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Entropy and information energy arithmetic operations for fuzzy numbers

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Abstract

There have been several tipical methods being used tomeasure the fuzziness (entropy) of fuzzy sets. Pedrycz is the original motivation of this paper. Recently, Wang and Chiu [FSS103(1999) 443-455] and Pedrycz [FSS 64(1994) 21-30] showed the relationship(addition, subtraction, multiplication) between the entropies of the resultant fuzzy number and the original fuzzy numbers of same type. In this paper, using Lebesgue-Stieltjes integral, we generalize results of Wang and Chiu [FSS 103(1999) 443-455] concerning entropy arithmetic operations without the condition of same types of fuzzy numbers. And using this results and trade-off relationship between information energy and entropy, we study more properties of information energy of fuzzy numbers.

Key words : Entropy; Fuzzy numbers; Measure of fuzzinesss; Arithmetic operations.

1. Introduction

The entropy of a fuzzy set is a measure of fuzziness of the fuzzy set. Since Shannon and Weaver[11] used of entropy in information theory in 1964, there have been several typical methods used to measure the fuzziness of fuzzy sets in possibility theory (see De Luca and Termini[2], Kaufmann[7], Yager[17]). And there are lots of literature talking about the entropies of fuzzy sets (see [1, 3, 4, 6, 9, 16]). One rather interesting application of the entropy measure for a fuzzy set was recently introduced by Pedrycz[10]. Pedrycz showed the entropy of triangular fuzzy number change when the interval size of the support is changed. Wang and Chiu[13] extended the result of Pedrycz[10] to the other types of fuzzy sets. Wang and Chiu also considered the relationship between the entropies of the resultant three common types of fuzzy numbers and the original three common types of fuzzy numbers through arithmetic operations including addition, subtraction and multiplication. For the triangular fuzzy numbers, these results has been proposed by Wang and Chiu[12]. Energy of a fuzzy set was first proposed by De Luca and Termini[2], and it stands for the scale cardinality of the fuzzy set. The information energy[3] can be seen as the information to be obtained from a fuzzy set. It seems that information energy and entropy have a trade-off relationship. Dumitrescu[5] and Yu[18] defined the information energy which can be seen as the information to be obtained from a fuzzy set. Wang and Chiu[14] studied some properties of information energy and considered the relationship between the information energy and entropy of a fuzzy set. Hong and Kim[6] introduced a simple new method on calculating the entropy of the image fuzzy sets gotten by the extension principle without calculating its membership function. In this paper, we will have far-reaching generalizations of the results of Pedrycz[9] and Wang and Chiu[12. 13]. Indeed, we will show the relationship between the entropies of the resultant fuzzy number and the original fuzzy numbers without any restriction of types of fuzzy numbers. And using this results and trade-off relationship between information energy and entropy, we study more properties of information energy of fuzzy numbers.

2. Definitions

Let X be a bounded real interval. Consider a fuzzy set and its associated membership function $A:X\to[0,1]$. As studied in [2], the entropy of A determined at a fixed element x is defined as

$$H(A(x))=h(A(x)) \tag{2.1}$$

where $h:[0,1]\to[0,1]$ is monotonically increasing in $[0, \frac{1}{2}]$ and monotonically decreasing on the other half of the unit interval; moreover, $h(u)=0$, as $u=0$ and 1; and $h(\frac{1}{2})=1$. Typical examples of $h(x)$ include

$$h(u)=u\log_2 u, \tag{2.2}$$

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$$h(u)=\begin{cases}2u & \text{if } u\in[0,\frac{1}{2}]\\ 2(1-u) & \text{if } u\in[\frac{1}{2},1];\end{cases} \quad (2.3)$$

$$h(u)=4u(1-u), \quad (2.4)$$

and

$$h(u)=-u\ln u-(1-u)\ln(1-u) \quad (2.5)$$

where (2.5) is called Shannon's function [11]. To determine a global entropy $H(A)$ of the fuzzy set A independent of x, the above

expression has to be aggregated over the entire universe of discourse X as follows[10]:

$$H(A)=\int_X h(A(x))p(x)dx,$$

where $p(x)$ is the probability density function of the available data in X. It is known that the larger $H(A)$ is, the more is the fuzziness of the fuzzy set A. The information energy of A at a fixed element x is defined by [18] as

$$e(A(x))=2(A^2(x)+(A^c)^2(x))-1 \quad (2.6)$$

where A^c is the standard complement of A, i.e., $A^c(x)=1-A(x)$. To determine the global information energy measure of a fuzzy set A independent of x, we can integrate over the universe of discourse X as follows:

$$E(A)=\int_X e(A(x))p(x)dx.$$

For the membership function of the fuzzy set $A(x)$, A^α denotes the α-cut of A, i.e. $A^\alpha=\{x|A(x)\geq\alpha, x\in X\}$; $A^{\alpha'}$ is the strong α-cut of A, i.e., $A^{\alpha'}=\{x|A(x)>\alpha, x\in X\}$. If $\alpha=0$, $A^{0'}$ is called the "support" of A.

