Bull. Korean Math. Soc. **42** (2005), No. 4, pp. 739–756

QUASI-VARIATIONAL AND MINIMAX INEQUALITIES AND COLLECTIVELY FIXED POINT RESULTS FOR S-KKM MAPS

Donal O'Regan, Naseer Shahzad, and Ravi P. Agarwal

Abstract. The paper presents new collectively fixed point theorems, minimax and quasi-variational inequalities for maps in the S-KKM class.

1. Introduction

The paper discusses maps in the S-KKM class. In Section 3, we prove new collectively fixed point results for such maps. Some quasi-equilibrium results are obtained in section 4. Various coincidence results, analytic alternatives and minimax inequalities are established in Section 5. Our results improve, extend and complement a number of known results in the literature, in particular those in [1, 2, 5, 9] and the references therein.

2. Preliminaries

Let X and Y be Hausdorff topological vector spaces. Recall a polytope P in X is any convex hull of a nonempty finite subset of X. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (the nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;
(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued;

Received June 04, 2004.
2000 Mathematics Subject Classification: 47H10.
Key words and phrases: collectively fixed point theorems, minimax, quasi-variational inequalities, quasi-equilibrium, S-KKM class.

(iii) for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

DEFINITION 2.1. $F \in \mathcal{U}_c^\kappa(X,Y)$ (i.e., F is $\mathcal{U}_c^\kappa$–admissible) if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

DEFINITION 2.2. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S, T : X \to 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map S, the family

$$\{\overline{S(x)} : x \in X\}$$

has the finite intersection property. We let

$$\text{KKM}(X,Y) = \{T : X \to 2^Y : T \text{ has the KKM property}\}.$$

REMARK 2.1. If X is a convex space, then $\mathcal{U}_c^\kappa(X,Y) \subset \text{KKM}(X,Y)$ (see [7]).

DEFINITION 2.3. Let X be a nonempty set, Y a nonempty convex subset of a Hausdorff topological vector space and Z a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$, $F : X \to 2^Z$ are three set-valued maps such that $T(\text{co}(S(A))) \subseteq F(A)$ for each nonempty finite subset A of X, then F is called a generalized S-KKM map w.r.t. T. If the map $T : X \to 2^Z$ is such that for any generalized S-KKM w.r.t. T map F, the family

$$\{\overline{F(x)} : x \in X\}$$

has the finite intersection property, then T is said to have the S-KKM property. The class

$$S\text{-KKM}(X,Y,Z) = \{T : Y \to 2^Z : T \text{ has the } S\text{-KKM property}\}.$$

REMARK 2.2. Note that S-KKM$(X,Y,Z) = $ KKM(X,Z), whenever $X = Y$ and S is the identity mapping $\mathbf{1}_X$. Moreover, KKM(Y,Z) is a proper subset of S-KKM(X,Y,Z) for any $S : X \to 2^Y$. S-KKM(X,Y,Z) also includes other important classes of multimaps (see [5, 6] for examples).

REMARK 2.3. Let X be a convex space, Y a convex subset of a Hausdorff locally convex space, and Z a normal space. Suppose $s : Y \to Y$ is surjective, $F \in s$-KKM(Y,Y,Z) is closed, and $f \in \mathcal{C}(X,Y)$. Then $F \circ f \in \mathbf{1}_X$-KKM$(X,X,Z)$ (see [6]).

REMARK 2.4. Let X be a convex subset of a Hausdorff topological space, Y a convex space, and Z, W topological spaces and $S : X \to 2^Y$. If $F : S\text{-KKM}(X, Y, Z)$ and $f \in \mathcal{C}(Z, W)$, then $f \circ F \in S\text{-KKM}(X, Y, W)$ (see [6]).

Let (E, d) be a pseudometric space. For any $C \subseteq E$, let $B(C, \epsilon) = \{x \in E : d(x, C) \leq \epsilon\}$, here $\epsilon > 0$ The measure of noncompactness of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$, where

$$Q(M) = \{\epsilon > 0 : M \subseteq B(A, \epsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let C be a subset of a locally convex Hausdorff topological vector space E, and let P be a defining system of seminorms on E. Suppose $F : C \to 2^E$. Then F is called countably P-concentrative mapping if $F(C)$ is bounded, and for $p \in P$ and each countably bounded subset S of C, we have $\alpha_p(F(S)) \leq \alpha_p(S)$, and for $p \in P$ for each countably bounded non-p-paracompact subset S of C (i.e., S is not precompact in the pseudonormed space (E, p)) we have $\alpha_p(F(S)) < \alpha_p(S)$; here $\alpha_p(.)$ denotes the measure of noncompactness in the pseudonormed space (E, p).

