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APPROXIMATING COMMON FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we deal with approximations of common fixed points of the iterative sequences with errors for three asymptotically nonexpansive mappings in a uniformly convex Banach space. Our results generalize and improve the corresponding results of Khan and Takahashi, Schu, Takahashi and Tamura, and others.

Let C be a nonempty subset of a real Banach space E, $T : C \to C$ be a mapping and $F(T)$ denote the set of fixed points of the mapping T. The mapping T is said to be *asymptotically nonexpansive* if, for a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. The mapping T is said to be *uniformly L-Lipschitzian* if there exists a positive number L such that

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

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for all $x, y \in C$. The mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|$$

for all $x \in C$ and $y \in F(T)$.

Note that every uniformly L-Lipschitzian mapping with $L = 1$ is nonexpansive and, in 1972, Goebel and Kirk[3] introduced initially the concept of asymptotically nonexpansive mappings, which are more general than nonexpansive and quasi-nonexpansive mappings, and proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point in C. Moreover, the fixed point set $F(T)$ is closed and convex. Since 1972, many authors have studied weak and strong convergence problems of the iterative sequences (with errors) for asymptotically nonexpansive mapping types in Hilbert spaces and Banach spaces (see [1]).

In [2], Das and Debata considered the iterative sequence $\{x_n\}$ for two quasi-nonexpansive self-mappings S and T of C defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - a_n)x_n + a_n S y_n, \quad n \geq 1, \\ y_n = (1 - b_n)x_n + b_n T x_n, \quad n \geq 1, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are some sequences in [0,1].

In [10], Takahashi and Tamura proved some convergence theorems of the above sequence for two nonexpansive mappings.

Recently, Khan and Takahashi considered the problems of approximating common fixed points of two asymptotically nonexpansive self-mappings S and T of C through weak and strong convergence of the iterative sequence $\{x_n\}$ defined by

$$(A) \qquad \begin{cases} x_1 \in C, \\ x_{n+1} = (1 - a_n)x_n + a_n S^n y_n, \quad n \geq 1, \\ y_n = (1 - b_n)x_n + b_n T^n x_n, \quad n \geq 1, \end{cases}$$

where $\{a_n\}$ and $\{b_n\}$ are some sequences in [0,1].

Let C be a nonempty subset of a real Banach space E and $T_i : C \to C$ be three asymptotically nonexpansive mappings for $i = 1, 2, 3$. Consider the following iterative sequence $\{x_n\}$ with errors defined by

$$(B)\qquad \begin{cases} x_1 \in C, \\ z_n = (1-\gamma_n-\nu_n)x_n + \gamma_n T_1^n x_n + \nu_n u_n, & n \geq 1, \\ y_n = (1-\beta_n-\mu_n)x_n + \beta_n T_2^n z_n + \mu_n v_n, & n \geq 1, \\ x_{n+1} = (1-\alpha_n-\lambda_n)x_n + \alpha_n T_3^n y_n + \lambda_n w_n, & n \geq 1, \end{cases}$$

where $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$, $\sum_{n=1}^{\infty} \lambda_n < +\infty$, $\sum_{n=1}^{\infty} \mu_n < +\infty$ and $\sum_{n=1}^{\infty} \nu_n < +\infty$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in C.

Note that the sequence defined by (B) deduces to the sequence defined by (A).

In this paper, we deal with approximations of common fixed points of the iterative sequences $\{x_n\}$ with errors defined by (B) for three asymptotically nonexpansive mappings in a uniformly convex Banach space. Our results generalize and improve the corresponding results of Das and Debata[2], Khan and Takahashi[5, 6], Schu[8, 9], Takahashi and Tamura[10], and others.

We need the following lemmas for our main results in this paper.

