Journal of Mechanical Science and Technology, Vol. 19, No. 1, ⟨Special Edition⟩ pp. 227~236, 2005 **227**

Recent Developments in Multibody Dynamics

Werner Schiehlen*
Institute B of Mechanics, University of Stuttgart, 70550 Stuttgart, Germany

Multibody system dynamics is based on classical mechanics and its engineering applications originating from mechanisms, gyroscopes, satellites and robots to biomechanics. Multibody system dynamics is characterized by algorithms or formalisms, respectively, ready for computer implementation. As a result simulation and animation are most convenient. Recent developments in multibody dynamics are identified as elastic or flexible systems, respectively, contact and impact problems, and actively controlled systems. Based on the history and recent activities in multibody dynamics, recursive algorithms are introduced and methods for dynamical analysis are presented. Linear and nonlinear engineering systems are analyzed by matrix methods, nonlinear dynamics approaches and simulation techniques. Applications are shown from low frequency vehicles dynamics including comfort and safety requirements to high frequency structural vibrations generating noise and sound, and from controlled limit cycles of mechanisms to periodic nonlinear oscillations of biped walkers. The fields of application are steadily increasing, in particular as multibody dynamics is considered as the basis of mechatronics.

Key Words : History of Multibody Dynamics, Mechanical Modelling, Kinematics, Newton-Euler Equations, Equations of Motion, Recursive Formalisms, Linear Vibrations, Nonlinear Analysis, Vehicle Vibrations and Control, Structural Vibrations, Contact, Mechanisms, Biped Walker

1. History and Recent Activities

The roots of multibody dynamics date back to the origins of analytical mechanics starting with the Principia of Newton (1687), Corporum Rigidarum by Euler (1776) and Mecanique Analytique by Lagrange (1788). Even more important for the computational aspects of multibody dynamics are the contributions of D'Alembert (1743) in his Traité de Dynamique, Jourdain (1909) with his Analogue at Gauss' Principle and the work of Kane and Levinson (1985). Multibody dynamics was also promoted at the beginning of the 20th century by the theory of gyroscopes, see e.g. Grammel (1920), and mechanism theory by the early work of Wittenbauer (1923). During the middle of the last century spacecraft and biomechanics pushed the development of multibody dynamics as documented by Roberson and Wittenburg (1967) and Huston and Passerello (1971).

Multibody dynamics as a new branch of mechanics was set up in 1977 by a IUTAM Symposium chaired by Magnus (1978). Twenty years later MULTIBODY SYSTEM DYNAMICS was established as the first scientific journal fully devoted to multibody dynamics. After many collo quia, symposia and conferences in Europe and North America, the First Asian Conference on Multibody Dynamics took place in 2002. And in 2003 an ASME Technical Committee on Multibody Systems and Nonlinear Dynamics was formed with the task to organize biannual conferences starting in 2005.

Recent research topics may be listed as follows.

(1) Datamodels from CAD (standardization, coupling)

* **E-mail** : schiehlen@mechb.uni-stuttgart.de
TEL : +497116856388; **FAX** : +497116856400
Institute B of Mechanics, University of Stuttgart, 70550 Stuttgart, Germany. (Manuscript **Received** November 29, 2004; **Revised** December 15, 2004)

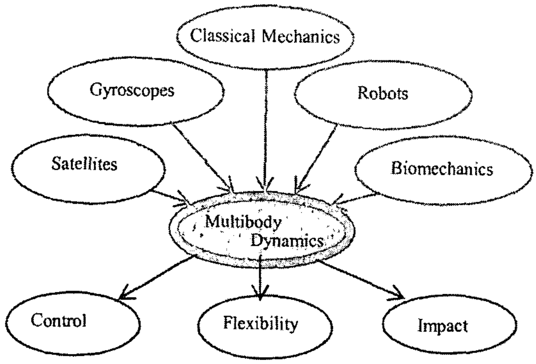


Fig. 1 Focus of multibody systems

(2) Parameter identification
(3) Real time simulation
(4) Contact and impact problems (Impact)
(5) Extension to electronics and mechatronics (Control)
(6) Dynamic strength analysis (Flexibility)
(7) Optimization of design and control
(8) Integration codes
(9) Challenging applications in biomechanics, robotics and vehicle dynamics.

In particular, elastic or flexible multibody systems, respectively, contact and impact problems and actively controlled mechatronic systems represent key issues for researchers worldwide. The focus of multibody systems is shown in Figure 1.

2. Fundamental Dynamics

In this section the essential steps for generation the equations of motion in multibody dynamics will be summarized.

2.1 Mechanical modelling

First of all the engineering or natural system has to be replaced by the elements of the multibody system approach : rigid and/or flexible bodies, joints, gravity, springs, dampers and position and/or force actuators The system constrained by bearings and joints is disassembled as free body system using an appropriate number of inertial, moving reference and body fixed frames for the mathematical description.

2.2 Kinematics

A system of p rigid bodies holds $f=6p$ degrees of freedom characterized by translation vectors and rotation tensors as

$$\boldsymbol{r}_i=[r_{i1}\ r_{i2}\ r_{i3}]^T,\ \boldsymbol{S}_i(\alpha_i,\ \beta_i,\ \gamma_i),\ i=1(1)p \quad (1)$$

Thus, the position vector x of the free system can be written as

$$\boldsymbol{x}=[r_{11}\ r_{12}\ r_{13}\ r_{21}\ \cdots\ \alpha_p\beta_p\gamma_p]^T \quad (2)$$

The system's position remains as

$$\boldsymbol{r}_i=\boldsymbol{r}_i(\boldsymbol{x}),\ \boldsymbol{S}_i=\boldsymbol{S}_i(\boldsymbol{x}) \quad (3)$$

Assembling the system by q holonomic, rheonomic constraints reduces the number of degrees of freedom to $f=6p-q$. The corresponding constraint equations may be written in explicit or implicit form, respectively, as

$$\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{y},\ t)\ \text{or}\ \boldsymbol{\Phi}(\boldsymbol{x},\ t)=\boldsymbol{0} \quad (4)$$

where the position vector $\boldsymbol{y}$ summarizes the f generalized coordinates of the holonomic system

$$\boldsymbol{y}(t)=[y_1\ y_2\ y_3\ \cdots\ y_f]^T \quad (5)$$

Then, for the system's position it remains

$$\boldsymbol{r}_i=\boldsymbol{r}_i(\boldsymbol{y},\ t),\ \boldsymbol{S}_i=\boldsymbol{S}_i(\boldsymbol{y},\ t) \quad (6)$$