Let Q be a subset of a Hausdorff topological space X. We let $\overline{Q}$ (respectively, ∂Q, $\text{int}(Q)$) denote the closure (respectively, boundary, interior) of Q.

DEFINITION 2.4. Let Z and W be subsets of Hausdorff topological vector spaces E_1 and E_2 and F a set-valued map. We say that $F \in PK(Z, W)$ if W is convex, and there exists a map $S : Z \to W$ with

$$Z = \bigcup\{\text{int} S^{-1}(w) : w \in W\}, \ \text{co}(S(x)) \subset F(x) \ \text{ for } x \in Z$$

and $S(x) \neq \emptyset$ for each $x \in Z$; here $S^{-1}(w) = \{z : w \in S(z)\}$.

REMARK 2.5. Suppose Z is paracompact, W is convex, and $F \in PK(Z, W)$. Then there exists a continuous (single valued) mapping $f : Z \to W$ such that $f(x) \in F(x)$ for each $x \in Z$ (see [9]).

A nonempty subset X of a Hausdorff topological vector space E is said to be admissible if for every compact subset K of X and every neighborhood V of 0, there exists a continuous map $h : K \to X$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. X is called q-admissible if any nonempty compact convex subset Ω of X is admissible.

The following results [3, 5, 8] will be needed in the sequel.

THEOREM 2.1. *Let Ω be an admissible convex subset of a Hausdorff topological vector space E and X a nonempty subset of Ω. Suppose $s : X \to \Omega$ is surjective and $F \in s\text{-}KKM(X,\Omega,\Omega)$ is compact and closed. Then F has a fixed point in Ω.*

THEOREM 2.2. *Let Ω be a q-admissible closed convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s\text{-}KKM(\Omega,\Omega,\Omega)$ is closed with the following property holding:*

$$A \subseteq \Omega,\ A = \overline{\text{co}}\,(\{x_0\} \cup F(A)) \ \text{ implies } \ A \ \text{ is compact.} \tag{2.1}$$

Then F has a fixed point in Ω.

THEOREM 2.3. *Let Ω be a q-admissible closed convex subset of a Hausdorff topological vector space E and $x_0 \in \Omega$. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s\text{-}KKM(\Omega,\Omega,\Omega)$ is closed and satisfies the following properties:*

$$A \subseteq \Omega,\ A = \text{co}\,(\{x_0\} \cup F(A)) \ \text{ implies } \ \overline{A} \ \text{ is compact} \tag{2.2}$$

and

$$F(\overline{A}) \subseteq \overline{F(A)} \ \text{ for any relatively compact subset } \ A \ \text{ of } \ \Omega. \tag{2.3}$$

Then F has a fixed point in Ω.

THEOREM 2.4. *Let Ω be a q-admissible closed convex subset of a Hausdorff topological vector space E and $x_0 \in \Omega$. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s\text{-}KKM(\Omega,\Omega,\Omega)$ is closed, maps compact sets into relatively compact sets, satisfies (2.3) and suppose the following properties hold:*

$$\begin{cases} A \subseteq \Omega,\ A = \text{co}\,(\{x_0\} \cup F(A)) \ \text{ with } \ \overline{A} = \overline{C} \\ \text{and } \ C \subseteq A \ \text{ countable, implies } \ \overline{A} \ \text{ is compact,} \end{cases} \tag{2.4}$$

$$\begin{cases} \text{for any relatively compact subset } \ A \ \text{ of } \ \Omega \ \text{ there} \\ \text{exists a countable set } \ B \subseteq A \ \text{ with } \ \overline{B} = \overline{A} \end{cases} \tag{2.5}$$

and

$$\text{if } \ A \ \text{ is a compact subset of } \ \Omega, \ \text{ then } \ \overline{\text{co}}\,(A) \ \text{ is compact.} \tag{2.6}$$

Then F has a fixed point in Ω.

REMARK 2.5. It is worth noting that if E is metrizable then (2.5) holds, and if E is quasicomplete then (2.6) holds.

REMARK 2.6. Following arguments similar to those given in [3, Theorem 2.3], one can remove (2.3) in Theorem 2.3 and Theorem 2.4 for certain subclasses of s-KKM maps. We also refer the reader to the next theorem.

