Solving Dynamic Equation Using Combination of Both Trigonometric and Hyperbolic Cosine Functions for Approximating Acceleration

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This paper introduces a numerical method for integration of the linear and nonlinear differential dynamic equation of motion. The variation of acceleration in two time steps is approximated as a combination of both trigonometric cosine and hyperbolic cosine functions with weighted coefficient. From which all necessary formulae are elaborated for the direct integration of the governing equation. A number of linear and nonlinear dynamic problems with various degrees of freedom are analysed using both the suggested method and Newmark method for the comparison. The numerical results show high advantages and effectiveness of the new method.

Key Words : Numerical Method, Approximation, Time Step, Accuracy

1. Introduction

At the present, there are a number of numerical methods for solving dynamic problems in the time domain (Clough and Penzien, 1993 ; Daniel, 1997 ; Fung, 1998 ; Hahn, 1991 ; Newmark, 1959), but the Newmark method is most widely used (Daniel, 1997; Fung, 1998). This method assumes acceleration by linear function or constant in each time step (Newmark, 1959). Recently, two numerical methods are introduced in 1999, 2000 by the authors of this article (Quoc and Phuoc, 2000 ; Quoc and Hai, 1999), in which interpolate acceleration by trigonometric cosine and hyperbolic cosine functions in two time steps. In these me-thods, the numerical results given good accuracy (Quoc and Phuoc, 2000 ; 2002), and the stability was conditionally stable (Phuoc and Quoc, 2003).

The objective of this article is to develop the numerical method of assuming acceleration by nonlinear functions for solving general dynamic equations approximating acceleration by the combination of both trigonometric cosine and hyperbolic cosine functions with weighted coefficient. The numerical examples including single and multi degrees of freedom systems are studied with various weighted coeffici-ent.

2. Formulation

The governing equation of motion of the structures under dynamic loads is given as

$$\begin{aligned} M\ddot{v}(t) + C(\dot{v}(t), v(t))\,\dot{v}(t) \\ + K(\dot{v}(t), v(t))\,v(t) = P(t) \end{aligned} \quad (1)$$

where the matricesand M, $C(\dot{v}(t), v(t))$ are $K(\dot{v}(t), v(t))$ mass, damping, and stiffness matrices of the structure, respectively ; the vectors $\ddot{v}(t)$, $\dot{v}(t)$ and $v(t)$ are acceleration, velocity,

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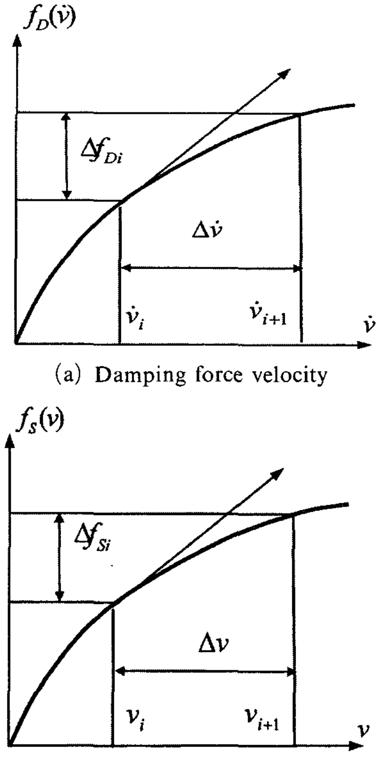

(a) Damping force velocity

(b) Elastic force displacement

Fig. 1 Determination of nonlinear influence coefficients of damping and stiffness matrices

and displacement vectors, respectively ; and $P(t)$ is applied load vector.

In the case nonlinear dynamic problems, the components of the damping and stiffness matrices C_{ij} ; K_{ij} are determined for each time step based on the given relations as in Fig. 1. It is convenient to use the initial tangent (Clough and Penzien, 1993 ; Quoc and Phuoc, 2000) as follows

$$\begin{aligned} C_{ij}(t+\Delta t) &\cong \left(\frac{df_{Di}}{d\dot{v}_j}\right)_t \equiv C_{ij}(t) \\ K_{ij}(t+\Delta t) &\cong \left(\frac{df_{Si}}{dv_j}\right)_t \equiv K_{ij}(t) \end{aligned} \tag{2}$$

The acceleration vectors at the moments $t-\Delta t$, t, $t+\Delta t$ are $\ddot{v}_{i-1}$, $\ddot{v}_i$, $\ddot{v}_{i+1}$, respectively ; the velocity vectors at the moments t, $t+\Delta t$ are $\dot{v}_i$, $\dot{v}_{i+1}$, respectively ; the displacement vectors at the moments t, $t+\Delta t$ are v_i, v_{i+1} respectively.