By differentiation the translational and rotational velocity vectors are found

$$\begin{aligned}\boldsymbol{v}_i=\dot{\boldsymbol{r}}_i&=\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{y}^T}\dot{\boldsymbol{y}}+\frac{\partial \boldsymbol{r}_i}{\partial t}\\&=\boldsymbol{J}_{Ti}(\boldsymbol{y},\ t)\dot{\boldsymbol{y}}+\bar{\boldsymbol{v}}_i(\boldsymbol{y},\ t)\end{aligned} \quad (7)$$

$$\begin{aligned}\boldsymbol{\omega}_i=\dot{\boldsymbol{s}}_i&=\frac{\partial \boldsymbol{s}_i}{\partial \boldsymbol{y}^T}\dot{\boldsymbol{y}}+\frac{\partial \boldsymbol{s}_i}{\partial t}\\&=\boldsymbol{J}_{Ri}(\boldsymbol{y},\ t)\dot{\boldsymbol{y}}+\bar{\boldsymbol{\omega}}_i(\boldsymbol{y},\ t)\end{aligned} \quad (8)$$

where $\boldsymbol{s}$ means a vector of infinitesimal rotations following from the corresponding rotation tensor, see, e.g., Schiehlen (1997). Further, the Jacobian matrices $\boldsymbol{J}_{Ti}$ and $\boldsymbol{J}_{Ri}$ for translation and rotation are defined by Eqs. (7) and (8).

The system may be subject to additional r nonholonomic constraints which do not affect the $f=6p-q$ positional degrees of freedom. But they reduce the velocity dependent degrees of freedom to $g=f-r=6p-q-r$. The corresponding constraint equations can be written explicitly or implicitly, too,

$$\dot{\boldsymbol{y}}=\dot{\boldsymbol{y}}(\boldsymbol{y},\ \boldsymbol{z},\ t)\ \text{or}\ \boldsymbol{\Psi}(\boldsymbol{y},\ \dot{\boldsymbol{y}},\ t)=\boldsymbol{0} \quad (9)$$

where the g generalized velocities are summarized by the vector

$$z(t)=[z_1\ z_2\ z_3 \cdots z_g]^T \tag{10}$$

For the system's translational and rotational velocities it follows from Eqs. (7) to (9)

$$\boldsymbol{v}_i=\boldsymbol{v}_i(\boldsymbol{y},\ \boldsymbol{z},\ t) \text{ and } \boldsymbol{\omega}_i=\boldsymbol{\omega}_i(\boldsymbol{y},\ \boldsymbol{z},\ t) \tag{11}$$

By differentiation the acceleration vectors are obtained, e.g., the translational acceleration as

$$\begin{aligned}\boldsymbol{a}_i&=\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{z}^T}\dot{\boldsymbol{z}}+\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{y}^T}+\frac{\partial \boldsymbol{v}_i}{\partial t}\\ &=\boldsymbol{L}_{Ti}(\boldsymbol{y},\ \boldsymbol{z},\ t)\dot{\boldsymbol{z}}+\bar{\boldsymbol{v}}_i(\boldsymbol{y},\ \boldsymbol{z},\ t)\end{aligned} \tag{12}$$

A similar equation yields for the rotational acceleration. The Jacobian matrices L are related to the generalized velocities, for translations as well as for rotations.

2.3 Newton-Euler Equations

Newton's equations and Euler's equations are based on the velocities and accelerations from Section 2.2 as well as on the applied forces and torques, and the constraint forces and torques acting on all the bodies. The reactions or constraint forces and torques, respectively, can be reduced to a minimal number of generalized constraint forces also known as Lagrange's multipliers. In matrix notation the following equations are obtained, see also Schiehlen (1997).

Free body system kinematics and holonomic constraint forces:

$$\begin{gathered}\bar{\boldsymbol{M}}\ddot{\boldsymbol{x}}+\bar{\boldsymbol{q}}^c(\boldsymbol{x},\ \dot{\boldsymbol{x}},\ t)=\bar{\boldsymbol{q}}^e(\boldsymbol{x},\ \dot{\boldsymbol{x}},\ t)+\bar{\boldsymbol{Q}}\boldsymbol{g}\\ \bar{\boldsymbol{Q}}=-\boldsymbol{\Phi}_x^T\end{gathered} \tag{13}$$

Holonomic system kinematics and constraints:

$$\bar{\boldsymbol{M}}\bar{\boldsymbol{J}}\ddot{\boldsymbol{y}}+\bar{\boldsymbol{q}}^c(\boldsymbol{y},\ \dot{\boldsymbol{y}},\ t)=\bar{\boldsymbol{q}}^e(\boldsymbol{y},\ \dot{\boldsymbol{y}},\ t)+\bar{\boldsymbol{Q}}\boldsymbol{g} \tag{14}$$

Nonholonomic system kinematics and constraints:

$$\bar{\boldsymbol{M}}\bar{\boldsymbol{L}}\dot{\boldsymbol{z}}+\bar{\boldsymbol{q}}^c(\boldsymbol{y},\ \boldsymbol{z},\ t)=\bar{\boldsymbol{q}}^e(\boldsymbol{y},\ \boldsymbol{z},\ t)+\bar{\boldsymbol{Q}}\boldsymbol{g} \tag{15}$$

On the left hand side of Eqs. (13) to (15) the inertia forces appear characterized by the inertia matrix $\bar{\boldsymbol{M}}$, the global Jacobian matrices $\bar{\boldsymbol{J}}$, $\bar{\boldsymbol{L}}$ and the vector $\bar{\boldsymbol{q}}^c$ of the Coriolis forces. On the right hand side the vector $\bar{\boldsymbol{q}}^e$ of the applied forces and the constraint forces composed by a global distribution matrix $\bar{\boldsymbol{Q}}$ and the vector of the generalized constraint forces $\boldsymbol{g}$ are found.

