REMARKS ON THE MINIMIZER
OF A $p$-GINZBURG-LANDAU TYPE

YUTIAN LEI

ABSTRACT. The author studies the asymptotic behavior of the radial minimizer for a variant of the $p$-Ginzburg-Landau type functional, in the case of $p$ larger than the dimension, when the parameter tends to zero. The $C^{1,\alpha}$ convergence of the radial minimizer is proved. And the estimation of the convergent rate of the minimizer is given.

1. Introduction

Let $G \subset \mathbb{R}^n (n \geq 2)$ be a bounded and simply connected domain with smooth boundary $\partial G$. $g(x) : \partial G \to S^1$ is smooth map satisfying $d = \text{deg}(g, \partial G)$. When $n = 2$, many papers studied the asymptotic behavior of minimizer $u_\varepsilon$ of the Ginzburg-Landau functional

$$E_\varepsilon^1(u) = \frac{1}{2} \int_G |\nabla u|^2 \, dx + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2 \, dx$$

on $H^1_0(G, \mathbb{R}^2)$ as $\varepsilon \to 0$. In particular, some subsequence $u_{\varepsilon_k}$ of the minimizer $u_\varepsilon$ satisfies

$$\lim_{k \to \infty} u_{\varepsilon_k} = u_* \quad \text{in} \quad C^{1,\alpha}_{\text{loc}}(\overline{G \setminus A}),$$

where $\alpha \in (0, 1)$, $u_*$ is a harmonic map (see [1]). Here $A$ is the set of the singularities of $u_*$. The papers [2] and [5] presented the properties of the radial minimizer of $E_\varepsilon^1(u)$ in the function class

$$V = \left\{ u(x) = f(r) \frac{x}{|x|} \in H^1(B, \mathbb{R}^2) ; f(1) = 1, r = |x| \right\}.$$
where \( B = \{ x \in \mathbb{R}^2; |x| < 1 \} \). The asymptotic behavior of the minimizer in \( H^1_g(G, \mathbb{R}^2) \) of the Ginzburg-Landau type functional

\[
E^2_\varepsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 \, dx + \frac{1}{4\varepsilon^2} \int_G |u|^2(1 - |u|^2)^2 \, dx,
\]

with the different penalization, was discussed extensively in [3]. The same result as (1.1) was also derived. Afterwards, it is researched asymptotic properties of the radial minimizer of

\[
E_\varepsilon(u, B) = \frac{1}{p} \int_B |\nabla u|^p \, dx + \frac{1}{4\varepsilon^p} \int_B |u|^2(1 - |u|^2)^2 \, dx, \quad (p > n)
\]

in \( W = \{ u(x) = f(r) \frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n); f(1) = 1, r = |x| \} \), where \( B = \{ x \in \mathbb{R}^n; |x| < 1 \} \). The following properties have been proved (see Theorems 3.6 and 4.3 in [4]):

(1.2) The radial minimizer \( u_\varepsilon \) is unique as long as \( \varepsilon \) is sufficiently small,

and the convergence which is weaker than (1.1)

(1.3) \( u_\varepsilon \to \frac{x}{|x|} \) in \( W^{1,p}_{\text{loc}}(\overline{B} \setminus \{0\}) \) as \( \varepsilon \to 0 \).

In this paper, we will prove that the radial minimizer of \( E_\varepsilon(u, B) \) also satisfies the convergent property as (1.1). To do this, the \( C^{1,\alpha} \) uniform estimation of the minimizer \( u_\varepsilon \) should be obtained. Indeed, it is difficult since the Euler-Lagrange equation, which the minimizer satisfies, is degenerate when \( p > 2 \). There may not be any classical solution to the equation. Hereby, we consider the regularized functional

\[
E^*_\varepsilon(u, B) = \frac{1}{p} \int_B (|\nabla u|^2 + \tau)^{p/2} \, dx + \frac{1}{4\varepsilon^p} \int_B |u|^2(1 - |u|^2)^2 \, dx, \quad (\tau \in (0, 1)).
\]

It is easy to see that the minimizer \( u^*_\varepsilon(x) = f^*_\varepsilon(r) \frac{x}{|x|} \) exists in \( W \). By the argument of the weak low semi-continuity we can deduce that

(1.4) \( \lim_{\tau \to 0} u^*_\varepsilon = \tilde{u}_\varepsilon \) in \( W^{1,p}(B, \mathbb{R}^n) \),

where \( \tilde{u}_\varepsilon \) is a radial minimizer of \( E_\varepsilon(u, B) \) in \( W \). Noticing (1.2), we know that as \( \varepsilon \) is sufficiently small, the limit \( \tilde{u}_\varepsilon \) must be the unique radial minimizer \( u_\varepsilon \). Hence, we may derive the \( C^{1,\alpha} \) convergence of the radial minimizer via establishing the \( C^{1,\alpha} \) estimation of \( u^*_\varepsilon \) (see Theorem 2.2).

