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MULTIOBJECTIVE FRACTIONAL PROGRAMMING WITH A MODIFIED OBJECTIVE FUNCTION

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Abstract. We consider multiobjective fractional programming problems with generalized invexity. An equivalent multiobjective programming problem is formulated by using a modification of the objective function due to Antczak. We give relations between a multiobjective fractional programming problem and an equivalent multiobjective fractional problem which has a modifed objective function. And we present modified vector saddle point theorems.

1. Introduction

Khan and Hanson[7] have used the ratio invexity concept to characterize optimality and duality results in fractional programming. This concept seems to be new and it introduces a modified kind of characterization in sufficient optimality conditions. The optimality conditions of Karush-Kuhn-Tucker type for a multiobjective programming problem and the saddle points of its vector-valued Lagrangian function have been studied by many authors ([2, 3, 4, 5, 6, 8, 9, 10, 11]). But in most of these, an assumption of convexity on the functions involving was made. The aim of this paper is to show how one can obtain optimality conditions for Pareto optimality by constructing an equivalent multiobjective programming problem for a multiobjective fractional programming problem with generalized invexity. The equivalent multiobjective programming problem is obtained by a modification of the objective function due to Antczak[1]. Furthermore, a Lagrangian function is introduced for a constructed multiobjective fractional programming problem and modified vector saddle point results are presented.

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2. Preliminaries

Throughout this paper, we will use the following conventions for vectors in R^n:

$$x < y \Leftrightarrow x_i < y_i, \ \ i = 1, 2, \cdots, n;$$
$$x \leqq y \Leftrightarrow x_i \leqq y_i, \ \ i = 1, 2, \cdots, n;$$
$$x \leq y \Leftrightarrow x_i \leqq y_i \ \text{ but } \ x \neq y \ \ i = 1, 2, \cdots, n;$$
$$x \not< y \ \text{ is the negation of } \ x < y.$$

We consider the following multiobjective programming problem (MP)

$$(MP) \qquad \text{Minimize} \quad f(x) \qquad \text{subject to} \quad g(x) \leqq 0,$$

where $f = (f_1, \cdots, f_k) : X \to R^k$ and $g = (g_1, \cdots, g_m) : X \to R^m$ are differentiable functions on a nonempty open set $X \subset R^n$.

Consider the following multiobjective fractional programming problem (MFP):

$$(MFP) \qquad \text{Minimize} \quad \left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_k(x)}{g(x)} \right)$$
$$\text{subject to} \quad h_j(x) \leqq 0, \ \ j = 1, \cdots, m,$$

where f_i, g and h_j are continuously differentiable over $X \subset R^n$. Let $S = \{x \in X : h_j(x) \leqq 0, \ \ j = 1, \cdots, m\}$ denote the set of all feasible solutions and $I(x) = \{i : h_i(x) = 0\}$ for any $x \in X$. We assume that $f(x) \geqq 0$ for all $x \in X$ and $g(x) > 0$ for all $x \in X$ whenever g is not linear.

Optimization of (MP) is finding (weakly) efficient solutions defined as follows;

DEFINITION 2.1. (1) A point $\bar{x} \in S$ is said to be an efficient solution for (MFP) if there exists no other feasible point $x \in S$ such that $f(x) \leq f(\bar{x})$.

(2) A point $\bar{x} \in S$ is said to be a weakly efficient solution for (MFP) if there exists no other feasible point $x \in S$ such that $f(x) < f(\bar{x})$.

Now we define the concepts of invexity for vector-valued function.

DEFINITION 2.2. Let $f : X \to R^k$ be a differentiable function on a nonempty open set $X \subset R^n$.