Each of the Eqs. (13) to (15) represents $6p$ scalar equations. However, the number of unknowns is different. In Eq. (13) there are $6p+q$ unknowns resulting from the vectors $\boldsymbol{x}$ and $\boldsymbol{g}$. In Eq. (14) the number of unknowns is exactly $6p=f+q$ by the vectors y and $\boldsymbol{g}$, while in Eq. (15) the number of unknowns is $12p-q$ due to the additional velocity vector $\boldsymbol{z}$ and an extended constraint vector $\boldsymbol{g}$. Obviously, the Newton-Euler equations have to be supplemented for the simulation of motion.

2.4 Equations of motion

The equations of motion are complete sets of equations to be solved by vibration analysis and/or numerical integration. There are two approaches used resulting in differential-algebraic equations (DAE) or ordinary differential equations (ODE), respectively.

For the DAE approach the implicit constraint equations (4) are differentiated twice and added to the Newton-Euler equations (13) resulting in

$$\begin{bmatrix}\bar{\boldsymbol{M}} & \boldsymbol{\Phi}_x^T\\ \boldsymbol{\Phi}_x & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{x}}\\ \boldsymbol{g}\end{bmatrix}=\begin{bmatrix}\boldsymbol{q}^e-\boldsymbol{q}^c\\ -\boldsymbol{\Phi}_t-\dot{\boldsymbol{\Phi}}_x\dot{\boldsymbol{x}}\end{bmatrix} \tag{16}$$

Eqs. (16) are numerically unstable due to a double zero eigenvalue originating from the differentiation of the constraints. During the last decade great progress was achieved in the stabilization of the solutions of Eqs. (16) well documented by Eich-Soellner and Fuhrer (1998).

The ODE approach is based on the elimination of the constraint forces using the orthogonality of generalized motions and constraints, $\bar{\boldsymbol{J}}^T\bar{\boldsymbol{Q}}=\boldsymbol{0}$, also known as D'Alembert's principle for holonomic systems. Then, it remains a minimal number of equations

$$\boldsymbol{M}(\boldsymbol{y},\ t)\ddot{\boldsymbol{y}}+\boldsymbol{k}(\boldsymbol{y},\ \dot{\boldsymbol{y}},\ t)=\boldsymbol{q}(\boldsymbol{y},\ \dot{\boldsymbol{y}},\ t) \tag{17}$$

The orthogonality may also be used for nonholonomic systems, $\bar{\boldsymbol{L}}^T\bar{\boldsymbol{Q}}=\boldsymbol{0}$, corresponding to Jourdain's principle and Kane's equations. However, the explicit form of the nonholonomic constsraints (9) has to be added,

$$\begin{aligned}\dot{\boldsymbol{y}}&=\dot{\boldsymbol{y}}(\boldsymbol{y},\ \boldsymbol{z},\ t)\\ \boldsymbol{M}(\boldsymbol{y},\ \boldsymbol{z},\ t)\dot{\boldsymbol{z}}+\boldsymbol{k}(\boldsymbol{y},\ \boldsymbol{z},\ t)&=\boldsymbol{q}(\boldsymbol{y},\ \boldsymbol{z},\ t)\end{aligned} \tag{18}$$

Eqs. (17) and (18) can now be solved by any standard time integration code.

The equations presented can also be extended to flexible bodies. For the analysis of small structural vibration a floating frame of reference is used while for large deformations the absolute nodal coordinate formulation turned out to be very efficient. For more details see, e.g., Shabana (1998, 2003).

3. Recursive Algorithms

For time integration of holonomic systems the mass matrix in Eqs. (16) or (17), respectively, has to be inverted what is numerically costly for systems with many degrees of freedom,

$$\ddot{y}(\boldsymbol{t})=M^{-1}(y,\ \boldsymbol{t})\left[q(y,\ \dot{y},\ \boldsymbol{t})-k(y,\ \dot{y},\ t)\right] \tag{19}$$

Recursive algorithms avoid this matrix inversion. The fundamental requirement, however, is a chain or tree topology of the multibody system as shown in Figure 2. Loop topologies are not included. Contributions on recursive algorithms are due, e.g., to Hollerbach (1980), Bae and Haug (1987), Brandl, Johanni and Otter (1988), Schiehlen (1991).

3.1 Kinematics

Recursive kinematics use the relative motion between two neighboring bodies and the related constraints as shown in Figure 3. The absolute translational and rotational velocity vector $\boldsymbol{w}_i$ of body i is related to the absolute velocity vector $\boldsymbol{w}_{i-1}$ of body $i-1$ and the generalized coordinates $\boldsymbol{y}_i$ of the joint i between this two bodies. It yields

$$\underbrace{\begin{bmatrix}\nu_{0i}\\ \omega_i\end{bmatrix}}_{\boldsymbol{w}_i}=\underbrace{S^{i,i-1}\begin{bmatrix}E & -\tilde{r}_{0i-1,0i}\\ 0 & E\end{bmatrix}}_{C_i}\underbrace{\begin{bmatrix}\nu_{0i-1}\\ \omega_{i-1}\end{bmatrix}}_{\boldsymbol{w}_{i-1}}+\underbrace{S^{i,i-1}\begin{bmatrix}J_{Ti}\\ J_{Ri}\end{bmatrix}}_{J_i}\dot{\boldsymbol{y}}_i \tag{20}$$

Using the fundamentals of relative motion of rigid bodies, it remains for the absolute acceleration

$$\boldsymbol{b}_i=\boldsymbol{C}_i\boldsymbol{b}_{i-1}+\boldsymbol{J}_i\ddot{\boldsymbol{y}}_i+\beta_i(\dot{\boldsymbol{y}}_i,\ \boldsymbol{w}_{i-1}) \tag{21}$$

where the vector $\boldsymbol{b}_i$ summarizes the translational and rotational accelerations of body i as well.