In addition, we also concern with the convergent rate of \( |u_\varepsilon| \to 1 \) in \( W^{1,p}(B \setminus \{0\}) \) when \( \varepsilon \to 0 \). In fact, if \( T > 0 \), we can obtain firstly that

\[
E_\varepsilon(u_\varepsilon, B \setminus B_T(0)) - \frac{1}{p} \int_{B \setminus B_T(0)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon |\varepsilon|^{-p+1} \quad (\text{cf. Theorem 3.1}).
\]
Next, by improving the exponent \([p] - p + 1 \) of \(\varepsilon\) step by step, we can see at last the convergent rate in \(W^{1,p}([T,1])\):
\[
\int_T^1 r^{n-1}[f'(r)^p + \frac{1}{\varepsilon p}(1 - f^2_{\varepsilon})^2]dr \leq C\varepsilon^p,
\]
where \(f_{\varepsilon}(r) = |u_{\varepsilon}(x)|\) (cf. Theorem 3.2).

2. \(C^{1,\alpha}\) convergence

Since \(u_{\varepsilon}\) is a minimizer, it is not difficult to see that \(f_{\varepsilon} = |u_{\varepsilon}|\) solves
\begin{equation}
(2.1) \quad \left( rA^{(p-2)/2}f_{rr} + \frac{n-1}{r}A^{(p-2)/2}f = \frac{1}{2\varepsilon^p} rf \left( 4f^2 - 3f^4 - 1 \right), \right.
\end{equation}
and \(|f_{\varepsilon}| \leq 1\) in \([0,1]\) by the maximum principle, where \(A = f^2 + \frac{(n-1)f^2}{r^2} + \tau\). By the same argument of (3.8) and (4.2) in [4], we also see that for any \(T > 0\), there exists \(C > 0\) which is independent of \(\varepsilon\) and \(\tau\), such that
\begin{equation}
|f_{\varepsilon}| \geq 29/30 \quad \text{in} \quad [T,1],
\end{equation}
\begin{equation}
(2.3) \quad \int_T^1 A^{p/2}dr \leq C.
\end{equation}

**Proposition 2.1.** Denote \(u_{\varepsilon} = u = f(r)\frac{\varphi}{|x|}\). Then for any compact subset \(K \subset (0,1)\), there exists a positive constant \(C\) which does not depend on \(\varepsilon\) and \(\tau\), such that \(\|f\|_{C^{1,\alpha}(K,R)} \leq C, \quad \forall \beta \in (0,1/2)\).

**Proof.** Take \(T > 0\) so small that \(K_{3T} \subset K \subset K_{2T} \subset K_T = (T,1-T)\). Let \(\zeta \in C_{0}^\infty(K_T, [0,1])\) satisfy that \(\zeta = 0\) on \([0,1]\) \(\setminus K_T\), \(\zeta = 1\) on \(K_{2T}\), and \(|\zeta_r| \leq C(T)\) on \((0,1)\). Multiplying (2.1) with \(r^{-1}\) and differentiating, then multiplying by \(f_r\zeta^2\) and integrating on \((0,1)\), we have
\[
-\int_0^1 \left( A^{(p-2)/2}f_{rr} \right) (f_r\zeta^2) dr - \int_0^1 \left( r^{-1}A^{(p-2)/2}f_r \right) (f_r\zeta^2) dr \\
+ (n-1) \int_0^1 \left( r^{-2}A^{(p-2)/2}f \right) (f_r\zeta^2) dr \\
= \frac{1}{2\varepsilon^p} \int_0^1 \left[ f \left( 4f^2 - 3f^4 - 1 \right) \right] (f_r\zeta^2) dr.
\]
Integrating by parts and noting
\[
-\frac{1}{2\varepsilon^p} \int_{K_T} f^2 (f_r)^2 (12f^2 - 8)\zeta^2 dr \leq 0,
\]
we get
\[
\int_{K_T} \left( A^{\frac{p-2}{2}} f_r^r \right) r \left( f_r \zeta^2 \right)_r \, dr + \int_{K_T} A^{\frac{p-2}{2}} \left( f_r \zeta^2 \right)_r \left[ \frac{f_r}{r} - \frac{(n-1)f}{r^2} \right] \, dr \leq \frac{1}{2\varepsilon^p} \int_{K_T} (4f^2 - 3f^4 - 1) f_r^2 \zeta^2 \, dr.
\]

