PRIME RADICALS OF SKEW LAURENT POLYNOMIAL RINGS

Juncheol Han

ABSTRACT. Let $R$ be a ring with an automorphism $\sigma$. An ideal $I$ of $R$ is $\sigma$-ideal of $R$ if $\sigma(I) = I$. A proper ideal $P$ of $R$ is $\sigma$-prime ideal of $R$ if $P$ is a $\sigma$-ideal of $R$ and for $\sigma$-ideals $I$ and $J$ of $R$, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. A proper ideal $Q$ of $R$ is $\sigma$-semi prime ideal of $Q$ if $Q$ is a $\sigma$-ideal and for a $\sigma$-ideal $I$ of $R$, $I^2 \subseteq Q$ implies that $I \subseteq Q$. The $\sigma$-prime radical is defined by the intersection of all $\sigma$-prime ideals of $R$ and is denoted by $P_{\sigma}(R)$. In this paper, the following results are obtained: (1) For a principal ideal domain $R$, $P_{\sigma}(R)$ is the smallest $\sigma$-semi prime ideal of $R$; (2) For any ring $R$ with an automorphism $\sigma$ and for a skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$, the prime radical of $R[x, x^{-1}; \sigma]$ is equal to $P_{\sigma}(R)[x, x^{-1}; \sigma]$.

1. Introduction and some definitions

Throughout this paper, $R$ will denote an associative ring with identity, $\sigma$ will be an automorphism of $R$. A left (resp. right, two-sided) ideal $I$ of $R$ is called a left (resp. right, two-sided) $\sigma$-ideal if $\sigma(I) = I$. An ideal $P$ of $R$ is called $\sigma$-prime ideal if $P \neq R$ is a $\sigma$-ideal and for $\sigma$-ideals $I$, $J$ of $R$, $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$. An ideal $Q$ of $R$ is called $\sigma$-semi prime ideal if for any $\sigma$-ideal $I$, $I^2 \subseteq Q$ implies that $I \subseteq Q$. $R$ is called a $\sigma$-prime (resp. $\sigma$-semi prime) ring if $(0)$ is a $\sigma$-prime (resp. $\sigma$-semi prime) ideal. For more things about these terminologies, refer to [3], [5], and [6]. Note that every $\sigma$-prime ideal of $R$ is $\sigma$-semi prime ideal, and every prime (resp. semi prime) ring is $\sigma$-prime (resp. $\sigma$-semi prime).

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Recall that the prime radical (in other words, lower nil radical) of $R$ (denoted by $P(R)$) is the intersection of all prime ideals of $R$. We can define $\sigma$-prime radical (in other words, $\sigma$-lower nil radical) of $R$ (denoted by $P_\sigma(R)$) by the intersection of all $\sigma$-prime ideals of $R$. In Section 2, we will investigate some properties of $P_\sigma(R)$, in particular, we will show that $P_\sigma(R)$ is the smallest $\sigma$-semiprime ideal of principal ideal domain $R$.

Recall that the skew polynomial ring $R[x;\sigma]$ is a ring of polynomials in $x$ with coefficients in $R$ and subject to the relation $xa = \sigma(a)x$, for all $a \in R$. The skew Laurent polynomial ring $R[x, x^{-1};\sigma]$ is a localization of $R[x;\sigma]$ with respect to the set of powers of $x$ and so $R[x, x^{-1};\sigma]$ consists of $\sum_{i=-\infty}^{\infty} a_i x^i$ with only finitely many nonzero terms (these are called the skew Laurent polynomials). In [6], A. Moussavi has found some results on semiprimivity of $P(R[x;\sigma])$ for a left Noetherian ring with the ascending chain condition on the right annihilators and a ring monomorphism $\sigma$ of $R$ and he proved that Jacobson radical of $P(R[x;\sigma])$ is equal $N(R)[x;\sigma]$ ($N(R)$ is the nilpotent radical of $R$) if such a ring $R$ is semiprime or $\sigma$-prime. In [2], D. A. Jordan has obtained conditions which are sufficient for $R[x, x^{-1};\sigma]$ primitive. In [3], D. A. Jordan also has obtained some results on the primitivity of $R[x, x^{-1};\sigma]$ for a commutative Noetherian ring with an automorphism $\sigma$. In [5], A. Leroy and J. Matczuk have found the necessary and sufficient conditions for the primitivity of $R[x, x^{-1};\sigma]$ for a Noetherian P.I. ring with an automorphism $\sigma$. In Section 3, we will show that the prime radical of a skew Laurent polynomial ring $R[x, x^{-1};\sigma]$ is equal to the $P_\sigma(R)[x, x^{-1};\sigma]$.