For the total system one gets for the absolute acceleration in matrix notation

$$\boldsymbol{b}=\boldsymbol{C}\boldsymbol{b}+\boldsymbol{J}\ddot{\boldsymbol{y}}+\beta \tag{22}$$

where the geometry matrix $\boldsymbol{C}$ is a lower block-sub-diagonal matrix and the Jacobian $\boldsymbol{J}$ is a block-diagonal matrix as follows

$$\boldsymbol{C}=\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ C_2 & 0 & 0 & \cdots & 0\\ 0 & C_3 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & 0 & C_p & 0\end{bmatrix},\ \boldsymbol{J}=\begin{bmatrix}J_1 & 0 & 0 & \cdots & 0\\ 0 & J_2 & 0 & \cdots & 0\\ 0 & 0 & J_3 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \cdots\\ 0 & 0 & 0 & 0 & J_p\end{bmatrix} \tag{23}$$

From Eq. (23) it follows the non-recursive form of the absolute accelerations as

$$\boldsymbol{b}=(\boldsymbol{E}-\boldsymbol{C})^{-1}\boldsymbol{J}\ddot{\boldsymbol{y}}+\bar{\beta} \tag{24}$$

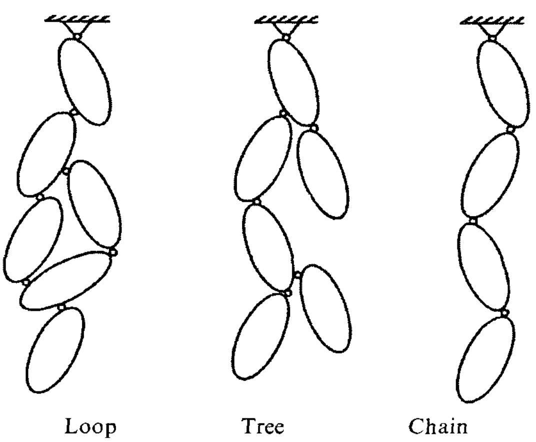


Fig. 2 Topology of multibody systems

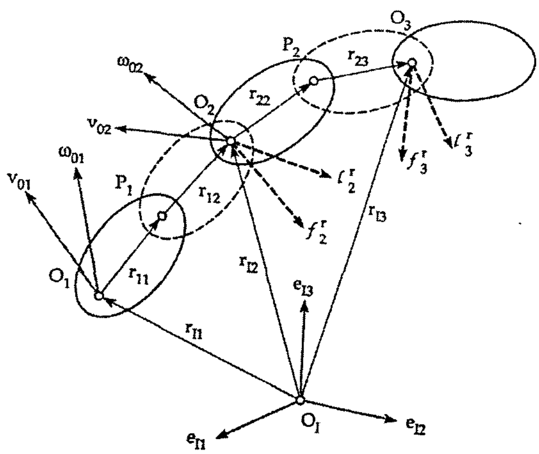


Fig. 3 Three-body system with two joints

where the global Jabobian matrix $\bar{J}$ is found again, see Section 2.2,

$$\bar{J}=(E-C)^{-1}J=\begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ C_2J_1 & J_2 & 0 & \cdots & 0 \\ C_3C_2J_1 & C_3J_2 & J_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & J_p \end{bmatrix} \quad (25)$$

Due to the chain topology the global Jacobian matrix is a lower triangular matrix.

3.2 Newton-Euler equations

Newton's and Euler's equations are now written for body i in its body fixed frame at the joint position O_i using the absolute accelerations and the external forces $\boldsymbol{q}_i$ acting on the body with holonomic constraints:

$$\underbrace{\begin{bmatrix} \boldsymbol{m}_i\boldsymbol{E} & \boldsymbol{m}_i\tilde{\boldsymbol{r}}_{OiCi}^T \\ \boldsymbol{m}_i\tilde{\boldsymbol{r}}_{OiCi} & \boldsymbol{I}_{Oi} \end{bmatrix}}_{\boldsymbol{M}_i=\boldsymbol{const}}\underbrace{\begin{bmatrix} \boldsymbol{a}_{Oi} \\ \alpha_i \end{bmatrix}}_{\boldsymbol{b}_i}+\underbrace{\begin{bmatrix} \boldsymbol{m}_i\tilde{\omega}_i\omega_i\boldsymbol{r}_{OiCi} \\ \tilde{\omega}_i\boldsymbol{I}_{Oi}\omega_i \end{bmatrix}}_{\boldsymbol{k}_i}=\underbrace{\begin{bmatrix} \boldsymbol{f}_i \\ \boldsymbol{l}_{Oi} \end{bmatrix}}_{\boldsymbol{q}_i} \quad (26)$$

Moreover, the external forces are composed of applied forces $\boldsymbol{q}_i^{(e)}$ and constraints forces $\boldsymbol{q}_i^{(r)}$ where the generalized constraint forces of the joint i and joint $i-1$ appear:

$$\begin{aligned} \boldsymbol{q}_i &= \boldsymbol{q}_i^{(e)}+\boldsymbol{q}_i^{(r)} \\ \boldsymbol{q}_i^{(r)} &= \boldsymbol{Q}_i\boldsymbol{g}_i-\boldsymbol{C}_{i+1}^T\boldsymbol{Q}_{i+1}\boldsymbol{g}_{i+1} \end{aligned} \quad (27)$$

3.3 Equations of motion

For the total system a set of 18 scalar equations remains from Eqs. (22), (26) and (27)

$$\boldsymbol{b}=\bar{\boldsymbol{J}}\dot{\boldsymbol{y}}+\bar{\boldsymbol{\beta}} \quad (28)$$

$$\bar{\boldsymbol{M}}\boldsymbol{b}+\bar{\boldsymbol{k}}=\boldsymbol{q}^{(e)}+\boldsymbol{q}^{(r)} \quad (29)$$

$$\boldsymbol{q}^{(r)}=(\boldsymbol{E}-\boldsymbol{C})^T\boldsymbol{Q}\boldsymbol{g}=\bar{\boldsymbol{Q}}\boldsymbol{g} \quad (30)$$

with 18 unknowns in the vectors $\boldsymbol{b}$, $\boldsymbol{y}$, $\boldsymbol{q}^{(r)}$, $\boldsymbol{g}$.

