/k, \beta')$ be the crossed product algebra then $B = k(\Gamma_\beta)$ is a radical k-algebra, so $[B] \in \mathrm{Rad}(k)$.

In connection with the inflation map $H^2(G_0, \Omega_0) \cong H^2(G/H, \Omega^H) \overset{\inf}{\to} H^2(G, \Omega)$, the composition map $(\inf \cdot v_{G \to G/H})$ on $H^2(G, \Omega)$ defines

$$\bar{\alpha}^{|H|} = (\inf \cdot v_{G \to G/H})(\bar{\alpha}) = \inf(\bar{\beta})$$

[8, (5.3.3)]. Hence $\inf\beta$ is cohomologous to $\alpha^{|H|}$. Thus due to [6, (29,13), (29,16)], we have the following isomorphisms of crossed product algebras:

$$\begin{aligned} K \otimes [B] &= K \otimes \left[(k(\Omega_0)/k, \beta')\right] \\ &= \left[(K(\Omega_0)/K, \beta')\right] \\ &= \left[(K(\Omega)/K, \inf\beta')\right] \\ &= \left[(K(\Omega)/K, \alpha'^{|H|})\right] \\ &= \left[(K(\Omega)/K, \alpha')\right]^{|H|} \\ &= [A]^{|H|} \\ &= [S]^{|H|}. \end{aligned} \qquad \square$$

In particular when $|H| = 1$ (i.e., $K(\Omega) = K(\Omega_0)$), a radical K-algebra can be extended from a radical k-algebra where k is the maximal radical extension in K.

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