The acceleration function in two time steps is assumed by the combination of both trigonometric and hyperbolic cosine functions as shown in Fig. 2, expressed as

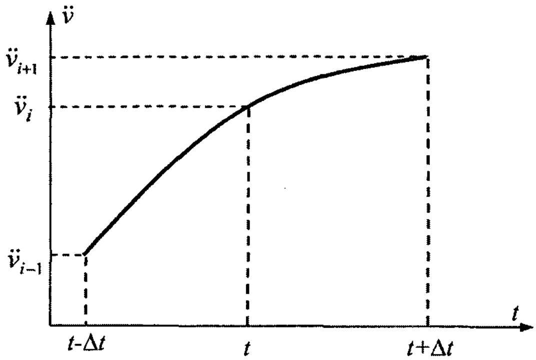

Fig. 2 Approximation function of acceleration

$$\begin{aligned} \ddot{v}(t+\tau) &= \ddot{v}_i - \beta + \alpha\frac{\tau}{\Delta t} \\ &+\beta\left[r\cos\left(\frac{\pi\tau}{2\Delta t}\right) + (1-r)\cosh\left(\frac{\tau}{\Delta t}\right)\right] \end{aligned} \tag{3}$$

where r is weighted coefficient of trigonometric and hyperbolic cosine functions.

The values of acceleration at the moments $\tau=-\Delta t$ and $\tau\Delta t$ are determined by

$$\begin{aligned} \ddot{v}(t-\Delta t) &= \ddot{v}_{i-1} = \ddot{v}_i - \beta - \alpha + \beta(1-r)\cos(1) \\ \ddot{v}(t+\Delta t) &= \ddot{v}_{i+1} = \ddot{v}_i - \beta + \alpha + \beta(1-r)\cosh(1) \end{aligned} \tag{4}$$

From eq. (4), the values of β and α are determined as follows

$$\begin{aligned} \beta &= \frac{\ddot{v}_{i+1} + \ddot{v}_{i-1} - 2\ddot{v}_i}{2\cosh(1) - 2r\cosh(1) - 2} \\ \alpha &= \frac{\ddot{v}_{i+1} - \ddot{v}_{i-1}}{2} \end{aligned} \tag{5}$$

When β, α are substituted into eq. (3), the acceleration equation (3) becomes

$$\begin{aligned} \ddot{v}(t+\tau) &= -\frac{\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i}{2\cosh(1)-2r\cosh(1)-2} + \ddot{v}_i + \frac{\ddot{v}_{i+1}-\ddot{v}_{i-1}}{2}\frac{\tau}{\Delta t} \\ &+ \frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)\,r}{2\cosh(1)-2r\cosh(1)-2}\cos\left(\frac{\pi\tau}{2\Delta t}\right) \\ &+ \frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)(1-r)}{2\cosh(1)-2r\cosh(1)-2}\cosh\left(\frac{\tau}{2\Delta t}\right) \end{aligned} \tag{6}$$

Integration of eq. (6) results in the velocity equation can be written as

$$\dot{v}(t+\tau)=-\frac{\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i}{2\cosh(1)-2r\cosh(1)-2}+\dot{v}_i\ddot{v}_i\tau+\frac{\ddot{v}_{i+1}-\ddot{v}_{i-1}}{2}\frac{\tau^2}{2\Delta t}+\frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)2r}{2\cosh(1)-2r\cosh(1)-2}\frac{\Delta t}{\pi}\sin\left(\frac{\pi\tau}{2\Delta t}\right)+\frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)(1-r)}{2\cosh(1)-2r\cosh(1)-2}\Delta t\sinh\left(\frac{\tau}{\Delta t}\right) \quad (7)$$

Similarly, the displacement equation could be obtained as

$$v(t+\tau)=v_i+\dot{v}_i\tau+\ddot{v}_i\frac{\tau^2}{2}+\frac{\ddot{v}_{i+1}-\ddot{v}_{i-1}}{2}\frac{\tau^3}{6\Delta t}-\frac{\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i}{2\cosh(1)-2r\cosh(1)-2}\frac{\tau^2}{2}-\frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)\left(\cos\frac{\pi\tau}{2\Delta t}-1\right)}{2\cosh(1)-2r\cosh(1)-2}\frac{r4\Delta t^2}{\pi^2}+\frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)\left(\cosh\frac{\tau}{\Delta t}-1\right)}{2\cosh(1)-2r\cosh(1)-2}(1-r)\Delta t \quad (8)$$