Definition 1. Suppose we have any two fuzzy sets A_1 and A_2 with the support $A_1^{0'}=(a_1,b_1)\subset X$ and $A_2^{0'}=(a_2,b_2)\subset X$, respectively. If

$$A_1(\tilde{x})=A_2(\tilde{x}) \quad (2.7)$$

where

$$\tilde{x}=a_1+c_1,\ \tilde{x}=a_2+\left(\frac{c_1}{b_1-a_1}\right)(b_2-a_2),$$
$$\tilde{x}\in[a_1,b_1] \text{ and } \tilde{x}\in[a_2,b_2];$$

or

$$A_1(\tilde{x})=A_2(\tilde{x}), \quad (2.8)$$

where $\tilde{x}=a_1+c_1$, $\tilde{x}=b_2-\left(\frac{c_1}{b_1-a_1}\right)(b_2-a_2)$, $\tilde{x}\in[a_1,b_1]$ and $\tilde{x}\in[a_2,b_2]$. Then we call A_1 and A_2 to be the "same type of fuzzy sets".

It is also noted that if $b_2=-a_1$, $a_2=-b_1$ then let A_2 be called the "image" of A_1 and be denoted by A_1^-, where A_1 and A_2 also satisfy (2.7).

Let us recall the notations of arithmetic operations on fuzzy number. A fuzzy set A is called a "fuzzy number" if it is convex and normal. Suppose A, $B$are two fuzzy numbers; we have [8]

$$(A\otimes B)^\alpha=A^\alpha\otimes B^\alpha,\ \alpha\in(0,1]; \quad (2.9)$$

moreover

$${}_\alpha A(x)=\begin{cases}\alpha & x\in A^\alpha,\\ 0 & x\notin A^\alpha.\end{cases}$$

Then by the decomposition theorem [8], we have

$$A\otimes B=\bigcup_{\alpha\in[0,1]}{}_\alpha(A\otimes B), \quad (2.10)$$

where $\otimes$ denotes any arithmetic operation which may be addition, subtraction, multiplication, and division. $\bigcup$ denotes the standard fuzzy union. We consider here three basic arithmetic operations: addition(i.e. $A+B$), subtraction(i.e. $A-B$), and a simple multiplication(i.e. $k\cdot A$ where k is a constant).

3. A relation between Riemann-Stieltjes and Lebesgue integral

There is a remarkably simple and useful representation of Lebesgue integrals over subsets of R in terms of Riemann-Stieltjes integrals. Let

$$w(\alpha)=w_{f,E}(\alpha)=|\{x\in E|f(x)>\alpha\}|,$$

where f is a measurable function on E, $-\infty<\alpha<+\infty$ and $|\cdot|$ is the Lebesgue measure of the set $\cdot$. We call $w_{f,E}$ the distribution function of f on E. Some properties of w were given in [15]. The following result is the essential tool of this paper.

Lemma 1 [15]. If $a<f\leq b$ (a and b finite) in E and ϕ is continuous on $[a,b]$, then

$$\int_E\phi(f)=-\int_a^b\phi(\alpha)dw(\alpha).$$

The next result is the very useful formula for integration by parts

Lemma 2 [15]. If $\int_a^b f d\phi$ exists, then so does $\int_a^b\phi\, df$, and

$$\int_a^b f d\phi=[f(b)\phi(b)-f(a)\phi(a)]-\int_a^b\phi df.$$

We write $\{f>\alpha\}=\{x\in E|f(x)>\alpha\}$, ect.

Let $\otimes$ be the addition "+" and the support

$A_i^{0'} = (a_i, b_i) \subset X$.

Lemma 3 [7]. Let A_i, $i=1,2,\cdots,n$ be fuzzy numbers. Then

$$|\{A_1+A_2+\cdots+A_n > \alpha\}| = |\{A_1 > \alpha\}| + \cdots + |\{A_n > \alpha\}|$$

for any $\alpha \in (0,1]$.

Let $\otimes$ denote the subtraction " $-$ "

Lemma 4. Let A_1 and A_2 be two fuzzy numbers. Then, for any $\alpha \in (0,1]$

$$|\{A_1 - A_2 > \alpha\}| = |\{A_1 > \alpha\}| + |\{A_2 > \alpha\}|.$$

Proof.
$$\begin{aligned} |\{A_1 - A_2 \geq \alpha\}| &= |\{A_1 + A_2^- \geq \alpha\}| \\ &= |\{A_1 \geq \alpha\}| + |\{A_2^- \geq \alpha\}| \\ &= |\{A_1 \geq \alpha\}| + |\{A_2 \geq \alpha\}| . \end{aligned}$$

The following lemma is trivial.