THEOREM 2.5. *Let Ω be a q-admissible closed convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $s : \Omega \to \Omega$ is surjective and $F \in \mathcal{A}(\Omega, \Omega, \Omega)$ is a closed map satisfying* (2.3) *and assume the following properties hold:*

(2.7)
$$\begin{cases} \text{for any relatively compact, convex subset } A \text{ of } \Omega \text{ we have} \\ F^* \in s\text{-}KKM(s^{-1}(\overline{A}), \overline{A}, \overline{A}) \text{ if } F^*(x) \neq \emptyset \text{ for } x \in \overline{A}; \\ \text{here } F^*(x) = F(x) \cap \overline{A} \text{ for } x \in \overline{A}. \end{cases}$$

Then F has a fixed point in Ω.

THEOREM 2.6. *Let Ω be a nonempty closed convex subset of a Fréchet space E* (P is a defining system of seminorms). *Suppose $s : \Omega \to \Omega$ is surjective and $F \in s$-KKM(Ω, Ω) is a closed countably P-concentrative map. Then F has a fixed point in Ω.*

THEOREM 2.7. *Let Q be a closed convex subset of a metrizable locally convex topological vector space E with $0 \in Q$. Suppose $s : Q \to Q$ is surjective and $F \in s$-KKM$(Q, Q, E)$is a closed compact map with the following condition satisfied:*

$$\text{(2.8)} \quad \begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } x \in \lambda F(x) \text{ and } 0 \leq \lambda < 1, \text{ then} \\ \{\lambda_j F(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then F has a fixed point in Q.

In the following result, by a countably condensing map F we mean $\alpha(F(B)) < \alpha(B)$ for all countably bounded sets B of C with $\alpha(B) \neq 0$ and $\alpha(F(D)) \leq \alpha(D)$ for all countably bounded sets D of C, where $\alpha(.)$ is the Kuratowski measure of noncompactness.

THEOREM 2.8. *Let Q be a closed convex subset of a Hilbert space E with $0 \in Q$. Suppose $s : Q \to Q$ is surjective and $F \in s$-KKM(Q, Q, E) is a closed countably condensing map and assume* (2.8) *holds. Then F has a fixed point in Q.*

3. Collectively fixed points

We begin with the following result.

THEOREM 3.1. *Let I be an index set and let $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space E_i. For each $i \in I$,* *let K_i be a nonempty compact subset of X_i and suppose $F_i : X = \Pi_{i \in I} X_i \to 2^{K_i}$. In addition, assume X is an admissible convex subset of the Hausdorff topological vector space $E = \Pi_{i \in I} E_i$. Suppose $s : X \to X$ is surjective and $F \in s$-KKM(X, X, X) is a closed map; here $F(x) = \Pi_{i \in I} F_i(x)$ for $x \in X$. Then there exists $x \in K = \Pi_{i \in I} K_i$ with $x_i \in F(x_i)$ for each $i \in I$* (here x_i denotes the projection of x on X_i).

Proof. Clearly F is compact since K is compact. Now Theorem 2.1 guarantees that there exists $x \in X$ with $x \in F(x)$. Notice $x \in K$ since $F(X) \subseteq K$.

COROLLARY 3.2. *Let I be an index set and let $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space E_i. For each $i \in I$, let K_i be a nonempty compact subset of X_i and suppose $F_i : X = \Pi_{i \in I} X_i \to 2^{K_i}$. In addition, assume X is an admissible convex subset of the Hausdorff topological vector space $E = \Pi_{i \in I} E_i$. Suppose $F \in$ KKM(X, X) is a closed map; here $F(x) = \Pi_{i \in I} F_i(x)$ for $x \in X$. Then there exists $x \in K = \Pi_{i \in I} K_i$ with $x_i \in F(x_i)$ for each $i \in I$.*

Proof. Since $F \in s$-KKM(X, X, X) for any map $s : X \to X$, Theorem 3.1 guarantees that there exists $x \in K$ with $x \in F(x)$.

COROLLARY 3.3. *Let I be an index set and let $\{X_i\}_{i \in I}$ be a family of sets each in a Hausdorff topological vector space E_i. For each $i \in I$, let K_i be a nonempty compact subset of X_i and suppose $F_i : X = \Pi_{i \in I} X_i \to 2^{K_i}$. In addition, assume X is an admissible convex subset of the Hausdorff topological vector space $E = \Pi_{i \in I} E_i$. Suppose $F \in \mathcal{U}_c^{\kappa}(X, X)$ is a closed map; here $F(x) = \Pi_{i \in I} F_i(x)$ for $x \in X$. Then there exists $x \in K = \Pi_{i \in I} K_i$ with $x_i \in F(x_i)$ for each $i \in I$.*

Proof. Since $F \in \mathcal{U}_c^{\kappa}(X, X) \subset$ KKM(X, X) is a closed map by Remark 2.1, now Corollary 3.2 guarantees that there exists $x \in K$ with $x \in F(x)$.