At the time $\tau=\Delta t$ the velocity and displacement vectors at the end time step are given as

$$\dot{v}_{i+1}=\dot{v}_i+\ddot{v}_i\Delta t+\frac{\ddot{v}_{i+1}-\ddot{v}_{i-1}}{4}\Delta t-\frac{\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i}{2\cosh(1)-2r\cosh(1)-2}\Delta t+\frac{\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i}{2\cosh(1)-2r\cosh(1)-2}r\frac{2\Delta t}{\pi}+\frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)\sinh(1)}{2\cosh(1)-2r\cosh(1)-2}(1-r)\Delta t \quad (9)$$

$$v_{i+1}=v_i+\dot{v}_i\Delta t+\ddot{v}_i\frac{\Delta t^2}{2}+\frac{\ddot{v}_{i+1}-\ddot{v}_{i-1}}{12}\Delta t^2-\frac{\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i}{2\cosh(1)-2r\cosh(1)-2}\frac{\Delta t^2}{2}+\frac{\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i}{2\cosh(1)-2r\cosh(1)-2}r\frac{4\Delta t}{\pi^2}+\frac{(\ddot{v}_{i+1}+\ddot{v}_{i-1}-2\ddot{v}_i)(\cosh(1)-1)(1-r)}{2\cosh(1)-2r\cosh(1)-2}\Delta t^2 \quad (10)$$

Substituting eqs. (9), (10) into eq. (1), the expression of the unknown $\ddot{v}_{i+1}$ can be obtained. Then, the velocity and displacement vectors at the end of time step are determined by eqs. (9) and (10), respectively. The step by step analysed by eqs. (9), (10), (1) may be repeated to compute the response for subsequent discrete times.

There are a number of criteria evaluation of the accuracy of numerical methods in structural dynamics (Quoc and Phuoc, 2002). In this article, the accuracy of the Newmark method and suggested method are evaluated based on the displacement criteria, with evaluation parameter percent amplitude error e is defined as follows

$$e=\left|\frac{A-A_{ex}}{A_{ex}}100\%\right| \quad (11)$$

in which A and A_{ex} are maximum of the response due to approximation and exact solutions, respectively.

3. Numerical Examples

In the following part a number numerical examples from linear to nonlinear with single or multi degrees of freedom systems are considered to clarified the effectiveness of the proposed method.

3.1 Example 1

Consider a single degree of freedom system under the action of a periodic load. The governing equation of motion is given as follows

$$\ddot{v}(t)+2\dot{v}(t)+8v(t)=100\sin 4t \quad (12)$$

with initial conditions at time $t=0$ are

$$v(0)=0,\ \dot{v}(0)=0$$

The period of external load is $T_p=1.57$s

The exact solution of eq. (12) is

$$v(t)=5e^{-2t}\cos 2t+10e^{-2t}\sin 2t-2.5\sin 4t-5\cos 4t \quad (13)$$

The numerical results are analysed by numerical methods including the Newmark method (linear acceleration method) and suggested method with the time step of 0.2s. The solutions are shown in Fig. 3, and the percent amplitude error e with various weighted coefficient is given in Fig. 4

It is seen that the accuracy of the suggested method is better than that of the Newmark method. From Fig. 4 shows that with $r=0.29$ then the percent amplitude error e will be reduced to zeros, and $r=0.3$ then $e=0.35\%$.