Now Eqs. (28) and (30) are inserted in Eq. (29) and the global orthogonality $\bar{\boldsymbol{J}}^T\bar{\boldsymbol{Q}}=\boldsymbol{0}$ is used again resulting in Eq. (17). The mass matrix is completely full, again, and the vector $\boldsymbol{k}$ depends not only on the generalized velocities but also on the absolute velocities,

$$M=\begin{bmatrix} J_1^t(M_1+C_2^t(M_2+C_3^tM_3C_3)C_2)J_1 & J_1^tC_2^t(M_2+C_3^tM_3C_3)J_2 & J_1^tC_2^tM_3J_3 \\ J_2^t(M_2+C_3^tM_3C_3)C_2J_1 & J_2^t(M_2+C_3^tM_3C_3)J_2 & J_2^tC_3^tM_3J_3 \\ J_3M_3C_3C_2J_1 & J_3^tM_3C_3J_2 & J_3^tM_3J_3 \end{bmatrix} \quad (31)$$

and

$$\boldsymbol{k}=\boldsymbol{k}(\boldsymbol{y},\ \dot{\boldsymbol{y}},\ \boldsymbol{w}) \quad (32)$$

However, the mass matrix shows now a characteristic structure which can be used for a Gauss transformation,

3.4 Recursion

There are three steps required to obtain the generalized accelerations.

(1) Forward recursion to get the absolute motion starting with $i=1$.

(2) Backward recursion using a Gauss transformation starting with $i=p$. As a result the system

$$\hat{\boldsymbol{M}}\ddot{\boldsymbol{y}}+\hat{\boldsymbol{k}}=\hat{\boldsymbol{q}} \quad (33)$$

is obtained where $\hat{\boldsymbol{M}}$ is a lower triangular matrix

$$\hat{\boldsymbol{M}}=\begin{bmatrix} J_1^t\widetilde{M}_1J_1 & 0 & 0 \\ J_2^t\widetilde{M}_2C_2J_1 & J_2^t\widetilde{M}_2J_2 & 0 \\ J_3^t\widetilde{M}_3C_3C_2J_1 & J_3^t\widetilde{M}_3C_3J_2 & J_3^t\widetilde{M}_3J_3 \end{bmatrix} \quad (34)$$

$$\widetilde{\boldsymbol{M}}_{i-1}=M_{i-1}+C_i^t(\widetilde{M}_i-\widetilde{M}_iJ_i(J_i^t\widetilde{M}_iJ_i)^{-1}J_i^t\widetilde{M}_i)C_i$$

the block elements of which follow from the recursion formula in Eq. (34).

(3) Forward recursion for the generalized accelearations starting with $i=1$.

The recursion requires some computational overhead. Therefore, the recursive algorithms are more efficient than the matrix inversion for more than $p=8-10$ bodies.

There are also some extensions of the recursive approach to loop topologies, see Bae and Haug (1987) and Saha and Schiehlen (2001).

4. Dynamic Analysis

The dynamical analysis of multibody systems is closely related to vibration theory. For engineering applications mechanical vibrations of holonomic, rheonomic systems are most important. The dynamical phenomena are classified according to the equations of nonlinear and linear motion.

Starting with Eqs. (17), nonlinear time-variant mechanical systems, even with $f=1$ degree of freedom, may show chaotic vibrations. For small motions Eqs. (17) can be linearized resulting in

$$M(t)\ddot{y}+P(t)\dot{y}+Q(t)y=h(t) \tag{35}$$

This system may feature parametrically excited vibrations due to the time-varying, often periodic matrices. In the case of time-invariant matrices with symmetric and skew-symmetric characteristics one gets

$$M\ddot{y}+(D+G)\dot{y}+(K+N)y=h(t) \tag{36}$$

a system which performs forced vibrations due to the external excitation on the right hand side. In the case of $h(t)=0$ only free vibrations remain. Furthermore, if the damping matrix D, the gyroscopic matrix G, and the circulatory matrix N are missing, a conservative system

$$M\ddot{y}+Ky=0 \tag{37}$$

with free undamped vibrations is found.

4.1 Linear vibration analysis

The special structure of Eqs. (36) and (37) simplifies the analysis. Marginal stability of Eqs. (37) is guaranteed if the stiffness matrix K is positive definitive. Free damped vibrations due to Eqs. (36) with $G=N=0$ are asymptotically stable if both, the stiffness matrix K is positive definite and the damping matrix D is positive definite or pervasively positive semidefinite, respectively, see Ref. [16]. Moreover, Eqs. (36) is asymptotically stable is all eigenvalues have a negative real part.

The general solution of Eqs. (37) reads as

$$y(t)=\Psi_1(t)y_0+\Psi_2(t)\dot{y}_0 \tag{38}$$

where the transition matrices $\Psi_1(t)$, $\Psi_2(t)$ are found from a real eigenvalue analysis of dimension f. The general solution of Eq. (36) is more easily written in the state space with the state vector $x(t)$ summarizing the system's state given by the generalized coordinates and their first time derivatives as

$$x(t)=\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \tag{39}$$

Then, the general solution reads simply

$$x(t)=\Phi(t)x_0 \tag{40}$$

where the state transition matrix $\Phi(t)$ follows from a complex eigenvalue problem of dimension $2f$.

Matrix methods for linear systems with harmonic excitation $h(t)$ lead to the concept of frequency response matrices while random excitation processes require spectral density matrices or covariance matrices, respectively. In the case of Eqs. (35) with periodically time-varying coefficients Floquet's theory allows closed form solutions, see Muller and Schiehlen (1985).

4.2 Nonlinear vibration analysis

Chaotic vibrations can be analyzed by time integration only resulting in a solution

$$y(t)=y(t\,;\,y_0,\,\dot{y}_0) \tag{41}$$

which is very sensitive to the initial conditions. Powerful characteristics of chaotic vibrations are the phase portrait, the power spectral density, the Ljapunov exponents and the dimensions. In addition to the chaotic vibrations periodic motions may also be found depending on the parameters of the system.

As an example some results of Bestle (1988) are presented here for the Duffing oscillator. Parameter *Set a* allows a periodic motion, often called a limit cycle, while *Set d* represents chaotic behaviour resulting in a strange attractor, Figure 4. The Ljapunov exponents for *Set a* are computed as $\sigma_1=0$, $\sigma_2=-0.10$, $\sigma_3=-0.10$ what means a periodic motion, for *Set d* on gets $\sigma_1=0.17$, $\sigma_2=0$, $\sigma_3=-0.37$. The positive Ljapunov exponent identifies a chaotic motion. The same behaviour is found from the dimension, *Set a* results in $D_L=1$, and for *Set d* one gets $D_L=2.46$.