THEOREM 3.4. *Let I be an index set and let $\{X_i\}_{i \in I}$ be a family of closed convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$, suppose $F_i : X = \Pi_{i \in I} X_i \to 2^{X_i}$. In addition assume X is a q-admissible closed convex subset of the Hausdorff topological vector space $E = \Pi_{i \in I} E_i$, $x_0 \in X$. Suppose $s : X \to X$ is surjective and $F \in s$-KKM(X, X, X) is a closed map and assume the following property holds:*

$$A \subseteq X, \ \ A = \overline{\text{co}}\,(\{x_0\} \cup F(A)) \ \text{implies } A \text{ is compact}; \tag{3.1}$$

here $F(x) = \Pi_{i \in I} F_i(x)$ for $x \in X$. Then there exists $x \in X$ with $x_i \in F_i(x)$ for each $i \in I$.

Proof. Since F satisfies (3.1), Theorem 2.2 guarantees that there exists $x \in X$ with $x \in F(x)$.

THEOREM 3.5. *Let I be an index set and let $\{X_i\}_{i\in I}$ be a family of closed convex sets each in a Fréchet space E_i. For each $i \in I$, assume $F_i : X = \Pi_{i\in I} X_i \to 2^{X_i}$. In addition, suppose $s : X \to X$ is surjective and $F \in s\text{-}KKM(X, X, X)$ is a closed countably P-concentrative map; here $E = \Pi_{i\in I} E_i$ with P a defining system of seminorms and $F(x) = \Pi_{i\in I} F_i(x)$ for $x \in X$. Then there exists $x \in X$ with $x_i \in F(x_i)$ for each $i \in I$.*

Proof. Apply Theorem 2.6 to F.

THEOREM 3.6. *Let I be an index set and let $\{X_i\}_{i\in I}$ be a family of closed convex sets each in a metrizable locally convex topological vector space E_i with $0 \in X = \Pi_{i\in I} X_i$. For each $i \in I$, assume $F_i : X = \Pi_{i\in I} X_i \to 2^{E_i}$. Suppose $s : X \to X$ is surjective and $F \in s\text{-}KKM(X, X, E)$ is a closed compact map with the following condition satisfied:*

$$(3.2)\quad \begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial X \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } x \in \lambda F(x) \text{ and } 0 \le \lambda < 1, \text{ then} \\ \{\lambda_j F(x_j)\} \subseteq X \text{ for } j \text{ sufficiently large;} \end{cases}$$

here $E = \Pi_{i\in I} E_i$ and $F(x) = \Pi_{i\in I} F_i(x)$ for $x \in X$. Then there exists $x \in X$ with $x_i \in F(x_i)$ for each $i \in I$.

Proof. Apply Theorem 2.7 to F.

THEOREM 3.7. *Let I be an index set and let $\{X_i\}_{i\in I}$ be a family of closed convex sets each in a Hilbert space H_i with $0 \in X = \Pi_{i\in I} X_i$. For each $i \in I$, suppose $F_i : X = \Pi_{i\in I} X_i \to 2^{H_i}$. In addition, suppose $s : X \to X$ is surjective and $F \in s\text{-}KKM(X, X, H)$ is a closed countably condensing map with (3.2) holding; here $H = \Pi_{i\in I} H_i$. Then there exists $x \in X$ with $x_i \in F(x_i)$ for each $i \in I$.*

Proof. Apply Theorem 2.8 to F.

4. Quasi-variational inequalities

In this section we formulate some results in Section 2 as quasi-variatio nal inequalities.

THEOREM 4.1. *Let E and Y be Hausdorff topological vector spaces, Q a closed convex subset of E, $G : Q \to K(Q)$ and $T : Q \to 2^C$ with* *closed values, where C is a closed convex subset of Y. In addition assume the following conditions hold:*

(4.1) $f : Q \times C \times Q \to \mathbf{R}$ *is an upper semicontinuous function*

(4.2) $\begin{cases} Q \times C \text{ is an admissible subset of the Hausdorff} \\ \text{topological vector space } E \times Y \end{cases}$

(4.3) $\begin{cases} F \in s\text{-}KKM(Q \times C, Q \times C, Q \times C) \text{ is a closed map;} \\ \text{here } s : Q \times C \to Q \times C \text{ is surjective} \end{cases}$

and

(4.4) G *and* T *are compact maps;*

here $F(x,y) = \Phi(x,y) \times T(x)$ *for* $(x,y) \in Q \times C$ *with*

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. *Then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