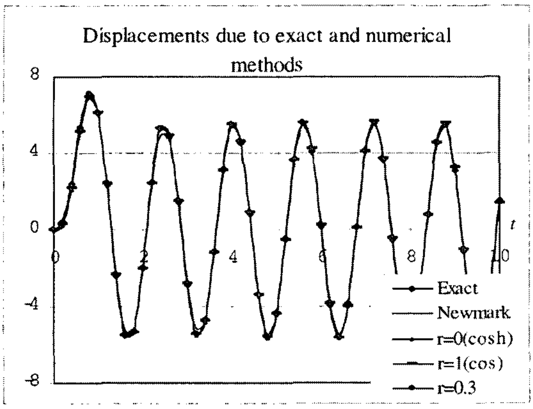

Fig. 3 Displacements with times step of 0.2s

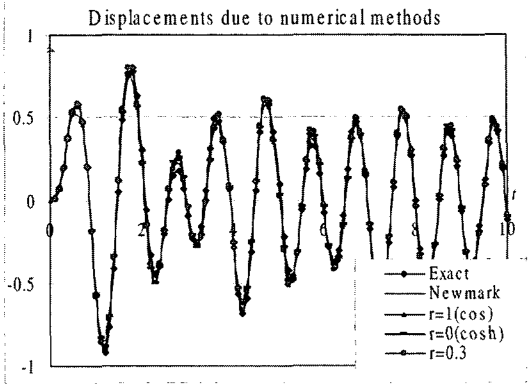

Fig. 5 Displacements with times step of 0.1s

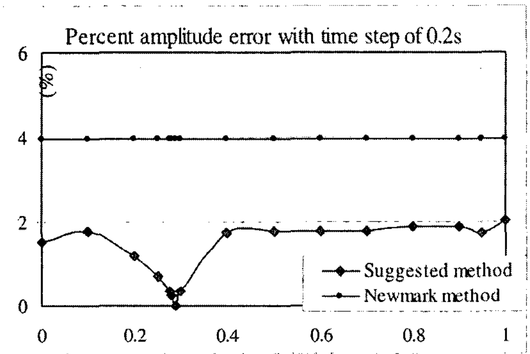

Fig. 4 The error with various weighted coefficients

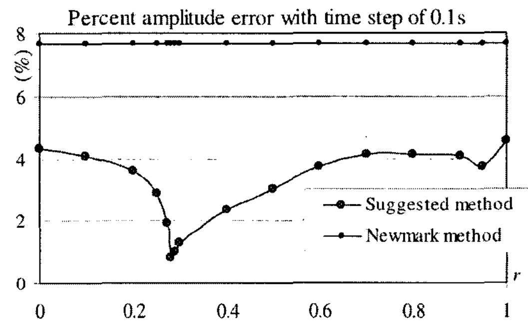

Fig. 6 The error with various weighted coefficients

3.2 Example 2

Consider a nonlinear single degree of freedom system under the action of a periodic load. The governing equation of motion is given as follows

$$\ddot{v}(t)+0.5\dot{v}(t)+20(1-0.3v^2(t))v(t) = 10\sin 2\pi t \tag{15}$$

with initial conditions at time $t=0$ are expressed as $v(0)=0$, $\dot{v}(0)=0$ The period of load is $T_p=$ 1s.

The exact solution is solved by numerical method with very small time step $\Delta t = T_p/100=0.01$s, and the results are converged. With time step of 0.1s, the responses are given in Fig. 5, and the value e with various r shows in Fig. 6.

From Figs. 5 and 6, it can be seen the accuracy of the suggested method is better than that of the Newmark method. With $r=0.28$ then $e=0.81\%$ and $r=0.3$ then $e=1.29\%$.

3.3 Example 3

The analysis of response of the three stories frame structure shown in Fig. 7 is modeled by three degree of freedom systems. The mass and stiffness of frame is expressed as Fig. 7, the damping ratios of first mode, second mode, and third mode are 5%, 4.34%, and 5%, respectively. The equation of motion can be expressed as follows

$$M\ddot{v}(t)+C\dot{v}(t)+Kv(t)=P(t) \tag{16}$$

with initial conditions at time $t=0$ are given as

$$\{v(0)\}=\{0\}$$
$$\{\dot{v}(0)\}=\{0\}$$

The mass, damping, and stiffness matrices, and load vector are determined as follows

$$M=\begin{bmatrix}1 & 0 & 0\\ 0 & 1.5 & 0\\ 0 & 0 & 2\end{bmatrix};\ C=\begin{bmatrix}2.094 & -0.99 & 0\\ -0.99 & 4.626 & -1.98\\ 0 & -.98 & 7.157\end{bmatrix}$$

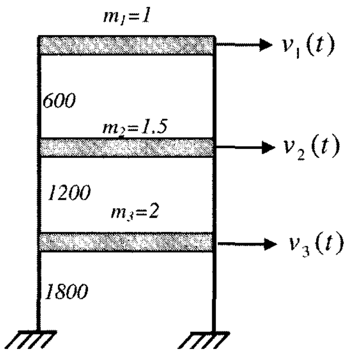

Fig. 7 Structural system with 3DOFS

$$K=600\begin{bmatrix}1 & -1 & 0\\ -1 & 3 & -2\\ 0 & -2 & 5\end{bmatrix};\ P(t)=\begin{Bmatrix}500\\ 1000\\ 1000\end{Bmatrix}\cos(50t)$$