**Example 1.1.** Let $\mathbb{Z}$ be the ring of integers. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be the upper $2 \times 2$ triangular matrix ring over $\mathbb{Z}$. Let $\sigma : R \to R$ be a map defined by $\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$. Then $\sigma$ is an automorphism of $R$ and $I = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ is a $\sigma$-ideal of $R$.

**Example 1.2.** Let $F$ be any field and $R = F[x]$ be the polynomial ring over $F$. Let $\sigma : R \to R$ be a map defined by $\sigma(f(x)) = f(-x)$ for all $f(x) \in R$. Then $\sigma$ is an automorphism of $R$ and $xR$ is a $\sigma$-prime ideal of $R$.

**Example 1.3.** Let $\mathbb{Z}$ be the ring of integers and let $R = \mathbb{Z} \times \mathbb{Z}$. Consider a map $\sigma : R \to R$ defined by $\sigma((a, b)) = (b, a)$ for all $(a, b) \in R$. 

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Then $\sigma$ is an automorphism of $R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of $R$, $I$ is not a $\sigma$-ideal of $R$ since $\sigma(I) = \{0\} \times \mathbb{Z} \neq I$.

2. $\sigma$-prime radical of a ring $R$

Since the prime radical of $R$ is the smallest semiprime ideal of $R$, we can also have the following question:

**Question.** For an automorphism $\sigma$ of a ring $R$, is $P_{\sigma}(R)$ the smallest $\sigma$-semiprime ideal of $R$?

It is clear that for an automorphism $\sigma$ of a ring $R$, $P_{\sigma}(R)$ is a $\sigma$-semiprime ideal of $R$. In this section, we will show that for any principal ideal domain $R$, the question is affirmative.

The definitions and the results in this section are obtained by the similar arguments on prime radical of ring $R$ in [4]. A nonempty subset $S$ of a ring $R$ is called a $\sigma$-m-system if, for any $a, b \in S$ such that $\sigma(a) \in (a), \sigma(b) \in (b)$, there exists $r \in R$ such that $arb \in S$.

**Proposition 2.1.** Let $R$ be a principal ideal domain with an automorphism $\sigma$. If $P \subseteq R$ is any $\sigma$-ideal of $R$, then the following are equivalent:

1. $P$ is $\sigma$-prime;
2. For any $a, b \in R$ such that $\sigma(a) \in (a), \sigma(b) \in (b), (a) \cdot (b) \subseteq P$ implies that $a \in P$ or $b \in P$;
3. For any $a, b \in R$ such that $\sigma(a) \in (a), \sigma(b) \in (b), aRb \subseteq P$ implies that $a \in P$ or $b \in P$.

**Proof.** (1) $\Rightarrow$ (2). Clear.
(2) $\Rightarrow$ (3). If $aRb \subseteq P$ such that $\sigma(a) \in (a), \sigma(b) \in (b)$, then $(a) \cdot (b) = RaRbR \subseteq RPR = P$. By (2), $a \in P$ or $b \in P$.
(3) $\Rightarrow$ (1). Clear. □

**Corollary 2.2.** Let $R$ be a principal ideal domain with an automorphism $\sigma$. Then $P$ is a $\sigma$-prime ideal of $R$ if and only if $R \setminus P$ is a $\sigma$-m-system.