Denote
\[
I = \int_{K_T} \zeta^2 \left( A^{(p-2)/2} f_{rr}^2 + (p-2) A^{(p-4)/2} f_r^2 f_{rr}^2 \right) \, dr.
\]

Noting
\[
A_r = 2 \left[ f_r f_{rr} + (n-1)(f f_{r} - r^{-2} - r^{-3} f^2) \right],
\]
and using Young inequality, we see that for any \( \delta \in (0, 1) \),
\[
I \leq \delta I + C(\delta, T) \int_{K_T} A^{p/2} \zeta_r^2 \, dr
\]
\[
+ \frac{1}{2\varepsilon^p} \left| \int_{K_T} f_r^2 (4f^2 - 3f^4 - 1) \zeta^2 \, dr \right|.
\]

From (2.1) and (2.2) and by Young inequality, it follows
\[
\frac{1}{2\varepsilon^p} \left| \int_{K_T} (4f^2 - 3f^4 - 1) f_r^2 \zeta^2 \, dr \right| \leq \delta I + C(\delta, T) \int_{K_T} A^{(p+2)/2} \zeta^2 \, dr
\]
with \( \delta \in (0, 1) \). Substituting this into (2.4) and choosing \( \delta \) sufficiently small, we have
\[
I \leq C \int_{K_T} A^{p/2} \zeta_r^2 \, dr + C \int_{K_T} A^{(p+2)/2} \zeta^2 \, dr.
\]

To estimate the second term of the right hand side, we take \( \phi = \zeta^{2/q} f_r^{(p+2)/q} \) in the embedding inequality
\[
\| \phi \|_{L^q} \leq C \| \phi_r \|_{L^1}^{1/q} \| \phi \|_{L^1}^{1/q} \,, \quad q \in (1 + \frac{2}{p}, 2).
\]

Applying Young inequality we see that for any \( \delta \in (0, 1) \),
\[
\int_{K_T} f_r^{p+2} \zeta^2 \, dr \leq C \left( \int_{K_T} \zeta^{2/q} |f_r|^{(p+2)/q} \, dr \right)
\]
\[
\times \left( \int_{K_T} \zeta^{2/q-1} |\zeta_r| |f_r|^{(p+2)/q} \right.
\]
\[
+ \delta I + C(\delta) \int_{K_T} A^{p+2/\alpha} \zeta^{4/q-2} \, dr \left.)^{q-1} \right.}
\]
Noting \( q \in (1 + \frac{2}{p}, 2) \), we can use the Holder inequality to estimate the right hand side of (2.7). Thus, from (2.3) it leads to \( \int_{K_T} f_r^{p+2} \zeta^2 dr \leq \delta I + C(\delta) \) for any \( \delta \in (0, 1) \). Substituting this into (2.5) and choosing \( \delta \) sufficiently small, we obtain \( \int_{K_T} A^{(p-2)/2} f_r^2 \zeta^2 dr \leq C \). Combining this with (2.3), we get \( \| A^{p/4} \zeta \|_{H^1(K_T)} \leq C \). Noticing that \( \zeta = 1 \) on \( K \), we see that \( \| A^{p/4} \|_{H^1(K)} \leq C \). Applying the embedding inequality, we know that for any \( \beta \leq 1/2, \| A^{p/4} \|_{C^\beta(K)} \leq C \). The inner estimation is set up.