(1) f is said to be invex with respect to η at $u \in X$ if, for all $x \in X$, there exists $\eta : X \times X \to R^n$ such that for all $i = 1, 2, \cdots, k$,

$$f_i(x) - f_i(u) \geqq \nabla f_i(u)\eta(x, u).$$

(2) f is said to be strictly invex with respect to η at $u \in X$ if, for all $x \in X$ with $x \neq u$, there exists $\eta : X \times X \to R^n$ such that for all $i = 1, 2, \cdots, k$,

$$f_i(x) - f_i(u) > \nabla f_i(u)\eta(x, u).$$

LEMMA 2.3. *If real valued functions $f(x)$ and $-g(x)$ are invex with respect to the same $\eta(x, y)$, then $f(x)/g(x)$ is an invex function with respect to $\bar{\eta}(x, y) = (g(y)/g(x))\eta(x, y)$.*

PROOF. Let $x, y \in X$. Since $f(x)$ and $-g(x)$ are invex with respect to the same $\eta(x, y)$, then we have

$$\begin{aligned}\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} &= \frac{f(x) - f(y)}{g(x)} - \frac{f(y)(g(x) - g(y))}{g(x)g(y)} \\ &\geqq \frac{g(y)}{g(x)}\left(\frac{\nabla f(y)}{g(y)} - \frac{f(y)\nabla g(y)}{g^2(y)}\right)\eta(x, y) \\ &= \frac{g(y)}{g(x)}\nabla\left(\frac{f(y)}{g(y)}\right)\eta(x, y).\end{aligned}$$

Therefore, $\frac{f(x)}{g(x)}$ is an invex function with respect to $\bar{\eta}(x, y) = \left(\frac{g(y)}{g(x)}\right)$ $\eta(x, y)$. □

3. Optimality conditions

In this section, we give Fritz John necessary conditions and Karush-Kuhn-Tucker necessary condition and establish sufficient conditions for efficient and weakly efficient solutions of (MFP).

Necessary optimality conditions a Karush-Kuhn-Tucker type for the multiobjective problems were obtained, for example, by Kanniappan[6], Weir[11]. Therefore, we are using the following necessary optimality conditions of Karush-Kuhn-Tucker type under some constraint qualification (CQ) (for example, Linear Independence)

The following Theorem 3.1, 3.2 and 3.3 are necessary conditions for a (weakly) efficient solution.

THEOREM 3.1 (FRITZ JOHN NECESSARY CONDITIONS). *If $\bar{x} \in S$ is a (weakly) efficient solution of (MFP), then there exists $\lambda_i, i = 1, \cdots, k$ and $\mu_j, j = 1, \cdots, m$, such that*

$$\sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(\bar{x})}{g(\bar{x})} + \sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} \mu_j h_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_m) \geq 0.$$

THEOREM 3.2 (KARUSH-KUHN-TUCKER NECESSARY CONDITIONS). *Assume that there exists $z^* \in X$ such that $\langle \nabla h_j(\bar{x}), z^* \rangle > 0, \ j \in I(\bar{x})$. If $\bar{x} \in S$ is a weakly efficient solution of (MFP), then there exists $\lambda_i \geqq 0, \ i = 1, \cdots, k$ and $\mu_j \geqq 0, j = 1, \cdots, m$, such that*

$$\sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(\bar{x})}{g(\bar{x})} + \sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} \mu_j h_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_m) \geqq 0, \ (\lambda_1, \cdots, \lambda_k) \neq (0, \cdots, 0).$$

THEOREM 3.3 (KARUSH-KUHN-TUCKER NECESSARY CONDITIONS). *Assume that $\nabla h_j(\bar{x}), \ j \in I(\bar{x})$ are linearly independent. If $\bar{x} \in S$ is an efficient solution of (MFP), then there exists $\lambda_i \geqq 0, \ i = 1, \cdots, k$ and $\mu_j \geqq 0, j = 1, \cdots, m$, such that*

$$\sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(\bar{x})}{g(\bar{x})} + \sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} \mu_j h_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_m) \geqq 0, \ (\lambda_1, \cdots, \lambda_k) \neq (0, \cdots, 0).$$

The following Theorem 3.4 is sufficient conditions for a weakly efficient solution.