Proof. It is clear that $\Phi(x,y)$ is nonempty and compact for each $(x,y) \in Q \times C$ (see [4, p.44]). Consequently, $F : Q \times C \to 2^{Q \times C}$ is compact from (4.4). Now Theorem 2.1 guarantees that there exists $(x_0, y_0) \in Q \times C$ with $x_0 \in G(x_0)$, $y_0 \in T(x_0)$ and $f(x_0, y_0, x_0) = M(x_0, y_0)$. As a result, we have

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

COROLLARY 4.2. *Let E and Y be Hausdorff topological vector spaces, Q a closed convex subset of E, $G : Q \to K(Q)$ and $T : Q \to 2^C$ with closed values, where C is a closed convex subset of Y. Suppose* (4.1), (4.2), (4.4) *hold and in addition assume the following condition holds:*

(4.5) $F \in \mathcal{U}_c^{\kappa}(Q \times C, Q \times C)$ *is a closed map;*

here $F(x,y) = \Phi(x,y) \times T(x)$ *for* $(x,y) \in Q \times C$ *with*

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. *Then there exists* $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

THEOREM 4.3. *Let E and Y be Hausdorff topological vector spaces, Q a closed convex subset of E, $G : Q \to K(Q)$ and $T : Q \to 2^C$ with closed values, where C is a closed convex subset of Y. Suppose (4.1) holds and in addition assume the following conditions are satisfied:*

$$(4.6)\quad \begin{cases} Q \times C \text{ is a } q\text{-admissible subset of the Hausdorff} \\ \text{topological vector space } E \times Y \text{ and assume } z_0 \in Q \times C \end{cases}$$

$$(4.7)\quad \begin{cases} F \in s\text{-}KKM(Q \times C, Q \times C, Q \times C) \text{ is a closed map;} \\ \text{here } s : Q \times C \to Q \times C \text{ is surjective.} \end{cases}$$

and

$$(4.8)\qquad A \subseteq Q \times C,\ \ A = \overline{co}\,(\{z_0\} \cup F(A)) \text{ implies } A \text{ is compact;}$$

here $F(x,y) = \Phi(x,y) \times T(x)$ *for* $(x,y) \in Q \times C$ *with*

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. *Then there exists* $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with

$$f(x_0,y_0,z) \le f(x_0,y_0,x_0) \text{ for all } x \in G(x_0).$$

Proof. Notice $\Phi(x,y)$ is nonempty and compact for each $(x,y) \in Q \times C$, $F : Q \times C \to 2^{Q \times C}$. Now Theorem 2.2 guarantees that there exists $(x_0,y_0) \in Q \times C$ with $x_0 \in G(x_0)$, $y_0 \in T(x_0)$ and $f(x_0,y_0,x_0) = M(x_0,y_0)$. As a result, we have

$$f(x_0,y_0,z) \le f(x_0,y_0,x_0) \text{ for all } z \in G(x_0).$$

In our next results, we consider $\mathcal{A}$ a subclass of KKM maps. Let X and Y be subsets of Hausdorff topological vector spaces. If $F \in \mathrm{KKM}(X,Y)$ is upper semicontinuous with nonempty compact values and satisfies property (C) (will be specified in examples), we say $F \in \mathcal{A}(X,Y)$. The class $\mathcal{A}$ must have the following property also:

For subsets X_1 and X_2 of Hausdorff topological vector spaces

$$(4.9)\qquad \begin{aligned} &F_1 \in \mathcal{A}(X_1 \times X_2, X_1),\ F_2 \in \mathcal{A}(X_1, X_2) \text{ implies} \\ &F_3 \in \mathrm{KKM}(X_1 \times X_2, X_1 \times X_2); \end{aligned}$$

here $F_3(x,y) = F_1(x,y) \times F_2(x)$. The class of acyclic maps (i.e., property (C) means the maps have acyclic values) and the class of admissible maps with respect to Gorniewicz (if the spaces are metric spaces) are examples of the class $\mathcal{A}$

THEOREM 4.4. *Let E and Y be Hausdorff topological vector spaces, Q a closed convex subset of E, $G : Q \to K(Q)$ and $T : Q \to K(C)$, where C is a closed convex subset of Y. Suppose (4.1) and (4.9) hold and in addition assume the following conditions are satisfied:*

(4.10) $G : Q \to 2^Q$ *is upper semicontinuous*

(4.11) $M : Q \times C \to Q$ *is lower semicontinuous* (*here* $M(x,y) = \max_{w \in G(x)} f(x,y,w)$)