In the following, we consider the estimation near the boundary point \( r = 1 \). Denote \( g(r) = f(r + 1) - 1 \). Set \( \tilde{g}(r) = g(r) \) as \( -1 \leq r \leq 0 \), \( \tilde{g}(r) = -g(-r) \) as \( 0 < r \leq 1 \). If denote \( f(r) = \tilde{g}(r - 1) + 1 \) in \([0, 2]\), then \( f(r) \) solves (2.1) in \([0, 2]\). Take \( R < \frac{1}{4} \). Assume that \( \zeta \in C^\infty(0, 1) \) satisfies \( \zeta = 1 \) when \( r \geq 1 - R; \zeta = 0 \) when \( r \leq 2R \). Differentiating (2.1) and multiplying by \( f_r \zeta^2 \), then integrating on \([R, 1]\), we have

\[
- \int_R^1 (A^{(p-2)/2} f_r)_{rr} (f_r \zeta^2) \, dr - \int_R^1 (r^{-1} A^{(p-2)/2} f_r) (f_r \zeta^2) \, dr \\
+ (n - 1) \int_R^1 (r^{-2} A^{(p-2)/2} f_r) (f_r \zeta^2) \, dr \\
= \frac{1}{\varepsilon^p} \int_R^1 [f(1 - f^2)(3f^2 - 1)]_r (f_r \zeta^2) \, dr.
\]

Integrating by parts leads to

\[
\int_R^1 (A^{(p-2)/2} f_r)_r (f_r \zeta^2)_r \, dr \\
+ \int_R^1 A^{(p-2)/2} (f_r \zeta^2)_r [r^{-1} f_r - (n - 1) r^{-2} f] \, dr \\
\leq \frac{1}{\varepsilon^p} \int_R^1 (1 - f^2)(3f^2 - 1) f_r^2 \zeta^2 \, dr \\
- \frac{1}{\varepsilon^p} \int_R^1 f^2(12f^2 - 8)(f_r)^2 \zeta^2 \, dr + |I(1) - I(R)|,
\]

where \( I(r) = (A^{(p-2)/2} f_r)_r + \frac{1}{r} A^{(p-2)/2} f_r - (n-1) r^{-2} A^{(p-2)/2} f \). From (2.1) it follows that \( I(r) = \frac{1}{\varepsilon^p} f(f^2 - 1)(3f^2 - 1) f_r \zeta^2 \). Noting \( f(1) = 1 \) and \( \zeta(R) = 0 \), we obtain \( I(1) = I(R) = 0 \). Substituting this into the inequality above, and by the same argument as the inner estimation, we also derive (2.5). Now, take \( \phi = \zeta^{2/q} f_r^{(p+2)/q} \) in the embedding
inequality
\[ \| \phi \|_{L^q} \leq C(\| \phi_r \|_{L^1} + \| \phi \|_{L^1})^{1 - 1/q} \| \phi \|_{L^1}^{1/q}, \quad q \in (1 + \frac{2}{p}, 2) \]

instead of (2.6) (in fact, (2.6) is not valid since \( \phi \neq 0 \) near \( r = 1 \)). Thus, (2.7) can be still derived. The rest proof is same as the proof of the inner estimation.

**Theorem 2.2.** Let \( u_\varepsilon = f_\varepsilon(r) \frac{\varphi}{|x|} \) be a radial minimizer of \( E_\varepsilon(u, B) \). Then for any compact subset \( K \subset \overline{B \setminus \{0\}} \), \( \lim_{\varepsilon \to 0} u_\varepsilon = \frac{\varphi}{|x|} \) in \( C^{1,\alpha}(K, \mathbb{R}^n) \) for all \( \alpha \in (0, 1/2) \).

**Proof.** For any compact subset \( K \subset \overline{B \setminus \{0\}} \), by using Proposition 2.1 we know that for some \( \beta \in (0, 1/2) \),
\[\| u_\varepsilon^T \|_{C^{1,\beta}(K)} \leq C = C(K) \]
with \( C > 0 \) independent of \( \varepsilon, \tau \). From this and the embedding theorem, we see that for some \( \beta_1 < \beta \), there exist \( u_\varepsilon^* \in C^{1,\beta_1}(K, \mathbb{R}^n) \) and a subsequence \( \tau_k \) of \( \tau \), such that as \( k \to \infty \),
\[ u_\varepsilon^{T_k} \to u_\varepsilon^* \quad \text{in} \quad C^{1,\beta_1}(K, \mathbb{R}^n). \]

Combining this with (1.4) and (1.2), we have \( w_\varepsilon^* = u_\varepsilon \).