THEOREM 3.4 (KARUSH-KUHN-TUCKER SUFFICIENT CONDITIONS). *Let $(\bar{x}, \lambda, \mu)$ satisfy the Karush-Kuhn-Tucker conditions as follows:*

$$\sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(\bar{x})}{g(\bar{x})} + \sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) = 0,$$

$$\sum_{j=1}^{m} \mu_j h_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_m) \geqq 0, \quad (\lambda_1, \cdots, \lambda_k) \neq (0, \cdots, 0).$$

If $(f_1, \cdots, f_k)$ and $-g$ are invex with respect to the same η and if $(h_1, \cdots, h_m)$ is invex with respect to $\bar{\eta}$, then $\bar{x}$ is a weakly efficient solution of (MFP).

PROOF. Suppose $\bar{x}$ is not a weakly efficient solution of (MFP). Then, for $x \in S$,

$$\frac{f_i(x)}{g(x)} < \frac{f_i(\bar{x})}{g(\bar{x})}, \quad \text{for all } i = 1, \cdots, k.$$

Since $f_i, \ i = 1, \cdots, k$, and $-g$ are invex with respect to the same η, by Lemma 2.3,

$$\frac{g(\bar{x})}{g(x)} \nabla \Big(\frac{f_i(\bar{x})}{g(\bar{x})}\Big) \eta(x, \bar{x}) < 0.$$

From $(\lambda_1, \cdots, \lambda_k) \geq 0$,

$$\sum_{i=1}^{k} \lambda_i \nabla \Big(\frac{f(\bar{x})}{g(\bar{x})}\Big) \bar{\eta}(x, \bar{x}) < 0.$$

Using Karush-Kuhn-Tucker conditions,

$$\sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) \bar{\eta}(x, \bar{x}) > 0. \tag{1}$$

Since $(h_1, \cdots, h_m)$ is invex with respect to $\bar{\eta}$,

$$\sum_{j=1}^{m} \mu_j h_j(x) - \sum_{j=1}^{m} \mu_j h_j(\bar{x}) \geqq \sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x}) \bar{\eta}(x, \bar{x}).$$

Then

$$\sum_{j=1}^{m} \mu_j \nabla h_j(\bar{x})\bar{\eta}(x,\bar{x}) \leqq 0.$$

This inequality contradicts (1). Hence $\bar{x}$ is a weakly efficient solution of (MFP). □

REMARK. If we replace the invexity hypothesis on one of f and $-g$ by strictly invexity in Theorem 3.4 or if we replace the invexity hypothesis of h by strictly invexity in Theorem 3.4, then Theorem 3.4 holds in the sense of an efficient solution.

4. An equivalent multiobjective fractional problem

Let $\bar{x}$ be a feasible solution of (MFP). We consider the following multiobjective fractional program $(MFP_{\bar{\eta}}(\bar{x}))$ given by

$$(MFP_{\bar{\eta}}(\bar{x}))\ \text{Minimize}\ \left(\bar{\eta}^T(x,\bar{x})\nabla\frac{f_1(\bar{x})}{g(\bar{x})},\cdots,\bar{\eta}^T(x,\bar{x})\nabla\frac{f_k(\bar{x})}{g(\bar{x})}\right)$$
$$\text{subject to}\quad h_j(x) \leqq 0,\ \ j=1,\cdots,m,$$

where $\bar{\eta}(x,\bar{x}) = \frac{g(\bar{x})}{g(x)}\eta(x,\bar{x})$. Let $S = \{x \in X : h_j(x) \leqq 0,\ j = 1,\cdots,m\}$ denote the set of all feasible solutions and $I(x) = \{i : h_i(x) = 0\}$ for any $x \in X$.

