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ON DERIVATIONS IN NONCOMMUTATIVE SEMIPRIME RINGS AND BANACH ALGEBRAS

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Abstract. Let R be a noncommutative semiprime ring. Suppose that there exists a derivation $d : R \to R$ such that for all $x \in R$, either $[[d(x), x], d(x)] = 0$ or $\langle\langle d(x), x\rangle, d(x)\rangle = 0$. In this case $[d(x), x]$ is nilpotent for all $x \in R$. We also apply the above results to a Banach algebra theory.

1. Introduction

Throughout this paper, R will represent an associative ring and the *Jacobson radical* of R will be denoted by $\mathrm{rad}(R)$. We write $[x, y]$ for the *Lie product* $xy - yx$ and $\langle x, y\rangle$ denotes the *Jordan product* $xy + yx$. Recall that R is *semiprime* if $aRa = \{0\}$ implies $a = 0$ and is *prime* if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

Let us introduce the background of our investigation. In 1955, I. M. Singer and J. Wermer obtained a fundamental result which started investigation into the ranges of derivations on Banach algebras [7]. The result states that every continuous derivation on a commutative Banach algebra maps into the Jacobson radical. In the same paper they conjectured that the assumption of continuity is not necessary. This is called the Singer-Wermer conjecture. In 1988, M. P. Thomas[8] proved the conjecture. Since then, a number of authors have presented many noncommutative versions of the Singer-Wermer theorem (see, e.g., [5] and references therein). In particular, B. D. Kim[3, 4] has showed the following result: let A be a noncommutative complex Banach algebra. Suppose that there exists a continuous linear Jordan derivation $d : A \to A$ such that for all $x \in A$, either $[d(x), x]d(x)[d(x), x] \in rad(A)$ or $d(x)[d(x), x]d(x) \in rad(A)$. In this case we have $d(A) \subseteq rad(A)$. Our

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goal in this note is to obtain another noncommutaive versions of the Singer-Wermer theorem.

2. Results

The following lemma is due to J. Vukman[10].

LEMMA 2.1. *Let R be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a + c)xb = 0$ is satisfied for all $x \in R$.*

First, let us prove the next two results in the ring theory in order to apply it to the Banach algebra theory. The proof of these purely algebraic results is elementary without any specific knowledge concerning prime ring.

THEOREM 2.2. *Let R be a noncommutative semiprime ring. Suppose that there exists a derivation $d : R \to R$ such that $[[d(x), x], d(x)] = 0$ holds for all $x \in R$. In this case $[d(x), x]$ is nilpotent for all $x \in R$.*

Proof. Suppose that the relation

$$[[d(x), x], d(x)] = 0,$$

that is,

$$d(x)^2x - 2d(x)xd(x) + xd(x)^2 = 0 \tag{2.1}$$

holds for all $x \in A$. The linearization of (2.1) leads to

$$p_1(x, y) + p_2(x, y) = 0, \quad x, y \in R, \tag{2.2}$$

where $p_k(x, y)$ is the sum of terms involving x and y such that

$$p_k(x, my) = m^k p_k(x, y), \quad k = 1, 2 \text{ and } m \in \mathbb{Z}.$$

Substituting $-y$ for y in (2.2), we obtain by comparing this new result with (2.2) that

$$\begin{aligned} 0 = p_1(x, y) = &-d(x)^2y - d(x)d(y)x - d(y)d(x)x \\ &+2d(x)xd(y) + 2d(x)yd(x) + 2d(y)xd(x) \\ &-xd(x)d(y) - xd(y)d(x) - yd(x)^2, \quad x, y \in R, \end{aligned}$$

which yields

$$\begin{aligned} &d(x)^2y + d(x)d(y)x + d(y)d(x)x \\ &\quad - 2d(x)xd(y) - 2d(x)yd(x) - 2d(y)xd(x) \\ &\quad + xd(x)d(y) + xd(y)d(x) + yd(x)^2 = 0, \quad x, y \in R. \end{aligned} \tag{2.3}$$

Substituting xy for y in (2.3), we then get

$$\begin{aligned} &d(x)^2xy + d(x)xd(y)x + d(x)^2yx + xd(y)d(x)x \\ &\quad + d(x)yd(x)x - 2d(x)x^2d(y) - 2d(x)xd(x)y - 2d(x)xyd(x) \\ &\quad - 2xd(y)xd(x) - 2d(x)yxd(x) + xd(x)xd(y) + xd(x)^2y \\ &\quad + x^2d(y)d(x) + xd(x)yd(x) + xyd(x)^2 = 0, \quad x, y \in R. \end{aligned}$$