**Proof.** It follows from the definition of $\sigma$-m-system and Proposition 2.1. □

For a $\sigma$-ideal $I$ in a ring $R$ with an automorphism $\sigma$, let $P_{\sigma}(R : I) = \{r \in R : \text{every } \sigma\text{-m-system containing } r \text{ meets } I\}$. Then we have the following theorem.
Theorem 2.3. Let $R$ be a principal ideal domain with an automorphism $\sigma$. Then for any $\sigma$-ideal $I$ of a ring $R$, $P_\sigma(R : I)$ equals to the intersection of all the $\sigma$-prime ideals containing $I$. In particular, $P_\sigma(R : I)$ is a $\sigma$-ideal of $R$.

Proof. Let $a \in P_\sigma(R : I)$ and $P$ be any $\sigma$-prime ideal of $R$ containing $I$. Then $R \setminus P$ is a $\sigma$-m-system $R \setminus P$ by Corollary 2.2. This $\sigma$-m-system cannot contain $a$, for otherwise $(R \setminus P) \cap I \neq \emptyset$, a contradiction. Therefore, we have $a \in P$. Conversely, assume that $a \notin P_\sigma(R : I)$. Then by definition, there exists a $\sigma$-m-system $S$ containing $a$ which is disjoint from $I$. Note that there exists a $\sigma$-prime ideal $P$ which is maximal in the set of all $\sigma$-ideals of $R$ disjoint from $S$ and containing $I$. Indeed, consider the set $\Gamma_\sigma$ of all $\sigma$-ideals of $R$ disjoint from $S$ and containing $I$. Then $\Gamma_\sigma$ is nonempty since $I \in \Gamma_\sigma$. Since $\Gamma_\sigma \neq \emptyset$, every $\sigma$-ideal in $\Gamma_\sigma$ is properly contained in $R$. Let $\Gamma_\sigma$ be partially ordered by inclusion. By Zorn's Lemma there is a $\sigma$-ideal $P$ of $R$ which is maximal in $\Gamma_\sigma$. Let $U, V$ be $\sigma$-ideals of $R$ such that $UV \subseteq P$. If $U \nsubseteq P$ and $V \nsubseteq P$, then each of the $\sigma$-ideals $P + U$ and $P + V$ properly contains $P$ and hence must meet $S$. Consequently, for some $p_i \in P$, $u \in U$ and $v \in V$, $p_1 + u = s_1 \in S$ and $P_2 + v = s_2 \in S$. Since $S$ is a $\sigma$-m-system, there exists an element $r \in R$ such that $s_1rs_2 \in S$. Thus $s_1rs_2 = p_1rP_2 + p_1rv + urP_2 + urv \in P + UV \subseteq P$, a contradiction since $s_1rs_2 \in S \cap P = \emptyset$. Therefore $U \subseteq P$ or $V \subseteq P$, and so $P$ is prime. Hence we have $a \notin P$, as desired.

A nonempty subset $S$ of a ring $R$ is called an $\sigma$-$n$-system if, for any $a \in S$ such that $(a)$ is $\sigma$-ideal of $R$ there exists $r \in R$ such that $ara \in S$.

Proposition 2.4. Let $R$ be a principal ideal domain with an automorphism $\sigma$. For any $\sigma$-ideal $Q$ of $R$, the following are equivalent:

1. $Q$ is $\sigma$-semiprime;
2. For any $a \in R$ such that $(a)$ is $\sigma$-ideal of $R$, $(a)^2 \subseteq Q$ implies that $a \in Q$;
3. For any $a \in R$ such that $(a)$ is $\sigma$-ideal of $R$, $aRa \subseteq Q$ implies that $a \in Q$.

Proof. It is similar to the proof as given in the Proposition 2.1.

Corollary 2.5. Let $R$ be a principal ideal domain with an automorphism $\sigma$. Then $P$ is a $\sigma$-semiprime ideal of $R$ if and only if $R \setminus P$ is a $\sigma$-$n$-system.
Proof. It follows from the definition of $\sigma$-$n$-system and Proposition 2.4.