A chaotic multibody system is represented by the chaos pendulum consisting of $p=3$ bodies with $f=3$ degrees of freedom, see Schiehlen (1999).

5. Vehicle Vibrations and Control

Vehicle dynamics is a major application field of multibody dynamics. The corresponding software tools have been highlighted by Kortum,

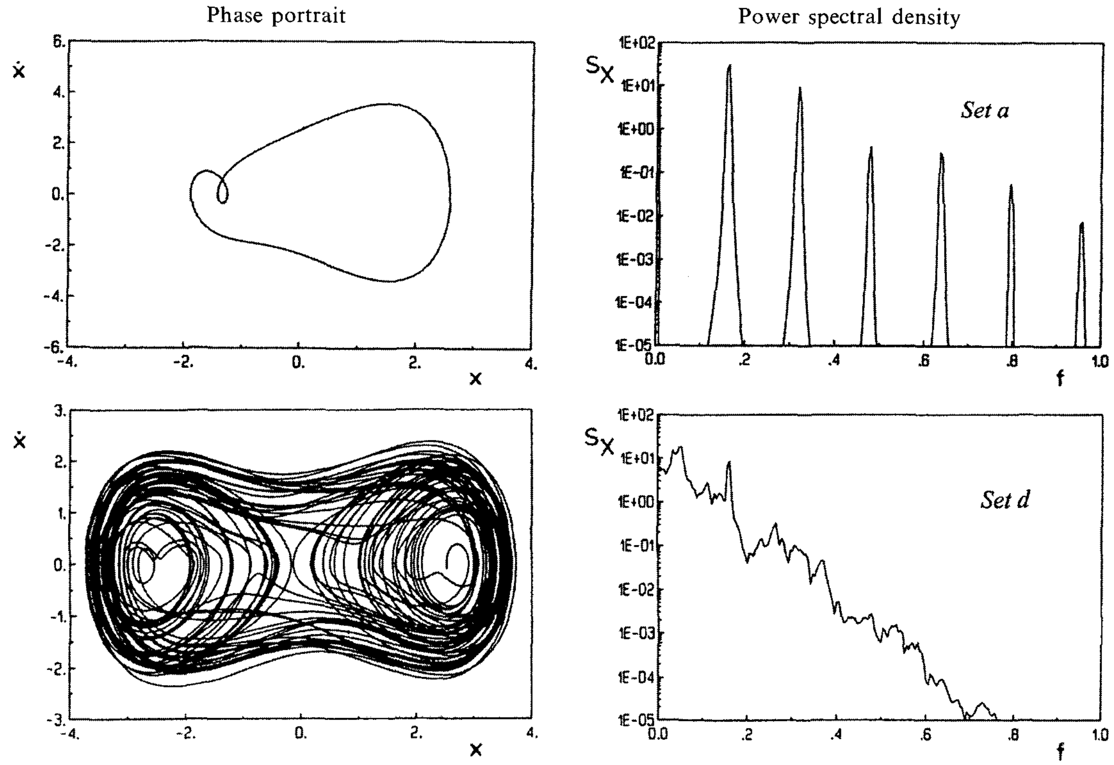


Fig. 4 Characteristics of a duffing oscillator (from Bestlerk (1988))

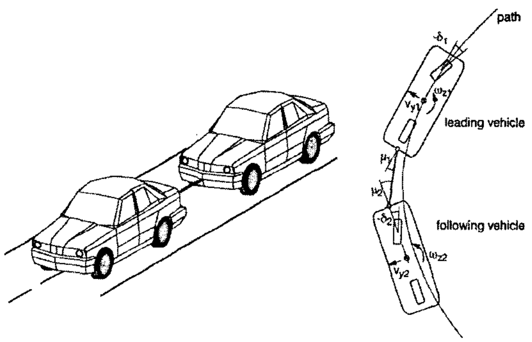


Fig. 5 Vehicle convoy as simulation and control design model

Arnold and Schiehlen (2001). These tools are most successfully used for detailed models representing the vehicle motion by simulation. For the control design such models are too complex, additional more simple models are helpful.

As an example the lateral dynamics of a vehicle convoy with the second vehicle following autonomously the leading vehicle is considered, Figure 5. The simulation model consists of $p=19$ bodies with $f=19$ degrees of freedom, McPherson front wheel strut, semi-trailing rear wheel suspension, Pacejka's magic formula tire model and driver models by Legouis and Donges. The control design model is restricted to a plane motion of lateral and yaw dynamics, the two tires of each axis are replaced by one tire in the middle of the axis (bicycle model), a linear tire model is used and the longitudinal velocity is constant. More details of the models, the corresponding equations and simulation results are available from Schiehlen and Petersen (1997).

6. Structural Vibrations and Contact

Structural vibrations occur often after collisions representing dynamical contact modelled as impacts between rigid and/or elastic bodies, respectively. Contact can be considered as a multiscale problem as shown in Schiehlen and Hu (2000). On the fast time scale the energy loss can be computed by an elastodynamic or finite element model, respectively. Then, from the momentum balance the coefficient of restitution is found and fed back to the multibody dynamics analysis. Using a linear motion of the two colliding bodies with masses m_1, m_2 it yields in the compression and the restitution phase

$$\begin{aligned} \Delta p_c &= m_1(v_1^- - v),\ \Delta p_c = m_2(v - v_2^-) \\ \Delta p_r &= m_1(v - v_1^+),\ \Delta p_r = m_2(v_2^+ - v) \end{aligned} \quad (42)$$

Poisson's law of momentum reads as

$$\Delta p = \Delta p_c + \Delta p_r = \Delta p_c(1+e) \quad (43)$$

From Eqs. (42) and (43) it follows the coefficient of restitution as

$$e = \frac{(m_1+m_2)\Delta p}{m_1 m_2(v_1^- - v_2^-)} - 1 \quad (44)$$

The coefficient of restitution depends on the shape of the bodies, their material and their relative velocity. Computational and experimental results are shown in Figure 6 for rods, plates, balls and beams made from aluminium.

The structural vibrations superimposed to the rigid body motion are shown in Figure 7. For more details see Schiehlen and Seifried (2003).