Applying (2.8) and the embedding theorem again, we know that for some \( \beta_2 < \beta \), there exist \( w^* \in C^{1,\beta_2}(K, \mathbb{R}^n) \) and a subsequence \( \tau_m \) of \( \tau_k \), such that as \( m \to \infty \),
\[ u_\varepsilon^{T_m} \to w^* \quad \text{in} \quad C^{1,\beta_2}(K, \mathbb{R}^n). \]

Set \( \alpha = \min(\beta_1, \beta_2) \). Thus, when \( m \to \infty \), using (2.9) and (2.10) we obtain
\[
\begin{align*}
\| u_{\varepsilon_m} - w^* \|_{C^{1,\alpha}(K, \mathbb{R}^n)} & \leq \| u_{\varepsilon_m} - u_{\varepsilon_m}^{T_m} \|_{C^{1,\alpha}(K, \mathbb{R}^n)} + \| u_{\varepsilon_m}^{T_m} - w^* \|_{C^{1,\alpha}(K, \mathbb{R}^n)} \\
& \leq o(1),
\end{align*}
\]

which, together with (1.3), implies \( w^* = \frac{\varphi}{|x|} \). Noting the uniqueness of the limit \( \frac{\varphi}{|x|} \), we deduce that the convergence (2.11) holds not only for the subsequence, but also for the whole \( u_\varepsilon \).
3. Analysis of the convergent rate

From (4.2) in [4], it was led to, for any compact subset of $K \subset (0,1]$, the convergent rate

\begin{equation}
\frac{1}{4\varepsilon^p} \int_K \left(1 - f_\varepsilon^2\right)^2 r^{n-1} dr \leq C.
\end{equation}

In this section, we shall present the better rate.

**Theorem 3.1.** Let $u_\varepsilon(x) = f_\varepsilon(r) \frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u,B)$. Then for any $T > 0$, there exists a constant $C > 0$ which is independent of $\varepsilon$, such that as $\varepsilon \to 0$,

\begin{equation}
\int_T^1 |f'_\varepsilon|^p r^{n-1} dr + \frac{1}{\varepsilon^p} \int_T^1 f_\varepsilon^2 (1 - f_\varepsilon^2)^2 r^{n-1} dr \leq C \varepsilon^{[p]-p+1}.
\end{equation}

\begin{align}
\frac{1}{p} \int_{B \setminus B_T(0)} |\nabla u_\varepsilon|^p + \frac{1}{4\varepsilon^p} \int_{B \setminus B_T(0)} |u_\varepsilon|^2 (1 - |u_\varepsilon|^2)^2 &
\rightarrow \frac{1}{p} \int_{B \setminus B_T(0)} |\nabla \frac{x}{|x|}|^p.
\end{align}

**Proof.** From [4, Theorem 4.2] it follows that

\begin{equation}
E_\varepsilon(f_\varepsilon; T) \leq \frac{1}{p} \int_T^1 (n-1)^{p/2} r^{n-p-1} dr + C \varepsilon^{[p]-p+1}.
\end{equation}

Here $E_\varepsilon(f; T) = \frac{1}{p} \int_T^1 |f'|^p r^{n-1} dr + \frac{1}{4\varepsilon^p} \int_T^1 f^2 (1 - f^2)^2 r^{n-1} dr$. On the other hand, Jensen’s inequality with $p > 2$ implies that

\begin{align}
E_\varepsilon(f_\varepsilon; T) &\geq \frac{1}{p} \int_T^1 |f'_\varepsilon|^p r^{n-1} dr + \frac{1}{p} \int_T^1 \left( (n-1) \frac{f_\varepsilon^2}{r^2} \right)^{p/2} r^{n-1} dr \\
&\quad + \frac{1}{4\varepsilon^p} \int_T^1 f_\varepsilon^2 (1 - f_\varepsilon^2)^2 r^{n-1} dr.
\end{align}