LEMMA 1. [8] *Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying the following:*

$$\limsup_{n\to\infty} \|x_n\| \leq r,\ \limsup_{n\to\infty} \|y_n\| \leq r,\ \lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

We recall that a Banach space E is said to satisfy *Opial's condition* [7] if, for any sequence $\{x_n\}$ in E, $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

LEMMA 2. [4] *Let E be a uniformly convex Banach space satisfying Opial's condition and C be a nonempty closed and convex subset of E. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demi-closed with respect to zero, i.e., for any sequence $\{x_n\}$ in C with $x_n \to x$ weakly and $x_n - Tx_n \to 0$, we have $x = Tx$.*

LEMMA 3. *Let E be a normed linear space and C be a nonempty closed and convex subset of E. Let $T_i : C \to C$ be uniformly L-Lipschitzian mappings for $i = 1, 2, 3$. Let $\{x_n\}$ be the sequence defined by (B). If*

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0, \quad \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0,$$

then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$.

Proof. Since T_i is uniformly L-Lipschitzian mappings for $i = 1, 2, 3$, we have

$$\begin{aligned}
&\|x_{n+1} - T_i x_{n+1}\| \\
&\le \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i x_{n+1}\| \\
&\le c_{n+1} + L\|T_i^n x_{n+1} - x_{n+1}\| \\
&\le c_{n+1} + L\big(\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^n x_{n+1}\|\big) \\
&\le c_{n+1} + L(L+1)\|x_{n+1} - x_n\| + Lc_n \\
&\to 0 \quad (n \to \infty),
\end{aligned}$$

where $c_n = \|x_n - T_i^n x_n\|$. This completes the proof.

REMARK 1. Lemma 3 generalizes the corresponding lemma of Schu [9] for one mapping. Further, if $F = \bigcap_{i=1}^3 F(T_i) \neq \emptyset$ and

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0 \quad (i = 1, 2, 3),$$

then we have $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

LEMMA 4. [11] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

THEOREM 5. *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $T_i : C \to C$ be three mappings for $i = 1, 2, 3$ satisfying*

$$\|T_1^n x - T_1^n y\| \le k_n^{(1)} \|x - y\|,$$

$$\|T_2^n x - T_2^n y\| \le k_n^{(2)} \|x - y\|,$$

$$\|T_3^n x - T_3^n y\| \leq k_n^{(3)} \|x - y\|$$

for all $x, y \in C$ *and* $n \geq 1$, *where* $\{k_n^{(i)}\} \subset [1, \infty)$ *for* $i = 1, 2, 3$ *satisfy* $k_n^{(i)} \to 1$ *as* $n \to \infty$ $(i = 1, 2, 3)$ *and* $\sum_{n=1}^{\infty} \left(\prod_{i=1}^{3} k_n^{(i)} - 1\right) < \infty$. *Let* $\{x_n\}$ *be the sequence defined by* (B). *If* $F = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$, *then we have the following:*

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F$.
(2) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \quad (i = 1, 2, 3)$.

Proof. For any $p \in F$, set

$$M = \max\left\{\sup_{n\geq 1} \|u_n - p\|, \sup_{n\geq 1} \|v_n - p\|, \sup_{n\geq 1} \|w_n - p\|\right\}.$$

By using (B), then we have

$$\begin{aligned} &\|z_n - p\| \\ &\leq (1 - \gamma_n - \nu_n)\|x_n - p\| + \gamma_n k_n^{(1)} \|x_n - p\| + \nu_n \|u_n - p\| \\ &\leq [1 + \gamma_n (k_n^{(1)} - 1)]\|x_n - p\| + M\nu_n \\ &\leq k_n^{(1)} \|x_n - p\| + M\nu_n \end{aligned}$$

and

$$\begin{aligned} &\|y_n - p\| \\ &\leq (1 - \beta_n - \mu_n)\|x_n - p\| + \beta_n k_n^{(2)} \|z_n - p\| + \mu_n \|v_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n k_n^{(1)} k_n^{(2)} \|x_n - v\| + k_n^{(2)} \nu_n M + \mu_n M \\ &\leq k_n^{(1)} k_n^{(2)} \|x_n - p\| + (\nu_n + \mu_n) M_1, \end{aligned}$$