7. Mechanisms and Biped Walkers

Robots and manufacturing systems as well as walking devices are characterized by mechanisms with some or all mechanical degrees of freedom controlled resulting in prescribed motions or rheonomic constraints, respectively.

These motions are usually periodic vibrations and due to the control effort for accelerating and decelerating of the bodies a considerable amount of energy may be consumed. By using storage springs, the motion may be adjusted to the limit cycle of periodic nonlinear vibrations.

The first example is a robot arm with $f=2$ degrees of freedom and the task of a horizontal

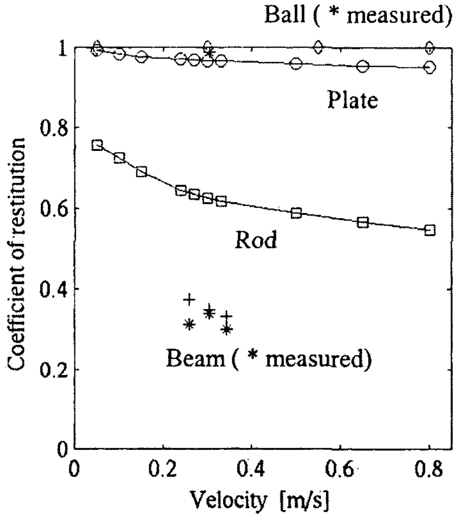


Fig. 6 Coefficient of restitution for bodies of different shape

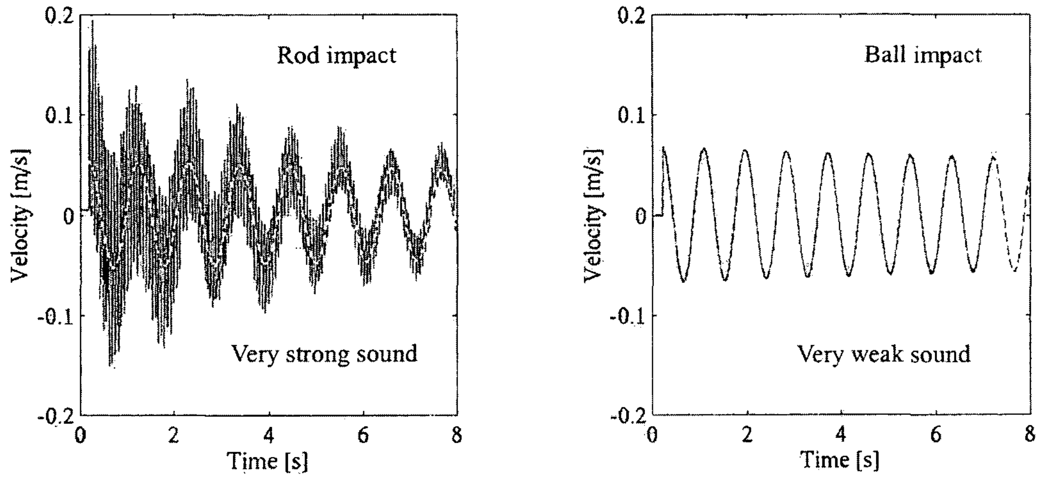


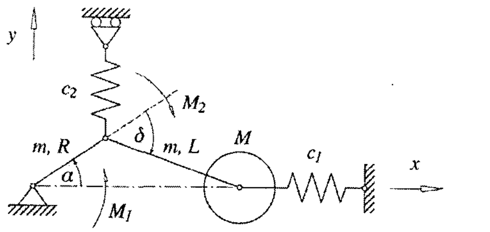
Fig. 7 Slow time scale simulation and related sound generation

motion featuring a limit cycle, Figure 8. The storage springs with stiffness c_1, c_2 support the motion in a natural way reducing the energy consumption as shown in Guse and Schiehlen (2002). Reduction of the energy consumption may reach more than 90%.

This principle can also be applied to walking machines. Passive walking devices are very efficient just powered by a small slope of the ground. In this case the potential energy is stored in the gravitational field by the vertical vibrations of the machine's centre of mass. The passive motion is then used as prescribed motion of a fully active walking machine, see Figure 9. The equation of motion of the active machine with $f=9$ degrees of freedom reads as

$$\boldsymbol{M}(\boldsymbol{y}, t)\ddot{\boldsymbol{y}}+\boldsymbol{k}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t) = \boldsymbol{q}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t)+\boldsymbol{W}(\boldsymbol{y}, t)\boldsymbol{g}+\boldsymbol{B}\boldsymbol{u} \tag{45}$$

where $\boldsymbol{W}(\boldsymbol{y}, t)\boldsymbol{g}$ represent the reaction forces due to the feet contact points and the locking knee, $\boldsymbol{B}$ is the control input matrix and $\boldsymbol{u}=[u_0\ u_1\ u_2\ u_3\ u_4\ u_5]^T$ means the control input vector. As shown in Gruber and Schiehlen (2002) the actively controlled biped model is as efficient as human walking what is superior to walking machine operating for comfort reasons without vertical vibrations of the centre of mass.



Rotating crank $x(t) = L + R\cos\omega t$, $y(t) = 0$

Oscillating crank $x(t) = a + b\cos\omega t$ between $\alpha=30°$ and $\alpha=150°$

$a = L\cos\left[\arcsin\left(-\frac{R}{2L}\right)\right]$, $y(t) = 0$

$b = R\frac{\sqrt{3}}{2}$,

Fig. 8 Robot arm with two prescribed horizontal motions

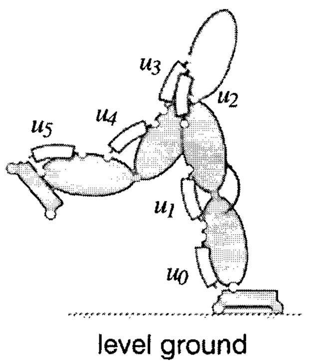


Fig. 9 Actively controlled biped walking machine

8. Conclusions

Multibody dynamics is an excellent foundation for multivariable vibration analysis and sophisticated control design. Recent research activities are devoted to large deformations in flexible multibody systems, to contact and impact problems requiring multi-time-scale modeling and all kinds of actively controlled mechanical systems often denoted as mechatronic systems.