THEOREM 4.1. *Let $\bar{x}$ be* (weakly) *efficient in* (MFP) *and assume that* (*there exists $z^* \in X$ such that $\langle \nabla h_j(\bar{x}), z^*\rangle > 0,\ \ j \in I(\bar{x})$*) $\nabla h_j(\bar{x})$*, $j \in I(\bar{x})$ are linearly independent. Futher, we assume that h is strictly invex with respect to $\bar{\eta}$ at $\bar{x}$ on S and $\eta(\bar{x},\bar{x}) = 0$. Then $\bar{x}$ is* (weakly) *efficient in $(MFP_{\bar{\eta}}(\bar{x}))$.*

PROOF. Since $\bar{x}$ is efficient in (MFP) and $\nabla h_j(\bar{x}),\ j \in I(\bar{x})$ are linearly independent, Karush-Kuhn-Tucker conditions are satisfied. Assume that $\bar{x}$ is not efficient for $(MFP_{\bar{\eta}}(\bar{x}))$. This implies that there exists $\hat{x}$ which is feasible for $(MFP_{\bar{\eta}}(\bar{x}))$ such that

$$\left(\bar{\eta}(\hat{x},\bar{x})\nabla\frac{f_1(\bar{x})}{g(\bar{x})},\cdots,\bar{\eta}(\hat{x},\bar{x})\nabla\frac{f_k(\bar{x})}{g(\bar{x})}\right)$$
$$\leq \left(\bar{\eta}(\bar{x},\bar{x})\nabla\frac{f_1(\bar{x})}{g(\bar{x})},\cdots,\bar{\eta}(\bar{x},\bar{x})\nabla\frac{f_k(\bar{x})}{g(\bar{x})}\right) = (0,\cdots,0).$$

Since $\lambda \geq 0$,

$$\sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(\bar{x})}{g(\bar{x})} \bar{\eta}(\hat{x}, \bar{x}) \leqq 0. \tag{2}$$

Since $\hat{x}$ is feasible and $\mu \geqq 0$, $\mu^T h(\hat{x}) \leqq 0$. Hence it follows that $\mu^T h(\hat{x}) \leqq \mu^T h(\bar{x})$. By assumption, that is, h is strictly invex with respect to $\bar{\eta}$ at $\bar{x}$,

$$\mu^T \nabla h(\bar{x}) \bar{\eta}(\hat{x}, \bar{x}) < 0. \tag{3}$$

By (2) and (3), we obtain

$$\bar{\eta}^T(\hat{x}, \bar{x}) \left[\sum_{i=1}^{k} \lambda_i \nabla \frac{f_i(\bar{x})}{g(\bar{x})} + \mu^T \nabla h(\bar{x}) \right] < 0.$$

This inequality contradicts $\sum_{i=1}^{k} \lambda_i \nabla \frac{f_k(\bar{x})}{g(\bar{x})} + \mu^T \nabla h(\bar{x}) = 0$. Hence $\bar{x}$ is efficient in $(MFP_{\bar{\eta}}(\bar{x}))$. In the similar method, we prove that $\bar{x}$ is weakly efficient in $(MFP_{\bar{\eta}}(\bar{x}))$. □

THEOREM 4.2. *Let $\bar{x}$ be a feasible point for $(MFP_{\bar{\eta}}(\bar{x}))$. Further, we assume that f and $-g$ are invex with respect to η at $\bar{x}$ and $\eta(\bar{x}, \bar{x}) = 0$. If $\bar{x}$ is efficient in $(MFP_{\bar{\eta}}(\bar{x}))$, then $\bar{x}$ is also efficient in (MFP).*

PROOF. Since f and $-g$ are invex with respect to η, $\frac{f_i}{g}$, for all $i = 1, \cdots, k$, is invex with respect to $\bar{\eta}$ and $\bar{\eta}(\bar{x}, \bar{x}) = 0$, where $\bar{\eta}(x, \bar{x}) = \frac{g(\bar{x})}{g(x)} \eta(x, \bar{x})$. Assume that $\bar{x}$ is not efficient in (MFP). Then there exists $\hat{x}$ feasible for (MFP) such that

$$\left(\frac{f_1(\hat{x})}{g(\hat{x})}, \cdots, \frac{f_k(\hat{x})}{g(\hat{x})} \right) \leq \left(\frac{f_1(\bar{x})}{g(\bar{x})}, \cdots, \frac{f_k(\bar{x})}{g(\bar{x})} \right).$$