where $M_1 = \sup_{n\geq 1}\{k_n^{(2)}\} M$, and

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\leq (1 - \alpha_n - \lambda_n)\|x_n - p\| + \alpha_n k_n^{(3)} \|y_n - p\| + M\lambda_n \\ &\leq (1 - \alpha_n + \alpha_n k_n^{(1)} k_n^{(2)} k_n^{(3)})\|x_n - p\| + (\lambda_n + \mu_n + \nu_n) M_2 \\ &\leq \left[1 + (k_n^{(1)} k_n^{(2)} k_n^{(3)} - 1)\right] \|x_n - p\| + (\lambda_n + \mu_n + \nu_n) M_2, \end{aligned}$$

where $M_2 = \sup_{n\geq 1}\{k_n^{(3)}\}M_1$. By Lemma 4, we have $\lim_{n\to\infty}\|x_n - p\|$ exists, says $r \geq 0$. In particular, $\{\|x_n - p\|\}$ is bounded and hence so are $\{\|y_n - p\|\}$ and $\{\|T_3^n y_n - p\|\}$. Observe that

$$\begin{aligned} x_{n+1} - p + {} & \lambda_n(T_3^n y_n - w_n) \\ & = (1 - \alpha_n - \lambda_n)(x_n - p) + (\alpha_n + \lambda_n)(T_3^n y_n - p). \end{aligned}$$

Noting that $\lambda_n(T_3^n y_n - w_n) \to 0$ as $n \to \infty$, then we have

$$\lim_{n\to\infty} \|x_{n+1} - p + \lambda_n(T_3^n y_n - w_n)\| = r,$$

which implies that

$$\lim_{n\to\infty} \|(1 - \alpha_n - \lambda_n)(x_n - p) + (\alpha_n + \lambda_n)(T_3^n y_n - p)\| = r,$$

while we have

$$\limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = r,$$

$$\begin{aligned} \limsup_{n\to\infty} \|T_3^n y_n - p\| &\leq \limsup_{n\to\infty} \|y_n - p\| \\ &\leq \limsup_{n\to\infty} [k_n^{(1)} k_n^{(2)} \|x_n - p\| + (\nu_n + \mu_n)M_1] \\ &\leq \lim_{n\to\infty} \|x_n - p\| \\ &= r. \end{aligned}$$

Thus it follows from Lemma 1 that $\lim_{n\to\infty}\|x_n - T_3^n y_n\| = 0$. Since

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - T_3^n y_n\| + \|T_3^n y_n - p\| \\ &\leq \|x_n - T_3^n y_n\| + k_n^{(3)} \|y_n - p\|, \end{aligned}$$

we have $\liminf_{n\to\infty}\|y_n - p\| \geq r$ and hence $\lim_{n\to\infty}\|y_n - p\| = r$. Observe that

$$\begin{aligned} y_n - p + {} & \mu_n(T_2^n z_n - v_n) \\ & = [1 - (\beta_n + \mu_n)](x_n - p) + (\beta_n + \mu_n)(T_2^n z_n - p). \end{aligned}$$

Noting that $\mu_n(T_2^n z_n - v_n) \to 0$ as $n \to \infty$, then we conclude that

$$\lim_{n\to\infty} \|y_n - p + \mu_n(T_2^n z_n - v_n)\| = r,$$

which implies that

$$\lim_{n\to\infty} \|[1-(\beta_n+\mu_n)](x_n-p)+(\beta_n+\mu_n)(T_2^n z_n-p)\| = r,$$

while we have

$$\begin{aligned}\limsup_{n\to\infty} \|T_2^n z_n - p\| &\le \limsup_{n\to\infty}(k_n^{(2)}\|z_n-p\|)\\ &\le \limsup_{n\to\infty}\|z_n-p\|\\ &\le \limsup_{n\to\infty}\|x_n-p\|\\ &= r.\end{aligned}$$

Therefore, by Lemma 1, we assert that $\lim_{n\to\infty}\|x_n - T_2^n z_n\| = 0$. Since

$$\begin{aligned}\|x_n-p\| &\le \|x_n-T_2^n z_n\| + \|T_2^n z_n - p\|\\ &\le \|x_n - T_2^n z_n\| + k_n^{(2)}\|z_n-p\|,\end{aligned}$$

we have $\liminf_{n\to\infty}\|z_n-p\| \ge r$ and hence $\lim_{n\to\infty}\|z_n-p\| = r$. Observe that

$$\begin{aligned}&z_n - p + \nu_n(T_1^n x_n - v_n)\\ &\quad = [1-(\gamma_n+\nu_n)](x_n-p)+(\gamma_n+\nu_n)(T_1^n x_n - p).\end{aligned}$$