In view of (2.1), this relation can be rewritten as

$$\begin{aligned} &d(x)xd(y)x + d(x)^2yx + xd(y)d(x)x + d(x)yd(x)x \\ &\quad - 2d(x)x^2d(y) - 2d(x)xyd(x) \\ &\quad - 2xd(y)xd(x) - 2d(x)yxd(x) \\ &\quad + xd(x)xd(y) + x^2d(y)d(x) \\ &\quad + xd(x)yd(x) + xyd(x)^2 = 0, \quad x, y \in R. \end{aligned} \tag{2.4}$$

Left-multiplying by x in (2.3) and subtracting the result from (2.4), we obtain

$$\begin{aligned} 0 = &-xd(x)^2y + [d(x), x]d(y)x + d(x)^2yx \\ &+d(x)y(d(x)x - 2xd(x)) + (x[d(x), x] - 2[d(x), x]x)d(y) \\ &+(3xd(x) - 2d(x)x)yd(x), \quad x, y \in R. \end{aligned} \tag{2.5}$$

Replacing y by yx in (2.5), we arrive at

$$\begin{aligned} 0 = &-xd(x)^2yx + [d(x), x]d(y)x^2 + [d(x), x]yd(x)x \\ &+d(x)^2yx^2 + d(x)yx(d(x)x - 2xd(x)) \\ &+(x[d(x), x] - 2[d(x), x]x)d(y)x \\ &+(x[d(x), x] - 2[d(x), x]x)yd(x) \\ &+(3xd(x) - 2d(x)x)yxd(x), \quad x, y \in R. \end{aligned} \tag{2.6}$$

Right-multiplying by x in (2.5) and subtracting the result from (2.6), we have

$$\begin{aligned} &[d(x), x]yd(x)x + d(x)y(3xd(x)x - d(x)x^2 - 2x^2d(x)) \\ &\quad + (x[d(x), x] - 2[d(x), x]x)yd(x) \\ &\quad + (2d(x)x - 3xd(x))y[d(x), x] = 0, \quad x, y \in R. \end{aligned} \tag{2.7}$$

Putting $f(x)y$ instead of y in (2.7), we have

$$\begin{aligned} &[d(x), x]d(x)yd(x)x + d(x)^2y(3xd(x)x - d(x)x^2 - 2x^2d(x)) \\ &+ (x[d(x), x] - 2[d(x), x]x)d(x)yd(x) \\ &+ (2d(x)x - 3xd(x))d(x)y[d(x), x] = 0, \quad x, y \in R. \end{aligned} \tag{2.8}$$

Left-multiplying by $d(x)$ in (2.7) and subtracting the result from (2.8), we obtain

$$\begin{aligned}(2.9)\qquad &[[d(x),x],d(x)]yd(x)x+[x[d(x),x]-2[d(x),x]x,d(x)]yd(x)\\ &+[2d(x)x-3xd(x),d(x)]y[d(x),x]=0,\quad x,y\in R.\end{aligned}$$

Calculating the relation (2.9) in view of (2.1), we have

$$(2.10)\qquad [d(x),x]^2yd(x)+[d(x),x]d(x)y[d(x),x]=0,\quad x,y\in R.$$

Substituting $y[d(x),x]$ for y in (2.10), we arrive at

$$[d(x),x]^2y[d(x),x]d(x)+[d(x),x]d(x)y[d(x),x]^2=0,\quad x,y\in R$$

which, from Lemma 2.1, yields the relation

$$(2.11)\qquad [d(x),x]^2y[d(x),x]d(x)=0,\quad x,y\in R.$$

Replacing y by $d(x)y[d(x),x]$ in (2.11), we get

$$[d(x),x]^2d(x)y[d(x),x]^2d(x)=0,\quad x,y\in R.$$

Since R is semiprime, we see that

$$(2.12)\qquad [d(x),x]^2d(x)=0,\quad x\in R.$$

By the hypothesis $[[d(x),x],d(x)]=0$, we also obtain

$$(2.13)\qquad d(x)[d(x),x]^2=0,\quad x\in R.$$

From (2.12), it follows that

$$\begin{aligned}&0=[[d(x),x]^2d(x),x]\\ (2.14)\qquad &\;\;=[d(x),x]^3+[[d(x),x],x][d(x),x]d(x)\\ &\;\;\;+[d(x),x][[d(x),x],x]d(x),\quad x\in R.\end{aligned}$$

Right-multiplying by $[d(x),x]^2$ in (2.14) and combining (2.13) with the result, we have

$$[d(x),x]^5=0,\quad x\in R,$$

which completes the proof of the theorem.