Since $\frac{f_i}{g}$, $i = 1, \cdots, k$, is invex with respect to $\bar{\eta}$ at $\bar{x}$ and $\eta(\bar{x}, \bar{x}) = 0$, we have

$$\begin{aligned} &\left(\bar{\eta}(\hat{x}, \bar{x}) \nabla \frac{f_1(\bar{x})}{g(\bar{x})}, \cdots, \bar{\eta}(\hat{x}, \bar{x}) \nabla \frac{f_k(\bar{x})}{g(\bar{x})} \right) \\ \leqq &\left(\frac{f_1(\hat{x})}{g(\hat{x})} - \frac{f_1(\bar{x})}{g(\bar{x})}, \cdots, \frac{f_k(\hat{x})}{g(\hat{x})} - \frac{f_k(\bar{x})}{g(\bar{x})} \right) \\ \leq &(0, \cdots, 0) \\ = &\left(\bar{\eta}(\bar{x}, \bar{x}) \nabla \frac{f_1(\bar{x})}{g(\bar{x})}, \cdots, \bar{\eta}(\bar{x}, \bar{x}) \nabla \frac{f_k(\bar{x})}{g(\bar{x})} \right), \end{aligned}$$

which contradicts that $\bar{x}$ is efficient in $(MFP_{\bar{\eta}}(\bar{x}))$ □

5. Saddle point criteria

Now we introduce a definition of an $\bar{\eta}$-Lagrange function for a multiobjective fractional programming problem $(MFP_{\bar{\eta}}(\bar{x}))$.

DEFINITION 5.1. An $\bar{\eta}$-Lagrange function is said to be a Lagrange function for a multiobjective fractional programming problem $(MFP_{\bar{\eta}}(\bar{x}))$

$$L_{\bar{\eta}}(x,\mu) = \left(\bar{\eta}^T(x,\bar{x})\nabla\frac{f_1(\bar{x})}{g(\bar{x})} + \mu^T h(x), \cdots, \bar{\eta}^T(x,\bar{x})\nabla\frac{f_k(\bar{x})}{g(\bar{x})} + \mu^T h(x)\right).$$

Here, we give a new definition of a weak vector saddle point for the introduced $\bar{\eta}$-Lagrange function in a multiobjective fractional programming problem $(MFP_{\bar{\eta}}(\bar{x}))$.

DEFINITION 5.2. A point $(\bar{x},\bar{\mu}) \in S \times R^m_+$ is said to be a weak vector saddle point for the $\bar{\eta}$-Lagrange function if
(i) $L_{\bar{\eta}}(\bar{x},\mu) \not> L_{\bar{\eta}}(\bar{x},\bar{\mu}), \ \forall \mu \in R^m_+$,
(ii) $L_{\bar{\eta}}(x,\bar{\mu}) \not< L_{\bar{\eta}}(\bar{x},\bar{\mu}), \ \forall x \in S$.

THEOREM 5.3. *We assume that f_i, $i = 1,\cdots,k$, and g are invex with respect to η at $\bar{x}$ with $\eta(\bar{x},\bar{x}) = 0$ and some (CQ) holds at $\bar{x}$ for (MFP). If $(\bar{x},\bar{\mu})$ is a weak vector saddle point for $L_{\bar{\eta}}$, then $\bar{x}$ is a weakly efficient solution in (MFP).*