**Theorem 2.6.** Let $R$ be a principal ideal domain with an automorphism $\sigma$. For any $\sigma$-ideal $Q$ of $R$, the following are equivalent:

1. $Q$ is a $\sigma$-semiprime ideal;
2. $Q$ is an intersection of $\sigma$-prime ideals;
3. $Q = P_\sigma(R : Q)$.

**Proof.** $(3) \Rightarrow (2)$. It follows from Theorem 2.3. since any $\sigma$-prime ideal is $\sigma$-semiprime.

$(2) \Rightarrow (1)$. It follows from the observation that every $\sigma$-prime ideal is $\sigma$-semiprime and the intersection of any $\sigma$-semiprime ideals is $\sigma$-semiprime.

$(1) \Rightarrow (3)$. Suppose that $Q$ is a $\sigma$-semiprime ideal. By definition of $\sigma$-$n$-system, $Q \subseteq P_\sigma(R : Q)$. We want to show that $P_\sigma(R : Q) \subseteq Q$. Let $a \notin Q$ and let $N = R \setminus Q$. Then $N$ is a $\sigma$-$n$-system containing $a$ by Corollary 2.5. Then there exists a $\sigma$-m-system $M \subseteq N$ such that $a \in M$. Indeed, consider a subset $M = \{a_1, a_2, a_3, \ldots \}$ defined inductively as follows: $a_1 = a$, $a_{i+1} = a_1r_ia_i \in N$ for some $r_i \in R$, where $i = 1, 2, \ldots$. We will show that $M$ is a $\sigma$-m-system. Let $a_i, a_j \in M$ be arbitrary. If $i \leq j$, then $a_{j+1} \in a_jR a_j \subseteq a_iR a_j$, which means $a_{j+1} \in M$. If $j \leq i$, then similarly $a_{i+1} \in M$. Hence there is a $\sigma$-$m$-system $M \subseteq N$ such that $a \in M$. Since $M$ is disjoint from $Q$, $a \notin P_\sigma(R : Q)$.

**Corollary 2.7.** Let $R$ be a principal ideal domain with an automorphism $\sigma$. Then $P_\sigma(R : I)$ is the smallest $\sigma$-semiprime ideal of $R$ which contains $I$.

**Proof.** It follows from the Theorem 2.6.

For a ring $R$ with an automorphism $\sigma$, $P_\sigma(R : (0))$ (simply denoted by $P_\sigma(R)$) is called the $\sigma$-prime radical of $R$. We can note that $P_\sigma(R)$ is the intersection of all $\sigma$-prime ideals of $R$ by Theorem 2.3 and clearly, it is a $\sigma$-semiprime ideal of $R$ and in particular, for any principal ideal domain $R$, it is the smallest $\sigma$-semiprime ideal of $R$ by Corollary 2.7.

**Proposition 2.8.** Let $R$ be a principal ideal domain with an automorphism $\sigma$. Then the following are equivalent:

1. $R$ is a $\sigma$-semiprime ring;
2. $P_\sigma(R) = (0)$;
3. $R$ has no nonzero nilpotent $\sigma$-ideal.
Proof. (1) \iff (2) and (3) \implies (1) are clear. It remains to show the implication (1) \implies (3). Let \(R\) be a \(\sigma\)-semiprime ring and \(I\) be a nilpotent \(\sigma\)-ideal. Then \(I^n = (0)\) and \(I^{n-1} \neq (0)\) for some positive integer \(n\). Suppose that \(R\) is a \(\sigma\)-semiprime ring. If \(n \geq 2\), then \((I^{n-1})^2 = I^{2n-2} \subseteq I^{2n} = (0)\) implies \(I^{n-1} = (0)\) since \(R\) is \(\sigma\)-semiprime, a contradiction. Thus \(n = 1\) and so \(I = (0)\). \(\square\)

3. Prime radicals of skew Laurent polynomial rings

For any \(\sigma\)-ideal \(I\) of a ring \(R\) with an automorphism \(\sigma\), we can have a reduced automorphism \(\overline{\sigma}\) on \(R/I\) defined by \(\overline{\sigma}(a + I) = \sigma(a) + I\) for all \(a + I \in R/I\). Then we can note that \(\overline{\sigma}\) is an automorphism of \(R/I\). Hence we can consider a skew Laurent polynomial ring \((R/I)[x, x^{-1}; \overline{\sigma}]\) with multiplication subject to the relation \(x\overline{a} = \overline{\sigma}(\overline{a})x\) for all \(\overline{a} = a + I \in R/I\).