Lemma 5 For a constant k and a fuzzy number A, we have

$$|\{kA > \alpha\}| = |k||\{A > \alpha\}|.$$

4. Entropy arithmetic operations

In this section we discuss the entropy arithmetic operations. We will give generalized results of Wang and Chiu[12, 13] with remarkably simple proof without any restrictions of types of fuzzy numbers.

The following result generalize Theorem 2 of Wang and Chiu [13].

Theorem 1. Let X be a bounded set in R and A_i, $i=1,\cdots,n$, be fuzzy numbers such that the support of $A_i \subset X$, $i=1,\cdots,n$ and the support of $\sum_{i=1}^{n} A_i \subset X$. Suppose that $p(x)=s$, where s is a constant over X, then we have

$$H(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} H(A_i).$$

Proof. Let $|\{\sum_{i=1}^{n} A_i > \alpha\}| = w(\alpha)$ and $|\{A_i > \alpha\}| = w_i(\alpha)$, $i=1,2,\cdots,n$, for $\alpha \in (0,1]$. Then by Lemma 1, and 2,

$$\begin{aligned} H(\sum_{i=1}^{n} A_i) &= \int_X h(\sum_{i=1}^{n} A_i(x)) p(x) dx \\ &= s\int_X h(\sum_{i=1}^{n} A_i(x))\, dx \\ &= -s\int_0^1 h(\alpha)\, dw(\alpha) \\ &= -s[\,(h(1)w(1) - h(0)w(0) - \int_0^1 w(\alpha) dh(\alpha)]. \end{aligned}$$

Now, since $h(0)=h(1)=0$,

$$H(\sum_{i=1}^{n} A_i) = s\int_0^1 w(\alpha) dh(\alpha).$$

And trivially, we have $H(A_i) = s\int_0^1 w_i(\alpha)\, dh(\alpha)$ $i=1,2,\cdots,n$. Hence, from the equality $w(\alpha) = \sum_{i=1}^{n} w_i(\alpha)$ by Lemma 3, the theorem follows immediately.

Noting the fact $|A > \alpha| = |A^- > \alpha|$ and the relation $A_1 - A_2 = A_1 + A_2^-$, we will have the following result which generalize Theorem 3 of Wang and Chiu [13], immediately.

Theorem 2. Under the same conditions of Theorem 3, we have

$$H(\sum_{i \in I_1} A_i - \sum_{j \in I_2} A_j) = \sum_{k=1}^{n} H(A_k)$$

where I_1 and I_2 are two crisp sets of digits and $I_1 \cup I_2 = \{1,2,\cdots,n\}$, $I_1 \cap I_2 = \emptyset$.

The following result, which generalize Theorem 4 of Wang and Chiu [13], is also immediate from Lemma 5,

Theorem 3. Let A be any fuzzy numbers with bounded support and k be any constant. Then

$$H(k \cdot A) = |k| \cdot H(A).$$

Corollary 1. Let A and B be fuzzy sets such that $|\{B > \alpha\}| = k|\{A > \alpha\}|$ for any $0 < \alpha < 1$ and some constant k. Suppose that $p(x) = s =$ constant over X, then

$$H(A) = kH(B).$$

Remark 1. For the fuzzy sets A_1 and A_2 stated in Definition 1, we trivially have that $|\{A_2 > \alpha\}| = k|\{A_1 > \alpha\}|$ for any $0 < \alpha < 1$ where $k = \frac{b_1 - a_1}{b_2 - a_2}$. Hence the result of Pedrycz[9] and the result of Chen and Wang[1] is a special case of Corollary 1.

Example 1. Consider three fuzzy numbers A, B and C of Type(i), Type(ii) and Type(iii), respectively in [13]as follows:

$$A = \begin{cases} \frac{1}{1+(x-3)^2} & x \in A^{0.01} = [3-\sqrt{99}, 3+\sqrt{99}], \\ 0 & \text{elsewhere;} \end{cases}$$

$$B(x) = \begin{cases} \exp(-|x-3|) & x \in A^{0.01} = [3+\ln(0.01), \\ & 3-\ln(0.01)], \\ 0 & \text{elsewhere;} \end{cases}$$

$$C(x) = \begin{cases} \frac{[1+\cos((x+3)\pi])}{2} & x \in [2,4], \\ 0 & \text{elsewhere.} \end{cases}$$

Let us take the entropy function as (2.3) and the entropy as (2.5) with $p(x)=s$, where s is a constant over X. Then, we have $H(A) \simeq 5.484s$, $H(B) \simeq 3.92s$,

$H(B)\simeq 3.92s$ and $H(C)=s$.. Now, we are going to find $H(A+B+C)$ and $H(A-B+C)$.As we can see that all these fuzzy numbers are of different types. Hence, we cannot get the entropies directly from Wang and Chiu's [13] results, but by Theorem 1 and Theorem 2, we have $H(A+B+C)=5.484s+3.92s+s\simeq 10.504s$ and similarly $H(A-B+C)\simeq 10.504s$.