Noting that $\nu_n(T_1^n x_n - u_n) \to 0$ as $n\to\infty$, then we conclude that

$$\lim_{n\to\infty}\|z_n - p + \nu_n(T_1^n x_n - u_n)\| = r,$$

which implies that

$$\lim_{n\to\infty}\|[(1-(\gamma_n+\nu_n)](x_n-p)+(\gamma_n+\nu_n)(T_1^n x_n-p)\| = r,$$

while we have

$$\limsup_{n\to\infty}\|T_1^n x_n - p\| \le \limsup_{n\to\infty}(k_n^{(1)}\|x_n-p\|) \le r.$$

Hence, by Lemma 1, we know that $\lim_{n\to\infty}\|x_n - T_1^n x_n\| = 0$. Noting that

$$\begin{aligned}\|z_n-x_n\| &\le \gamma_n\|x_n-T_1^n x_n\| + \nu_n\|x_n-u_n\|\\ &\le \gamma_n\|x_n - T_1^n x_n\| + \nu_n\|x_n-p\| - \nu_n M\end{aligned}$$

and

$$\begin{aligned}\|x_n - T_2^n x_n\| &\le \|x_n - T_2^n z_n\| + \|T_2^n z_n - T_2^n x_n\| \\ &\le \|x_n - T_2^n z_n\| + k_n^{(2)} \|z_n - x_n\|,\end{aligned}$$

then we have

$$\lim_{n\to\infty} \|x_n - T_2^n x_n\| = 0.$$

Similarly, we have also

$$\lim_{n\to\infty} \|x_n - T_3^n x_n\| = 0.$$

On the other hand, by Remark 1, it is clear that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Therefore, by Lemma 3, we can conclude that $\|x_n - T_i x_n\| \to 0$ as $n \to \infty$ for $i = 1, 2, 3$. This completes the proof.

Under additional assumptions, we can prove the following weak and strong convergence theorems:

THEOREM 6. *Let E be a uniformly convex Banach space satisfying Opial's condition and C, T_i $(i = 1, 2, 3)$ and $\{x_n\}$ be as in Theorem 5. If $F = \bigcap_{i=1}^{3} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a common fixed point of T_i $(i = 1, 2, 3)$.*

Proof. Let $p \in F$. Then, by Theorem 5, $\lim_{n\to\infty} \|x_n - p\|$ exists. Assume that $x_{n_i} \to u$ weakly and $x_{n_j} \to v$ weakly as $n \to \infty$. Then $u, v \in F$. We prove that $u = v$. If $u \neq v$, by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{i\to\infty} \|x_{n_i} - u\| \\ &< \lim_{i\to\infty} \|x_{n_i} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| \\ &< \lim_{j\to\infty} \|x_{n_j} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\|,\end{aligned}$$

which is a contradiction. Therefore, we have the conclusion. This completes the proof.

REMARK 2. Theorem 6 includes Theorem 1 of Khan and Takahashi[5] as a special case when $T_1 = I$, where I denotes the identity mapping, $\lambda_n = \mu_n = \nu_n = 0$.

THEOREM 7. *Let all the assumptions of Theorem 6 be satisfied except Opial's condition. In addition, assume that C is a compact convex subset* *of E. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_i for $i = 1, 2, 3$.*

Proof. Since $\{x_n\} \subset C$ and C is compact, we see that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to z$ as $j \to \infty$. Since T_i is continuous, we see that $T_i x_{n_j} \to T_i z$ as $j \to \infty$.

On the other hand, $x_{n_j} - T_i x_{n_j} \to 0$ as $j \to \infty$ and so we have $T_i z = z$, i.e., $z \in F$. However, $\lim_{n\to\infty} \|x_n - z\|$ exists and hence $x_n \to z$ as $n \to \infty$. This completes the proof.

REMARK 3. Theorem 7 improves and extends Theorem 2 of Khan and Takahashi[5] in several aspects.

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