THEOREM 2.3. *Let R be a noncommutative semiprime ring. Suppose that there exists a derivation $d:R\to R$ such that $\langle\langle d(x),x\rangle,d(x)\rangle=0$ holds for all $x\in R$. In this case $[d(x),x]$ is nilpotent for all $x\in R$.*

Proof. Suppose that the functional equation

$$\langle\langle d(x),x\rangle,d(x)\rangle=0$$

holds for all $x\in R$. In other words,

$$(2.15)\qquad d(x)^2x+2d(x)xd(x)+xd(x)^2=0$$

holds for all $x \in A$. The linearization of (2.13) leads to

$$p_1(x,y) + p_2(x,y) = 0, \quad x, y \in R, \tag{2.16}$$

where $p_k(x,y)$ is the sum of terms involving x and y such that

$$p_k(x, my) = m^k p_k(x,y), \quad k = 1,2 \quad \text{and} \quad m \in \mathbb{Z}.$$

Substituting $-y$ for y in (2.16), we obtain by comparing the result with (2.16) that

$$\begin{aligned} p_1(x,y) = {} & d(x)^2 y + d(x)d(y)x + d(y)d(x)x \\ & +2d(x)xd(y) + 2d(x)yd(x) + 2d(y)xd(x) \\ & +xd(x)d(y) + xd(y)d(x) + yd(x)^2 = 0, \quad x, y \in R. \end{aligned} \tag{2.17}$$

Putting xy instead of y in (2.17) and using (2.15), we then get

$$\begin{aligned} & d(x)xd(y)x + d(x)^2 yx + xd(y)d(x)x \\ & +d(x)yd(x)x + 2d(x)x^2 d(y) + 2d(x)xyd(x) \\ & +2xd(y)xd(x) + 2d(x)yxd(x) + xd(x)xd(y) \\ & +x^2 d(y)d(x) + xd(x)yd(x) + xyd(x)^2 = 0, \quad x, y \in A. \end{aligned} \tag{2.18}$$

Left-multiplying by x in (2.17) and subtracting the result from (2.18), we obtain

$$\begin{aligned} 0 = {} & -xd(x)^2 y + [d(x), x]d(y)x + d(x)^2 yx \\ & +d(x)y(d(x)x + 2xd(x)) + (x[d(x), x] + 2[d(x), x]x)d(y) \\ & +(2[d(x), x] + xd(x))yd(x), \quad x, y \in R. \end{aligned} \tag{2.19}$$

Substituting yx for y in (2.19), we arrive at

$$\begin{aligned} 0 = {} & -xd(x)^2 yx + [d(x), x]d(y)x^2 + [d(x), x]yd(x)x \\ & +d(x)^2 yx^2 + d(x)yx(d(x)x + 2xd(x)) \\ & +(x[d(x), x] + 2[d(x), x]x)d(y)x \\ & +(x[d(x), x] + 2[d(x), x]x)yd(x) \\ & +(2[d(x), x] + xd(x))yxd(x), \quad x, y \in R. \end{aligned} \tag{2.20}$$

Right-multiplying by x in (2.19) and subtracting the result from (2.20) we have

$$\begin{aligned} & [d(x), x]yd(x)x + d(x)y[x, d(x)x + 2xd(x)] \\ & +(x[d(x), x] + 2[d(x), x]x)yd(x) \\ & -(2[d(x), x] + xd(x))y[d(x), x] = 0, \quad x, y \in R. \end{aligned} \tag{2.21}$$

Putting $d(x)y$ instead of y in (2.21), we have

$$\begin{aligned}
&[d(x),x]d(x)yd(x)x + d(x)^2y[x,d(x)x+2xd(x)] \\
(2.22)\qquad &+(x[d(x),x]+2[d(x),x]x)d(x)yd(x) \\
&-(2[d(x),x]+xd(x))d(x)y[d(x),x] = 0, \quad x,y \in R.
\end{aligned}$$

Left-multiplying by $f(x)$ in (2.21) and subtracting the result from (2.22), we obtain

$$\begin{aligned}
(2.23)\qquad &[[d(x),x],d(x)]yd(x)x + [x[d(x),x]+2[d(x),x]x,d(x)]yd(x) \\
&-[2[d(x),x]+xd(x),d(x)]y[d(x),x] = 0, \quad x,y \in R.
\end{aligned}$$

Substituting $yd(x)$ for y in (2.23), we arrive at

$$\begin{aligned}
(2.24)\qquad &[[d(x),x],d(x)]yd(x)^2x + [x[d(x),x]+2[d(x),x]x,d(x)]yd(x)^2 \\
&-[2[d(x),x]+xd(x),d(x)]yd(x)[d(x),x] = 0, \quad x,y \in R.
\end{aligned}$$