Example 2. Consider the fuzzy set A and B as follows :

$$A(x)=\begin{cases} x & \text{if } 0\le x\le 1, \\ 2-x & \text{if } 1\le x\le 2, \\ 0 & \text{elsewhere}, \end{cases}$$

$$B(x)=\begin{cases} \sqrt{x} & \text{if } 0\le x\le 1, \\ 1-\sqrt{x-1} & \text{if } 1\le x\le 2, \\ 0 & \text{elsewhere}. \end{cases}$$

It is noted A and B are not of the same shape but $|\{A>\alpha\}|=|\{B>\alpha\}|=2(1-\alpha)$ for $0<\alpha<1$.

Hence by Corollary 1, if with $p(x)=s=\frac{1}{4}$ and $h(u)=4u(1-u)$ in (2.4), then

$$\begin{aligned} H(A)=H(B) &= s\int_0^1 w(\alpha)dh(\alpha) \\ &= \frac{1}{4}\int_0^1 2(1-\alpha)(4-8\alpha)d\alpha \\ &= 2\int_0^1 (1-\alpha)(1-2\alpha)d\alpha \\ &= \frac{1}{3}. \end{aligned}$$

5. Information energy arithmetic operations

We now consider the following function

$$\hbar(A(x))=1-e(A(x))=4A(x)-4A^2(x)$$

where $e(A(x))$ is defined in (2.6). Then $\hbar$ has the same form as (2.4) which satisfies the properties of entropy function. Hence, we have the following result.

Lemma 6. Assume $p(x)=s=$constant over X, then we have

$$H(A)=s|X|-E(A)=1-E(A).$$

By Lemma 6 and results in Section 4 we will have some results about energy information arithmetic operations.

Theorem 4. Under the conditions of Theorem 1, we have

$$E(\sum_{i=1}^{n} A_i)=1-n+\sum_{i=1}^{n} E(A_i).$$

Theorem 5. Under the conditions of Theorem 4, we have

$$E(\sum_{i\in I_1} A_i-\sum_{j\in I_2} A_j)=E(\sum_{k=1}^{n} A_k)$$

where I_1 and I_2 are two crisp sets of digits and $I_1\cup I_2=\{1,2,\cdots,n\}$, $I_1\cap I_2=\emptyset$.

Proof. By Lemma 6, and Theorem 1 we have

$$\begin{aligned} E(\sum_{i\in I_1} A_i-\sum_{j\in I_2} A_j) &= 1-H(\sum_{i\in I_1} A_i-\sum_{j\in I_2} A_j) \\ &= 1-H(\sum_{k=1}^{n} A_k) \\ &= 1-(1-E(\sum_{k=1}^{n} A_k) \\ &= E(\sum_{k=1}^{n} A_k). \end{aligned}$$

Theorem 6. Let A be any fuzzy number with bounded support and k be any constant. Then

$$E(k\cdot A)=|k|E(A)+(1-|k|).$$

Proof. By Lemma 6 and Theorem 3, we have

$$\begin{aligned} E(k\cdot A) &= 1-H(k\cdot A) \\ &= 1-|k|H(A) \\ &= 1-|k|(1-E(A)) \\ &= |k|E(A)+(1-|k|). \end{aligned}$$

Example 3. Let A and B be the same fuzzy numbers as in Example 1 in $X=[0,4]$ with $p(x)=s=\frac{1}{4}$. Then by Theorem 4 and 6, we have

$$E(A+B)=1-2+E(A)+E(B)=\frac{1}{3},$$

$$E(2A)=2E(A)+(1-2)=\frac{1}{3}.$$

6. Conclusion

This paper gives new methods of calculating entropy of the fuzzy numbers, after arithmetic operations (addition, subtraction, multiplication). Without any restriction of the types of fuzzy numbers, the entropy of the resultant fuzzy number can be obtained by calculating the individual entropy of the original fuzzy numbers. This methods have far-reaching generalizations of earlier results of Wang and Chiu [Fuzzy Sets and Systems, 103(1999) 443-455]. Furthermore, it has been shown the information energy variation on the fuzzy numbers with arithmetic operations (addition, subtraction, multiplication) using a trade-off relationship between information energy and the entropy of a fuzzy set.

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