Combining this with (3.4) yields

\begin{align}
\frac{1}{p} \int_T^1 \left( (n-1) \frac{f_\varepsilon^2}{r^2} \right)^{p/2} r^{n-1} dr \\
\leq E_\varepsilon(f_\varepsilon; T) \\
\leq C \varepsilon^{[p]-p+1} + \frac{1}{p} \int_T^1 (n-1)^{p/2} r^{n-p-1} dr.
\end{align}
By using (3.1) and Holder inequality we deduce
\[
\int_T^1 (n-1)^{p/2} r^{n-p-1}(1 - f^p_\varepsilon)dr \leq C\varepsilon^{p/2}.
\]
Substituting this into (3.6) leads to
\[
(3.7) \quad E_\varepsilon(f_\varepsilon;T) - \frac{1}{p} \int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1}dr \leq C\varepsilon^{[p]-p+1}.
\]
Noticing \( \int_{B\setminus B_T(0)} |\nabla \frac{x}{|x|}|^p dx = |S^{n-1}| \int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1}dr \), from (3.5) and (3.7), we can obtain (3.2) and (3.3).

**Theorem 3.2.** Let \( u_\varepsilon(x) = f_\varepsilon(r) \frac{x}{|x|} \) be a radial minimizer of \( E_\varepsilon(u,B) \). Then for any \( T > 0 \), there exists \( C > 0 \) which is independent of \( \varepsilon \), such that \( \int_T^1 r^{n-1}[(f_\varepsilon')^p + \frac{1}{\varepsilon^p}(1 - f_\varepsilon^2)^2]dr \leq Ce^p \).

**Proof.** From Jensen's inequality and (2.2) it follows
\[
E_\varepsilon(f_\varepsilon;T) \geq \frac{1}{p} \int_T^1 (f_\varepsilon')^p r^{n-1}dr + \frac{1}{8\varepsilon^p} \int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1}dr
\]
\[
+ \frac{1}{p} \int_T^1 \frac{(n-1)^{p/2}}{r^p} f_\varepsilon^p r^{n-1}dr.
\]
Combining this with (3.4) and using (3.1), we have
\[
(3.8) \quad \frac{1}{p} \int_T^1 (f_\varepsilon')^p r^{n-1}dr + \frac{1}{8\varepsilon^p} \int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1}dr \leq C\varepsilon^{[p]-p+1}.
\]
Noting (3.4) and (3.8), and applying the integral mean value theorem, we see that there exists \( T_1 \in [T,2T] \), such that
\[
(3.9) \quad \left[ \frac{1}{\varepsilon^p} (1 - f_\varepsilon^2)^2 \right]_{r=T_1} \leq C_1\varepsilon^{[p]-p+1}.
\]

Clearly, we may find a minimizer \( \rho_1 \) in \( W^1_{f_\varepsilon}((T_1,1),R^+ \cup \{0\}) \) of the functional
\[
E(\rho,T_1) = \frac{1}{p} \int_{T_1}^1 (\rho^2_r + 1)^{p/2}dr + \frac{1}{2\varepsilon^p} \int_{T_1}^1 (1 - \rho)^2dr.
\]

**Proposition 3.3.** (3.9) implies that \( E(\rho_1,T_1) \leq C\varepsilon^{F[1]} \). Here \( F[j] = \frac{[p]+1-p}{2^j} + \frac{(2^j-1)p}{2^j}, j = 0,1,\ldots \).
Proof. The minimizer \( \rho_1 \) solves the problem

\[
-\varepsilon^p (v^{(p-2)/2} \rho_r)_{r} = 1 - \rho \quad \text{on} \quad [T_1, 1],
\]

\[
\rho(T_1) = f_\varepsilon(T_1), \quad \rho(1) = f_\varepsilon(1) = 1,
\]

where \( v = \rho^2 + 1 \). Obviously, \( \rho \leq 1 \). Noting that \( \rho_1 \) is a minimizer, we deduce from (3.8) in [4] and (3.4) that

\[
E(\rho_1, T_1) \leq E(f_\varepsilon, T_1) \leq CE_\varepsilon(f_\varepsilon; T_1) \leq C.
\]

Take \( \zeta \in C^\infty(0, 1) \), \( \zeta = 1 \) on \( (0, T_1) \), \( \zeta = 0 \) near \( r = 1 \), and \( |\zeta_r| \leq C(T_1) \). Multiplying (3.10) with \( \zeta \rho_r \) and integrating over \([T_1, 1]\), we have

\[
v^{(p-2)/2} \rho_r^2 \bigg|_{r=T_1} + \int_{T_1}^{1} v^{(p-2)/2} \rho_r (\zeta_r \rho_r + \zeta \rho_{rr}) \, dr
\]

\[
= \frac{1}{\varepsilon^p} \int_{T_1}^{1} (1 - \rho) \zeta \rho_r \, dr.
\]