Right-multiplying by $d(x)$ in (2.23) and subtracting the result from (2.24), we obtain

$$\begin{aligned}
(2.25)\qquad &[[d(x),x],d(x)]yd(x)[d(x),x] \\
&+[xd(x)+2[d(x),x],d(x)]y[[d(x),x],d(x)] = 0, \quad x,y \in R.
\end{aligned}$$

From Lemma 2.1, it follows that for all $y \in R$,

$$(2.26)\qquad ([xd(x)+2[d(x),x],d(x)] + d(x)[d(x),x])y[[d(x),x],d(x)] = 0.$$

Since $[xd(x)+2[d(x),x],d(x)] + d(x)[d(x),x] = [[d(x),x],d(x)]$, the relation (2.26) can be reduced to

$$[[d(x),x],d(x)]y[[d(x),x],d(x)] = 0, \quad x,y \in R.$$

From semiprimeness of R, we conclude that

$$[[d(x),x],d(x)] = 0, \quad x \in R.$$

Hence, Theorem 2.2 gives the result of the theorem.

Let A be a complex Banach algebra and let $a \in A$. The spectral radius of a, denoted by $r(a)$, is defined by $r(a) = \inf\{\|a^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}$. If $r(a) = 0$, then a is said to be *quasinilpotent.*

We first need the following lemma [1, Theorem].

LEMMA 2.4. *Let A be a complex Banach algebra. Suppose that there exists a continuous linear derivation $d : A \to A$ such that $[d(x),x]$ is quasinilpotent for all $x \in A$. In this case we have $d(A) \subseteq \mathrm{rad}(A)$.*

By utilizing two results before, we now obtain the following results for the Banach algebra theory, i.e., noncommutative versions of the Singer-Wermer theorem. The method of the proof is similar to the one of J. Vukman[9].

THEOREM 2.5. *Let A be a noncommutative complex Banach algebra. Suppose that there exists a continuous linear derivation $d : A \to A$ such that $[[d(x), x], d(x)] \in \mathrm{rad}(A)$ for all $x \in A$. In this case we have $d(A) \subseteq \mathrm{rad}(A)$.*

Proof. Following the result of A. M. Sinclair[6], every continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Therefore for every primitive ideal $P \subseteq A$, we can define a linear derivation $d_P : A/P \to A/P$, where A/P is a factor Banach algebra, by $d_P(\hat{x}) = d(x) + P$, $\hat{x} = x + P$ for all $x \in A$. Suppose that A is noncommutative. We first observe that the assumption of the theorem $[[d(x), x], d(x)] \in \mathrm{rad}(A)$, $x \in A$ yields the relation $[[d(\hat{x}), \hat{x}], d(\hat{x})] = 0$, $\hat{x} \in A/P$. Since P is a primitive ideal, the factor algebra A/P is prime and so it is semiprime. From Theorem 2.2, it is immediate that $[d(\hat{x}), \hat{x}]$ is nilpotent and so it is quasinilpotent for all $\hat{x} \in A/P$. We also see that d_P is continuous since A/P is semisimple [2]. Now, all the assumptions of Lemma 2.4 are fulfilled. Thus we obtain that $d_P(A/P) \subseteq \mathrm{rad}(A/P)$. Again using the semisimplicity of A/P, we see that $d_P = 0$ on A/P. In case A/P is commutative, we can conclude that $d_P = 0$ on A/P as well since A/P is semisimple and since we know that there are no nonzero linear derivations on commutative semisimple Banach algebras. In both cases, we obtain $d(A) \subseteq P$ for any primitive ideal P. Since the intersection of all primitive ideals is the Jacobson radical $\mathrm{rad}(A)$, it follows that $d(A) \subseteq \mathrm{rad}(A)$. This completes the proof of the theorem.

THEOREM 2.6. *Let A be a noncommutative complex Banach algebra. Suppose that there exists a continuous linear derivation $d : A \to A$ such that $\langle\langle d(x), x\rangle, d(x)\rangle \in \mathrm{rad}(A)$ for all $x \in A$. In this case we have $d(A) \subseteq \mathrm{rad}(A)$.*

Proof. The derivation d_P on the semiprime factor algebra A/P in the proof of Theorem 2.5 also gives $\langle\langle d_P(\hat{x}), \hat{x}\rangle, d_P(\hat{x})\rangle = 0$ for all $\hat{x} \in A/P$ and so $[d(\hat{x}), \hat{x}]$ is nilpotent by Theorem 2.3. The remainder carry over the same argument as in the proof of Theorem 2.5. Hence the proof of the theorem is completed.

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