The exact solution of the response of the frame is found by mode displacement superposition method (Clough and Penzien, 1993). This method can be expressed as follows:

The circular frequencies and natural periods undamped of the frame structure are

$$\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_-\end{Bmatrix}=\begin{Bmatrix}14.522\\ 31.048\\ 46.100\end{Bmatrix}\text{rad/s};\ \begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix}=\begin{Bmatrix}0.4327\\ 0.2024\\ 0.1363\end{Bmatrix}\text{s}$$

The orthonormalized mode shape matrix is

$$\hat{\phi}=\begin{bmatrix}0.74265 & 0.63577 & 0.21037\\ 0.48164 & -0.38566 & -0.53475\\ 0.22417 & -0.43168 & 0.51323\end{bmatrix}$$

The displacement vector is obtained by summing the modal vectors as

$$v(t)=\hat{\phi}u(t) \tag{17}$$

where $u(t)=\{u_1(t),\ u_2(t),\ u_3(t)\}^T$ is defined as generalized coordinates vector.

The uncouple equations of motion may be expressed as

$$\ddot{u}_k(t)+2\xi_k\omega_k\dot{u}_k(t)+\omega_k^2u_k(t)=p_k(t)\quad k=1,\ 2,\cdot\ 3 \tag{18}$$

The differential equations of $u_k(t)$ can be established by substraction ξ_k and ω_k into eq.(18) as follows

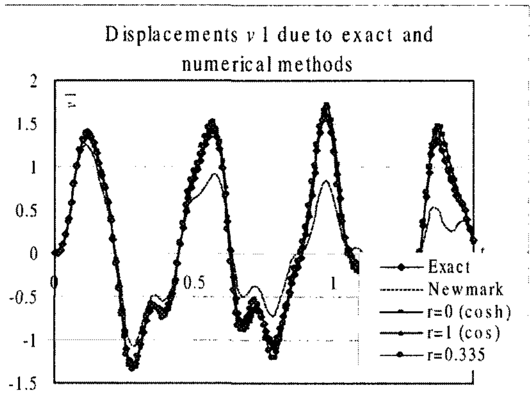

Fig. 8 Displacements with times step of 0.01s

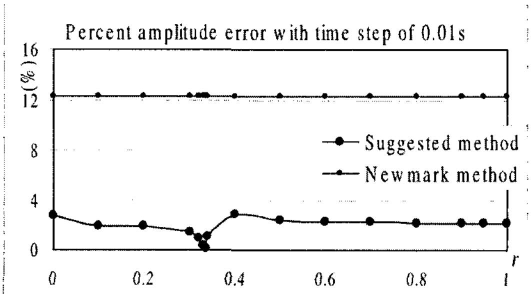

Fig. 9 The error with various weighted coefficients

$$\begin{aligned}&\ddot{u}_1(t)+1.452\dot{u}_1(t)+210.89u_1(t)=1077\cos(50t)\\ &\ddot{u}_2(t)+2.695\dot{u}_2(t)+963.98u_2(t)=-499.46\cos(50t)\\ &\ddot{u}_3(t)+4.610\dot{u}_3(t)+2125.21u_3(t)=83.67\cos(50t)\end{aligned} \tag{19}$$

With initial conditions $u_k(0)$ and $\dot{u}_k(0)$ the solution of the generalized coordinates can be found from eq. (19). Consequently, the exact solution of the problem is calculated from eq. (17).

The approximation solutions are analysed by numerical methods including the Newmark method and suggested method with the time step of 0.01s. The displacement v_1 compared with solution of mode superposition method (exact) are shown in Fig. 8, and the percent amplitude error e with various weighted coefficient is given in Fig. 9.

The accuracy of the suggested method is better than that of the Newmark method. From the Fig. 9, it can be seen that with $r=0.33$ then $e=0.39\%$ and $r=0.3$ then $e=1.38\%$.

4. Conclusions

From of this study, the following conclusions can be drawn :

(1) A numerical method for solving structural dynamic problems using combination of both trigonometric cosine and hyperbolic cosine functions with weighted coefficient for acceleration interpolation in two time steps has been presented.

(2) The numerical examples show that the accuracy of the suggested method with any value of the weighted coefficient r is better than that of the Newmark method.

(3) The best weighted coefficient r of the suggested method in the numerical examples ranges in area $0.28 \leq r \leq 0.34$ corresponding to the minimum error. With $r=0.3$ in all examples, the error is very near to the minimum error, so the value of $r=0.3$ may be used in application.

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