At first, by using (3.12) we obtain

\[
\left| \int_{T_1}^{1} v^{(p-2)/2} \rho_r (\zeta_r \rho_r + \zeta \rho_{rr}) \, dr \right|
\]

\[
\leq \int_{T_1}^{1} v^{(p-2)/2} |\zeta_r| (\rho_r)^2 \, dr + \frac{1}{p} \left| \int_{T_1}^{1} \left[ \left( v^{(p-2)/2} \zeta \right)_r - v^{(p-2)/2} \zeta_r \right] \, dr \right|
\]

\[
\leq C + \frac{1}{p} v^{p/2} \bigg|_{r=T_1}.
\]

Next, by applying (3.12), (3.11), and (3.9), we derive

\[
\frac{1}{\varepsilon^p} \left| \int_{T_1}^{1} (1 - \rho) \zeta \rho_r \, dr \right|
\]

\[
= \frac{1}{2\varepsilon^p} \left| \int_{T_1}^{1} \left[ ((1 - \rho)^2) \zeta_r - (1 - \rho)^2 \zeta_r \right] \, dr \right|
\]

\[
\leq \frac{1}{2\varepsilon^p} (1 - \rho)^2 \bigg|_{r=T_1} + \frac{C}{2\varepsilon^p} \int_{T_1}^{1} (1 - \rho)^2 \, dr
\]

\[
\leq C.
\]

Combining (3.13)–(3.15), we get \( v^{(p-2)/2} \rho_r^2 |_{r=T_1} \leq C + \frac{1}{p} v^{p/2} |_{r=T_1} \). Substituting this into \( v^{p/2} |_{r=T_1} = v^{(p-2)/2} (\rho_r^2 + 1) |_{r=T_1} \), and using Young.
inequality, we see that for any $\delta \in (0, 1/2)$, $v^{p/2}|_{r=T_1} \leq C(\delta) + (\frac{1}{p} + \delta)v^{p/2}|_{r=T_1}$. Choosing $\delta$ sufficiently small yields

(3.16) \[ v^{p/2}|_{r=T_1} \leq C. \]

Multiplying (3.10) with $(\rho - 1)$ and integrating over $[T_1, 1]$, we have

\[ \int_{T_1}^{1} \left[ v^{(p-2)/2} \rho_r (\rho - 1) \right] r \, dr = \int_{T_1}^{1} v^{(p-2)/2} \rho_r^2 r \, dr + \frac{1}{\varepsilon^p} \int_{T_1}^{1} (\rho - 1)^2 r \, dr. \]

Hence, by applying (3.16), (3.11), and (3.9), we obtain

\[ E(\rho_1, T_1) \leq C \left| \int_{T_1}^{1} \left[ v^{(p-2)/2} \rho_r (\rho - 1) \right] r \, dr \right| \]

\[ = C v^{(p-2)/2} |\rho_r| |\rho - 1| \bigg|_{r=T_1} \]

\[ \leq C \varepsilon^{F(1)}. \]

Proposition is proved.

**Proposition 3.4.** Proposition 3.3 implies that

\[ E_\varepsilon(f_\varepsilon; T_1) \leq C \varepsilon^{F[1]} + \frac{1}{p} \int_{T_1}^{1} \frac{(n-1)p/2}{r^{n+1}} \, dr. \]

**Proof.** Set $w_\varepsilon = f_\varepsilon$ if $r \in [0, T_1]$, $w_\varepsilon = \rho_1$ if $r \in [T_1, 1]$. Noticing that $u_\varepsilon$ is a minimizer, we have $E_\varepsilon(u_\varepsilon, B) \leq E_\varepsilon(w_\varepsilon \mathfrak{H}_{[\varepsilon]}, B)$. Hence

\[ E_\varepsilon(f_\varepsilon, T_1) \leq \frac{1}{p} \int_{T_1}^{1} \left( \rho_r^2 + \frac{n-1}{r^2} \rho^2 \right) r^{p/2} \, r^{n-1} dr \]

\[ + \frac{1}{4\varepsilon^p} \int_{T_1}^{1} \rho^2 (1 - \rho^2)^2 r^{n-1} \, dr \]

\[ \leq \frac{1}{p} \int_{T_1}^{1} \left( \frac{n-1}{r^2} \rho^2 \right) r^{p/2} \, r^{n-1} \, dr + CE(\rho_1, T_1). \]

Proposition 3.4 is seen by using Proposition 3.3.