(4.12) $\Phi \in \mathrm{KKM}(Q \times C, Q)$ *and satisfies property* (C)

(4.13)
$T \in \mathrm{KKM}(Q,C)$ *is upper semicontinuous and satisfies property* (C)

(4.14) $\left\{ \begin{array}{l} Q \times C \text{ is a } q\text{-admissible subset of the Hausdorff} \\ \text{topological vector space } E \times Y \text{ and assume } z_0 \in Q \times C \end{array} \right.$

and

(4.15) $A \subseteq Q \times C,\ A = \overline{co}\,(\{z_0\} \cup F(A))$ *implies* A *is compact;*

here $F(x,y) = \Phi(x,y) \times T(x)$ *for* $(x,y) \in Q \times C$ *with*

$$\Phi(x,y) = \{w \in G(x) : f(x,y,w) = M(x,y)\}$$

and $M(x,y) = \max_{w \in G(x)} f(x,y,w)$. *Then there exists* $(x_0,y_0) \in Q \times C$, $x_0 \in G(x_0)$ *and* $y_0 \in T(x_0)$ *with*

$$f(x_0,y_0,z) \le f(x_0,y_0,x_0) \text{ for all } z \in G(x_0).$$

Proof. First we show that Φ is closed. Let $\{(x_\alpha,y_\alpha,w_\alpha)\}$ be a net in graph(Φ) with $(x_\alpha,y_\alpha,w_\alpha) \to (x,y,w)$. Notice (4.1), (4.10), and [4, p.473] together imply that M is upper semicontinuous and so is continuous by (4.11). Consequently,

$$f(x,y,w) \ge \limsup f(x_\alpha,y_\alpha,w_\alpha) = \limsup M(x_\alpha,y_\alpha) = M(x,y).$$

Since G is upper semicontinuous (so G is closed [4, p.465]) and $w_\alpha \in G(x_\alpha)$, it follows that $w \in G(x)$ and $f(x,y,w) \ge M(x,y)$. As a result, $f(x,y,w) = M(x,y)$ and so $(x,y,w) \in$ graph(Φ). Thus Φ is closed. Next we show $\Phi \in \mathcal{A}(Q \times C, Q)$. For this, we need to show that Φ is upper semicontinuous. Notice

$$\Phi(x,y) = G(x) \cap \Lambda(x,y),$$

where

$$\Lambda(x,y) = \{w \in Q : f(x,y,w) = M(x,y)\}.$$

As above Λ is closed. Since $G : Q \to K(Q)$ is upper semicontinuous, then [4, p.470] implies Φ is upper semicontinuous. Hence $\Phi \in \mathcal{A}(Q \times C, Q)$ and so, by (4.9), $F \in \mathrm{KKM}(Q \times C, Q \times C)$.

Now Theorem 4.3 guarantees that there exists $(x_0, y_0) \in Q \times C$ with $x_0 \in G(x_0)$, $y_0 \in T(x_0)$ and

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } x \in G(x_0).$$

THEOREM 4.5. *Let E and Y be Hausdorff topological vector spaces, Q a closed convex subset of E, $G : Q \to K(Q)$ and $T : Q \to K(C)$ where C is a closed convex subset of Y. Suppose* (4.1), (4.9), (4.10), (4.11), (4.12), *and* (4.13) *hold and in addition assume the following conditions are satisfied*:

$$(4.15) \qquad \begin{cases} Q \times C \text{ is an admissible subset of the Hausdorff} \\ \text{topological vector space } E \times Y \end{cases}$$

and

$$(4.16) \qquad G \text{ and } T \text{ are compact maps;}$$

here $F(x, y) = \Phi(x, y) \times T(x)$ for $(x, y) \in Q \times C$ with

$$\Phi(x, y) = \{w \in G(x) : f(x, y, w) = M(x, y)\}$$

and $M(x, y) = \max_{w \in G(x)} f(x, y, w)$. Then there exists $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

Proof. As above $\Phi \in \mathcal{A}(Q \times C, Q)$ and so, by (4.9), $F \in \mathrm{KKM}(Q \times C, Q \times C, Q \times C)$. Now Theorem 4.1 guarantees that there exists $(x_0, y_0) \in Q \times C$ with $x_0 \in G(x_0)$, $y_0 \in T(x_0)$ and

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

REMARK 4.1. Using arguments similar to those given above, it is possible to obtain quasi-variational analogues of Theorems 2.3–2.8.