PROOF. Assume that $(\bar{x},\bar{\mu})$ is a saddle point for $L_{\bar{\eta}}$. Then by (i) of Definition 5.2, we have $L_{\bar{\eta}}(\bar{x},\mu) \not> L_{\bar{\eta}}(\bar{x},\bar{\mu})$. Since $\eta(\bar{x},\bar{x}) = 0$, we obtain

$$\mu^T h(\bar{x}) \leqq \bar{\mu}^T h(\bar{x}), \ \ \forall \mu \in R^m_+. \tag{4}$$

Suppose that $\bar{x}$ is not a weakly efficient solution in (MFP). Then there exists $\hat{x} \in S$ such that for all $i = 1,2,\cdots,k$,

$$\frac{f_i(\hat{x})}{g(\hat{x})} < \frac{f_i(\bar{x})}{g(\bar{x})}. \tag{5}$$

Since $\bar{x} \in S$ and $\bar{\mu} \in R^m_+, \bar{\mu}^T h(\bar{x}) \leqq 0$. In (4), let $\mu = 0$

$$\bar{\mu}^T h(\bar{x}) \geqq 0.$$

Hence

$$\bar{\mu}^T h(\bar{x}) = 0. \tag{6}$$

Since $\frac{f_i}{g}$, $i = 1, \cdots, k$ is invex with respect to $\bar{\eta}$, then from (5),

$$\bar{\eta}(\hat{x}, \bar{x}) \nabla \frac{f_i(\bar{x})}{g(\bar{x})} < 0. \tag{7}$$

Thus, by (6) and (7) and using the definition of $L_{\bar{\eta}}$, we get

$$\begin{aligned} L_{\bar{\eta}}(\hat{x}, \bar{\mu}) &= \left(\bar{\eta}(\hat{x}, \bar{x}) \nabla \frac{f_1(\bar{x})}{g(\bar{x})} + \bar{\mu}^T h(\bar{x}), \cdots, \bar{\eta}(\hat{x}, \bar{x}) \nabla \frac{f_k(\bar{x})}{g(\bar{x})} + \bar{\mu}^T h(\bar{x}) \right) \\ &< \left(\bar{\eta}(\bar{x}, \bar{x}) \nabla \frac{f_1(\bar{x})}{g(\bar{x})} + \bar{\mu}^T h(\bar{x}), \cdots, \bar{\eta}(\bar{x}, \bar{x}) \nabla \frac{f_k(\bar{x})}{g(\bar{x})} + \bar{\mu}^T h(\bar{x}) \right) \\ &= L_{\bar{\eta}}(\bar{x}, \bar{\mu}). \end{aligned}$$

This contradicts (ii) of Definition 5.2. Hence $\bar{x}$ is a weakly efficient solution in (MFP). □

THEOREM 5.4. *Let $\bar{x}$ be a weakly efficient solution in (MFP) at which (CQ) is satisfied. Further, we assume that h is invex with respect to $\bar{\eta}$ at $\bar{x}$ and $\eta(\bar{x}, \bar{x}) = 0$. Then there exists $\bar{\mu} \in R_+^m$ such that $(\bar{x}, \bar{\mu})$ is a weak vector saddle point for the $\bar{\eta}$-Lagrange function in a multiobjective fractional programming problem $(MFP_{\bar{\eta}}(\bar{x}))$.*

PROOF. Since $\bar{x}$ is a weakly efficient solution for (MFP), by Theorem 3.2, Karush-Kuhn-Tucker conditions hold. Assume $\sum_{i=1}^k \bar{\lambda}_i = 1$. Since h is invex with respect to $\bar{\eta}$ at $\bar{x}$ and $\bar{\mu} \in R_+^m$, it follows that the inequality

$$\bar{\mu}^T h(x) - \bar{\mu}^T h(\bar{x}) \geqq \bar{\mu}^T \nabla h(\bar{x}) \bar{\eta}(x, \bar{x})$$

holds for all $x \in S$. Then from Karush-Kunh-Tucker condtion,

$$\bar{\mu}^T h(x) - \bar{\mu}^T h(\bar{x}) \geqq -\sum_{i=1}^{k} \bar{\lambda}_i{}^T \nabla \frac{f_i(\bar{x})}{g(\bar{x})} \bar{\eta}(x, \bar{x}).$$