References

Bae, D. S. and Haug, E. J., 1987, "A Recursive Formulation for Constrained Mechanical System Dynamics: Part I, Open Loop Systems," *Mechanics of Structures and Machines*, Vol. 15, pp. 359~382.

Bae, D. S. and Haug, E. J., 1987, "A Recursive Formulation for Constrained Mechanical System Dynamics: Part II, Closed Loop Systems," *Mechanics of Structures and Machines*, Vol. 15, pp. 481~506.

Bestle, D., 1988, *Beurteilungskriterien fur Chaotische Bewegungen Nichtlinearer Schwingungssysteme*, Dusseldorf, VDI-Verlag.

Brandl, H., Johanni, R. and Otter, M., 1988, "A Very Efficient Algorithm for the Simulation of Robots and Similar Multibody Systems Without Inversion of the Mass Matrix," *Theory of Robots*, Kopacek, P., Troch, I. and Desoyer, K.(eds.), Oxford, Pergamon, pp. 95~100.

D'Alembert, J., 1743, *Traitě de Dynamique*, Paris.

Eich-Soellner, E. and Fuhrer, C., 1998, *Numerical Methods in Multibody Dynamics*, Stuttgart, Teubner-Verlag.

Euler, L., 1776, "Nova methodi motum corporum rigidarum determinandi," *Novi Commentarii Academiae Scientiarum Petropolitanae*, Vol. 20, pp. 208～233. See also Euler, L., *Opera Omnia*, Series II, Vol. 9, pp. 99～125.

Grammel, R., 1920, *Der Kreisel — Seine Theorie und seine Anwendungen*, First ed., Braunschweig, Vieweg.

Gruber, S. and Schiehlen, W., 2002, "Biped Walking Machines — A Challenge to Dynamics and Mechatronics," *Proc. 5th World Congr. Comp. Mech.*, Mang, H. A., Rammerstorfer, F. G. and Eberhardsteiner, J. (eds.), Vienna, Vienna University of Technology, Paper ID 81426, 11 pages (http://wccm.tuwien.ac.at).

Guse, N. and Schiehlen, W., 2002, "Efficient inverse dynamics control of multibody systems," *Proc. 6th Int. Conf. Motion Vibration Control*, Mizumo, T. and Suda, Y. (eds.), Tokyo, Japan Mech. Society, Vol. 1, pp. 502～507,

Hollerbach, J. M., 1980, "A Recursive Lagrangian Formulation of Manipulator Dynamics and Comparative Study of Dynamics Formulation Complexity," *IEEE Trans. Syst. Man. Cybern.*, Vol. 11, pp. 730～736.

Huston, R. L. and Passerello, C. E., 1971, "On the Dynamics of a Human Body Model," *Journal on Biomechanics*, Vol. 4, pp. 369～378.

Jourdain, P. E. B., 1909, "Note on an Analogue at Gauss' Principle of Least Constraint," *Quarterly Journal on Pure Applied Mathematics*, Vol. 40, pp. 153～197.

Kane, T. R. and Levinson, D. A., 1985, *Dynamics : Theory and Applications*, New York, McGraw-Hill.

Kortum, W., Schiehlen, W. and Arnold, M., 2001, "Software Tools : From Multibody System Analysis to Vehicle System Dynamics'," *Mechanics for a New Millenium*, Aref, H. and Phillips, J. W. (eds.), Dordrecht, Kluwer, pp. 225～238.

Lagrange, J. -L., 1788, *Mecanique Analytique*, Paris, L'Academie Royal des Sciences.

Magnus, K. (ed.), 1978, *Dynamics of Multibody Systems*, Berlin, Springer-Verlag.

Muller, P. C. and Schiehlen, W., 1985, *Linear Vibrations*, Dordrecht, Martinus Nijhoff Publ..

Newton, I., 1687, *Philosophiae Naturalis Principia Mathematica*, London, Royal Society.

Roberson, R. E. and Wittenburg, J., 1967, "A Dynamical Formalism for an Arbitrary Number of Interconnected Rigid Bodies, with Reference to the Problem of Satellite Attitude Control," *Proceedings 3rd Congr. Int. Fed. Autom. Control*, London, Butterworth, Vol. 1, Book 3, Paper 46 D.

Saha, S. K. and Schiehlen, W., 2001, "Recursive Kinematics and Dynamics for Parallel Structural Closed-Loop Multibody Systems," *Mechanics of Structures and Machines*, Vol. 29, pp. 143～175.

Schiehlen, W., 1991, *Computational Aspects in Multibody System Dynamics," Comp. Meth. Appl. Mech. Eng.*, Vol. 90, pp. 569～582.

Schiehlen, W., 1997, *Multibody System Dynamics : Roots and Perspectives*, Multibody System Dynamics Vol. 1, pp. 149～188.

Schiehlen, W. and Petersen, U., 1997, "Control Concepts for Lateral Motion of Road Vehicles in Convoy," *Interaction between Dynamics and Control in Advanced Mechanical Systems*, Van Campen, D. -H. (ed.), Dordrecht, Kluwer, pp. 345～354.

Schiehlen, W., 1999, "Control of Chaos for Pendulum Systems," *New Applications of Nonlinear and Chaotic Dynamics in Mechanics*, Moon, F. C. (ed.), Dordrecht, Kluwer, pp. 363～370.

Schiehlen, W. and Hu, B., 2000, "Contact Problems in MultiBody Dynamics," *Multibody Dynamics : Monitoring and Simulation Techniques II*, Rahnejat, H., Ebrahimi, M. and Whalley, R. (eds.), London, Prof. Eng. Publ., pp. 3～14.

Schiehlen, W. and Seifried, R., 2003, "Multiscale Impact Models : Multibody Dynamics and Wave Propagation," *Nonlinear Stochastic Dynamics*, Namachchivaya, N. Sri and Lin, Y. K. (eds.), Dordrecht, Kluwer, pp. 353～362.

Shabana, A. A., 1998, *Dynamics of Multibody Systems*, Second ed., Cambridge, University Press, Cambridge.

Shabana, A. A. (ed.), 2003, "Special Issue on Flexible Multi-body Dynamics," *Nonlinear Dynamics*, Vol. 34, pp. 1～232.

Wittenbauer, F., 1923, *Graphische Dynamik*, Berlin, Springer-Verlag.