5. Coincidence and minimax inequalities

We begin this section with some coincidence theorems.

THEOREM 5.1. *Let Ω be a paracompact admissible subset of a Hausdorff topological vector space and Y a convex subset of a Hausdorff topological vector space. In addition, we assume Y is a normal space.*

Suppose $s : Y \to Y$ is surjective and $F \in s\text{-}KKM(Y, Y, \Omega)$ is a compact closed map and $G \in PK(\Omega, Y)$ is a compact map. Then G and F^{-1} have a coincidence, that is, there exists $(x_0, y_0) \in \Omega \times Y$ with $y_0 \in G(x_0) \cap F^{-1}(x_0)$ (i.e., there exists $(x_0, y_0) \in \Omega \times Y$ with $y_0 \in G(x_0)$ and $x_0 \in F(y_0)$).

Proof. Remark 2.5 guarantees that there exists a continuous selection $g : \Omega \to Y$ of G. Notice $J = F \circ g \in \text{KKM}(\Omega, \Omega)$ is a compact closed map by Remark 2.3. Now Theorem 2.1 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in J(x_0)$. Set $y_0 = g(x_0)$. Then $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$.

THEOREM 5.2. *Let Ω be a convex admissible subset of a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological vector space. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s\text{-}KKM(\Omega, \Omega, Y)$ is a upper semicontinuous compact map with closed values and $G \in PK(Y, \Omega)$ is a compact map. Then F and G^{-1} have a coincidence.*

Proof. Remark 2.5 guarantees that there exists a continuous selection $g : Y \to \Omega$ of G. Notice $J = g \circ F \in s\text{-KKM}(\Omega, \Omega, \Omega)$ is a compact closed map by Remark 2.4. Now Theorem 2.1 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in J(x_0)$. So there exists $y_0 \in F(x_0)$ with $x_0 = g(y_0)$. Notice $x_0 \in G(y_0)$.

COROLLARY 5.3. *Let Ω be a convex admissible subset of a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological vector space. Suppose $F \in \mathcal{U}_c^{\kappa}(\Omega, Y)$ is an upper semicontinuous compact map with closed values and $G \in PK(Y, \Omega)$ is a compact map. Then F and G^{-1} have a coincidence.*

THEOREM 5.4. *Let Ω be a nonempty closed convex subset of a Fréchet space E* (P is a defining system of seminorms) *and Y a convex subset of E. Suppose $s : Y \to Y$ is surjective and $F \in s\text{-}KKM(Y, Y, \Omega)$ is a closed countably P-concentrative map and $G \in PK(\Omega, Y)$ is a countably P-concentrative map. Then G and F^{-1} have a coincidence.*

Proof. As above there exists a continuous selection $g : \Omega \to Y$ of G. Notice $J = F \circ g \in \text{KKM}(\Omega, \Omega)$ is a closed countably P-concentrative map. Now Theorem 2.6 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in J(x_0)$. Set $y_0 = g(x_0)$. Then $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$.

THEOREM 5.5. *Let Ω be a nonempty closed convex subset of a metrizable locally convex topological vector space E with $0 \in \Omega$ and Y a convex subset of E. Suppose $s : Y \to Y$ is surjective and $F \in s\text{-}KKM(Y, Y, E)$*

is a closed compact map and $G \in PK(\Omega, Y)$ is a compact map. Suppose the following condition holds:

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial\Omega \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } x \in \lambda FG(x) \text{ and } 0 \le \lambda < 1, \text{ then} \\ \{\lambda_j FG(x_j)\} \subseteq \Omega \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then G and F^{-1} have a coincidence.

Proof. As before there exists a continuous selection $g : \Omega \to Y$ of G. Notice $J = F \circ g \in \mathrm{KKM}(\Omega, E)$ is a compact closed map. Now Theorem 2.7 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in J(x_0)$. Set $y_0 = g(x_0)$. Then $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$.

THEOREM 5.6. *Let Ω be a nonempty closed convex subset of a metrizable locally convex topological vector space E with $0 \in \Omega$ and Y a subset of E. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s\text{-}KKM(\Omega, \Omega, Y)$ is a compact upper semicontinuous map with closed values and $G \in PK(Y, E)$ is a compact map. Suppose the following condition holds:*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial\Omega \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } x \in \lambda GF(x) \text{ and } 0 \le \lambda < 1, \text{ then} \\ \{\lambda_j GF(x_j)\} \subseteq \Omega \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then F and G^{-1} have a coincidence.