Lemma 3.1. Let \(R\) be a ring with an automorphism \(\sigma\) and let \(K, I\) be ideals of \(R\) such that \(R \supseteq K \supseteq I\). Then \(K\) is a \(\sigma\)-ideal of \(R\) if and only if \(K/I\) is a \(\overline{\sigma}\)-ideal of \(R/I\).

Proof. It follows from the definition of reduced automorphism \(\overline{\sigma}\). \(\square\)

Lemma 3.2. Let \(R\) be a ring with an automorphism \(\sigma\) and let \(I\) be an ideal of \(R\). Then \(I\) is a \(\sigma\)-semiprime ideal of \(R\) if and only if \(R/I\) is a \(\overline{\sigma}\)-semiprime ring.

Proof. (\(\Rightarrow\)) Suppose that \(I\) is a \(\sigma\)-semiprime ideal of \(R\). If \(K/I\) is any \(\overline{\sigma}\)-ideal of \(R/I\) such that \((K/I)^2 = (0)\), the zero ideal of \(R/I\). Then \(K^2 = I\). By Lemma 3.1, \(K\) is \(\sigma\)-ideal of \(R\). Since \(I\) is a \(\sigma\)-semiprime ideal, \(K = I\) and so \(K/I = (0)\), which means that \(R/I\) is a \(\overline{\sigma}\)-semiprime ring.

(\(\Leftarrow\)) Suppose that \(R/I\) is a \(\overline{\sigma}\)-semiprime ring. If \(Q\) is any \(\sigma\)-ideal of \(R\) such that \(Q^2 \subseteq I\), then \((0) = Q^2/I = (Q/I)^2\). Since \(R/I\) is a \(\overline{\sigma}\)-semiprime ring, \(Q/I = (0)\), so \(Q = I\). Hence \(I\) is a \(\sigma\)-semiprime ideal of \(R\). \(\square\)

Lemma 3.3. Let \(R\) be a ring with an automorphism \(\sigma\) and let \(I\) be a \(\sigma\)-ideal of \(R\). Then for such a reduced automorphism \(\overline{\sigma}\) on \(R/I\), we have \(R[x, x^{-1}; \overline{\sigma}]/I[x, x^{-1}; \overline{\sigma}] \simeq (R/I)[x, x^{-1}; \overline{\sigma}]\).

Proof. Define \(\theta : R[x, x^{-1}; \overline{\sigma}] \longrightarrow (R/I)[x, x^{-1}; \overline{\sigma}]\) by \(\theta(f(x)) = \sum_{i=m}^n \overline{\sigma}(\overline{a}_i)x^i\) for all \(f(x) = \sum_{i=m}^n a_ix^i \in R[x, x^{-1}; \sigma]\). It is straightforward to show that \(\theta\) is an epimorphism and the kernel of \(\theta\) is equal
to \( I[x, x^{-1}; \sigma] \). Hence we have the result by the First Fundamental Homomorphism Theorem.

**Lemma 3.4.** Let \( R \) be a ring with an automorphism \( \sigma \). Then \( R \) is \( \sigma \)-semiprime if and only if \( A = R[x, x^{-1}; \sigma] \) is semiprime.