Complete the proof of Theorem 3.2. Using Proposition 3.4 and (3.1), we can deduce that

\[ \int_{T_1}^{1} (f_\varepsilon')^p \rho r^{n-1} \, dr + \frac{1}{8\varepsilon^p} \int_{T_1}^{1} (1 - f_\varepsilon^2)^2 r^{n-1} \, dr \leq C \varepsilon^{F[1]} + C \varepsilon^{p/2} \leq C \varepsilon^{F[1]} \]

by the same derivation of (3.8). Comparing with (3.8), we find the rate is better than (3.8), since the exponent of $\varepsilon$ is improved from $F[0]$ to $F[1]$. 
Set $T_m \in [T_{m-1}, 2T]$. By the same argument above (whose idea is improving the exponent of $\varepsilon$ from $F[k]$ to $F[k+1]$), we know that there exists a sufficiently large integer $m$ satisfying $\frac{p}{2} + 1 \leq F[m]$, such that

$$\int_{T_m}^{1} (f'_{\varepsilon})^p r \, dr + \frac{1}{8\varepsilon^p} \int_{T_m}^{1} (1 - f_{\varepsilon}^2)^2 r^{n-1} \, dr \leq C \varepsilon^F[m] + C \varepsilon^{p/2} \leq C \varepsilon^{p/2}. \tag{3.17}$$

Similar to the derivations of (3.9), we see that there exists $T_{m+1} \in [T_m, 2T]$ such that

$$\left[ \frac{1}{2} \varepsilon^p \left( 1 - f_{\varepsilon}^2 \right)^2 \right]_{r=T_{m+1}} \leq C \varepsilon^{p/2}. \tag{3.18}$$

Obviously, the minimizer $\rho_2 \in W^{1,p}_{f_{\varepsilon}}((T_1, 1), R^+)$ of

$$E(\rho, T_{m+1}) = \frac{1}{p} \int_{T_{m+1}}^{1} (\rho^2 + 1)^{p/2} \, dr + \frac{1}{2\varepsilon^p} \int_{T_{m+1}}^{1} (1 - \rho)^2 \, dr$$

exists. By the analogous proof of Proposition 3.3, form (3.18) we can also obtain that

$$E(\rho_2, T_{m+1}) \leq v_{T_{m+1}} \rho_4(1 - \rho_1) \leq C(1 - \rho_2(T_{m+1})) \leq C \varepsilon^{G[1]},$$

where $G[j] = \frac{p}{2j} + \frac{(2j-1)p}{2j}$, $j = m + 1, m + 2, \cdots$. So, by the same proof of Proposition 3.4 we also conclude that

$$E_{\varepsilon}(f_{\varepsilon}; T_{m+1}) \leq C \varepsilon^{G[1]} + \frac{1}{p} \int_{T_{m+1}}^{1} \frac{(n - 1)^{p/2}}{r^{p-1}} \, dr.$$

Similar to the derivation of (3.8), using (3.17) we have

$$\int_{T_{m+1}}^{1} (f'_{\varepsilon})^p r^{n-1} \, dr + \frac{1}{8\varepsilon^p} \int_{T_{m+1}}^{1} (1 - f_{\varepsilon}^2)^2 r^{n-1} \, dr \leq C \varepsilon^{G[1]}.$$

By the same argument above (whose idea is improving the exponent of $\varepsilon$ from $G[k]$ to $G[k+1]$), we know that for any $k \in N$,

$$\int_{T_{m+k}}^{1} (f'_{\varepsilon})^p r^{n-1} \, dr + \frac{1}{8\varepsilon^p} \int_{T_{m+k}}^{1} (1 - f_{\varepsilon}^2)^2 r^{n-1} \, dr \leq C \varepsilon^{p/2 + \frac{(2k-1)p}{2k}}.$$

Letting $k \to \infty$, we can see the conclusion of Theorem.
References


Department of Mathematics, Nanjing Normal University, Nanjing, Jiangsu 210097, P.R.China

E-mail: lythxl@163.com