By assumption $\eta(\bar{x}, \bar{x}) = 0$, the inequality

$$\sum_{i=1}^{k} \bar{\lambda}_i^T \nabla \frac{f_i(\bar{x})}{g(\bar{x})} \bar{\eta}(x, \bar{x}) + \bar{\mu}^T h(x) \geqq \sum_{i=1}^{k} \bar{\lambda}_i^T \nabla \frac{f_i(\bar{x})}{g(\bar{x})} \bar{\eta}(\bar{x}, \bar{x}) + \bar{\mu}^T h(\bar{x})$$

holds for all $x \in S$. Since $\bar{\lambda} \geq 0$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$ and by the definition of the $\bar{\eta}$-Lagrange function, it follows that, for all $x \in S$

$$\bar{\lambda}^T L_{\bar{\eta}}(x, \bar{\mu}) \geqq \bar{\lambda}^T L_{\bar{\eta}}(\bar{x}, \bar{\mu}). \tag{8}$$

Assume that $L_{\bar{\eta}}(x, \bar{\mu}) < L_{\bar{\eta}}(\bar{x}, \bar{\mu})$ for all $x \in S$. Then $\bar{\lambda}^T L_{\bar{\eta}}(x, \bar{\mu}) < \bar{\lambda}^T L_{\bar{\eta}}(\bar{x}, \bar{\mu})$. This contradicts (8). Hence

$$L_{\bar{\eta}}(x, \bar{\mu}) \not< L_{\bar{\eta}}(\bar{x}, \bar{\mu}), \text{ for all } x \in S. \tag{9}$$

Since $\bar{x} \in S$, the inequality

$$\mu^T h(\bar{x}) \leqq \bar{\mu}^T h(\bar{x})$$

holds for all $\mu \in R_+^m$. Thus we obtain

$$\begin{aligned} &\left(\bar{\eta}(\bar{x}, \bar{x})\nabla \frac{f_1(\bar{x})}{g(\bar{x})} + \mu^T h(\bar{x}), \cdots, \bar{\eta}(\bar{x}, \bar{x})\nabla \frac{f_k(\bar{x})}{g(\bar{x})} + \mu^T h(\bar{x})\right) \\ \leqq &\left(\bar{\eta}^T(\bar{x}, \bar{x})\nabla \frac{f_1(\bar{x})}{g(\bar{x})} + \bar{\mu}^T h(\bar{x}), \cdots, \bar{\eta}(\bar{x}, \bar{x})\nabla \frac{f_k(\bar{x})}{g(\bar{x})} + \bar{\mu}^T h(\bar{x})\right) \end{aligned}$$

and

$$\bar{\lambda}^T L_{\bar{\eta}}(\bar{x}, \mu) \leqq \bar{\lambda}^T L_{\bar{\eta}}(\bar{x}, \bar{\mu}) \tag{10}$$

for all $\mu \in R_+^m$.

Assume that $L_{\bar{\eta}}(\bar{x}, \bar{\mu}) < L_{\bar{\eta}}(\bar{x}, \mu)$ for all $\mu \in R_+^m$. Then $\bar{\lambda}^T L_{\bar{\eta}}(\bar{x}, \bar{\mu}) < \bar{\lambda}^T L_{\bar{\eta}}(\bar{x}, \mu)$.

This contradicts (10). Hence

$$L_{\bar{\eta}}(\bar{x}, \bar{\mu}) \not< L_{\bar{\eta}}(\bar{x}, \mu), \text{ for all } \mu \in R_+^m. \tag{11}$$

Inequalities (9) and (11) mean that $(\bar{x}, \bar{\mu})$ is a weak vector saddle point for the $\bar{\eta}$-Lagrange function in a multiobjective fractional programming problem $(MFP_{\bar{\eta}}(\bar{x}))$. □

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