**Proof.** (\( \Rightarrow \)). Suppose that \( R \) is \( \sigma \)-semiprime. Let \( J \) be an ideal of \( A \) such that \( J^2 = (0) \). Consider an ideal of \( R \), \( J_0 \), the set of all leading coefficients of every \( f(x) \in J \). Then \( J_0 \) is a \( \sigma \)-ideal of \( R \). Indeed, for any \( f \in J \), by letting \( f = a_n x^n + \{ \text{terms of lower degrees} \} \) where \( a_n \in J_0 \) and by considering \( g = xf \in J \) (resp. \( h = x^{-1}f \in J \)) we have \( g = \sigma(a_n)x^{n+1} + \{ \text{terms of lower degrees} \} \) (resp. \( h = \sigma^{-1}(a_n)x^{n-1} + \{ \text{terms of lower degrees} \} \)), and so \( \sigma(a_n) \in J_0 \) (resp. \( \sigma^{-1}(a_n) \in J_0 \)). Since \( J_0^2 = (0) \), and so \( J_0 = (0) \) by the assumption. Continuing in this way, every coefficients of \( f(x) \) is equal to 0 for all \( f(x) \in J \). Thus \( J = (0) \), and so \( A \) is semiprime.

(\( \Leftarrow \)). Suppose that \( A \) is semiprime. Let \( I \) be a nonzero \( \sigma \)-ideal of \( R \). Then \( IA \) is a nonzero \( \sigma \)-ideal of \( A \). Since \( A \) is semiprime, \( (IA)^2 = I^2A \neq (0) \), and then \( I^2 \neq (0) \). Hence \( R \) is \( \sigma \)-semiprime. \( \square \)

**Theorem 3.5.** Let \( R \) be a ring with an automorphism \( \sigma \). Then the prime radical of \( R[x, x^{-1}; \sigma] \) is equal to \( P_\sigma(R)[x, x^{-1}; \sigma] \), i.e.,

\[
P(R[x, x^{-1}; \sigma]) = P_\sigma(R)[x, x^{-1}; \sigma].
\]

**Proof.** Let \( I = P_\sigma(R) \). Then \( I \) is the smallest \( \sigma \)-semiprime ideal of \( R \) by Corollary 2.7 and then \( R/I \) is \( \sigma \)-semiprime by Lemma 3.2. Thus \( (R/I)[x, x^{-1}; \sigma] \) is semiprime by Lemma 3.4 and so \( I[x, x^{-1}; \sigma] \) is a semiprime ideal of \( R[x, x^{-1}; \sigma] \) by Lemma 3.3. Hence we have \( I[x, x^{-1}; \sigma] \supseteq P(R[x, x^{-1}; \sigma]) \). To show the converse inclusion \( I[x, x^{-1}; \sigma] \subseteq P(R[x, x^{-1}; \sigma]) \), let \( P \) be any prime ideal of \( R[x, x^{-1}; \sigma] \). Then \( P \cap R \) is a \( \sigma \)-prime ideal of \( R \) by Proposition 1 in [2]. Since \( P \cap R \) is a \( \sigma \)-prime ideal of \( R \), \( I \subseteq P \cap R \subseteq P \), which implies that \( I[x, x^{-1}; \sigma] \subseteq P \), and so \( I[x, x^{-1}; \sigma] \subseteq P(R[x, x^{-1}; \sigma]) \). \( \square \)

**Remark.** We will have a question: For a ring with an automorphism \( \sigma \), what is the prime radical of skew polynomial ring \( R[x; \sigma] \)? We might have some partial answer to this question. We can note that the above Lemma 3.3 holds for skew polynomial ring \( R[x; \sigma] \), i.e., \( R[x; \sigma]/I[x; \sigma] \cong (R/I)[x; \sigma] \). In [1], A. W. Goldie and G. O. Michler have shown...
that for a Noetherian ring $R$, $I$ is $\sigma$-ideal of $R$ if and only if $\sigma(I) \subseteq I$. By using this result, we can also note that for a Noetherian ring $R$, the above Lemma 3.4 holds for skew polynomial ring $R[x; \sigma]$, i.e., $R$ is $\sigma$-semiprime if and only if $A = R[x; \sigma]$ is semiprime. Hence by the similar argument in the proof of Theorem 3.5 we have that for a Noetherian ring $R$ with an automorphism $\sigma$, the prime radical of $R[x; \sigma]$ is equal to $P_\sigma(R)[x; \sigma]$.

References


Department of Mathematics Education, Pusan National University, Pusan 609-735, Korea

E-mail: jchan@pusan.ac.kr