Proof. As above there exists a continuous selection $g : Y \to E$ of G. Notice $J = g \circ F \in s\text{-KKM}(\Omega, \Omega, E)$ is a compact closed map. Now Theorem 2.7 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in J(x_0)$. Thus there exists $y_0 \in F(x_0)$ with $x_0 = g(y_0)$. Notice $x_0 \in G(y_0)$.

THEOREM 5.7. *Let Ω be a nonempty closed convex subset of a Hilbert space E with $0 \in \Omega$ and Y a convex subset of H. Suppose $s : Y \to Y$ and $F \in s\text{-}KKM(Y, Y, H)$ is a closed countably condensing map and $G \in PK(\Omega, Y)$ is a countably condensing map. Suppose the following condition holds:*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial\Omega \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } x \in \lambda FG(x) \text{ and } 0 \le \lambda < 1, \text{ then} \\ \{\lambda_j FG(x_j)\} \subseteq \Omega \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then G and F^{-1} have a coincidence.

Proof. As before there exists a continuous selection $g : \Omega \to Y$ of G. Notice $J = F \circ g \in \mathrm{KKM}(\Omega, \Omega, H)$ is a closed countably condensing map. Now Theorem 2.8 guarantees that there exists $x_0 \in \Omega$ with $x_0 \in J(x_0)$. Set $y_0 = g(x_0)$. Then $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$.

We now prove new analytic alternatives.

THEOREM 5.8. *Let Ω be a paracompact admissible subset of a Hausdorff topological vector space and Y a convex subset of a Hausdorff topological vector space. In addition, we assume Y is a normal space. Let $f, g : \Omega \times Y \to \mathbf{R}$ be such that*

$$g(x,y) \le f(x,y) \text{ for all } (x,y) \in \Omega \times Y. \tag{5.1}$$

Fix $\alpha \in \mathbf{R}$ and let

$$G(x) = \{y \in Y : f(x,y) > \alpha\}$$

and

$$F(y) = \{x \in \Omega : g(x,y) \le \alpha\}.$$

Suppose $s : Y \to Y$ is surjective and $F \in s$-KKM(Y, Y, Ω) is a closed map. Also assume if $G(x) \ne \emptyset$ for every $x \in \Omega$, then $G \in PK(\Omega, Y)$. If F and G are compact maps then either

(A1) *there exists $z_0 \in \Omega$ with $f(z_0, y) \le \alpha$ for all $y \in Y$, or*

(A2) *there exists $(x_0, y_0) \in \Omega \times Y$ with $g(x_0, y_0) \le \alpha < f(x_0, y_0)$ hold.*

Proof. If $G(x) \ne \emptyset$ for every $x \in \Omega$, then $G \in PK(\Omega, Y)$. Now Theorem 5.1 guarantees that there exists $(x_0, y_0) \in \Omega \times Y$ with $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$ and so (A2) holds. If $G(x) \ne \emptyset$ for every $x \in \Omega$ does not hold, then there exists $z_0 \in \Omega$ with $G(z_0) = \emptyset$. This implies that $f(z_0, y) \le \alpha$ for every $y \in Y$ and so (A1) holds.

REMARK 5.1. If $g \equiv f$ in Theorem 5.8, then $(A2)$ does not hold.

REMARK 5.2. If we change $F \in s$-KKM(Y, Y, Ω) in Theorem 5.8 to if $F(y) \ne \emptyset$ for every $y \in Y$ then $F \in s$-KKM(Y, Y, Ω), then the conclusion in Theorem 5.8 is that either (A1), (A2) or (A3). there exists $w_0 \in Y$ with $g(x, w_0) > \alpha$ for all $x \in \Omega$ hold. Notice in this case if there exists $w_0 \in Y$ with $F(w_0) = \emptyset$, then $G(x) \ne \emptyset$ for every $x \in \Omega$ because if there exists $z_0 \in \Omega$ with $G(z_0) = \emptyset$, then we get a contradiction to (5.1).

Essentially the same reasoning as in Theorem 5.8 establishes the following result (here we use Theorem 5.2).

THEOREM 5.9. *Let Ω be a convex admissible subset of a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological vector space. Let $f, g : \Omega \times Y \to \mathbf{R}$ be such that* (5.1) *holds. Fix $\alpha \in \mathbf{R}$ and let*

$$G(y) = \{x \in \Omega : f(x,y) > \alpha\}$$

and

$$F(x) = \{y \in Y : g(x,y) \le \alpha\}.$$

Suppose $s : \Omega \to \Omega$ is surjective and $F \in s\text{-}KKM(\Omega, \Omega, Y)$ is upper semicontinuous with closed values. Also assume if $G(y) \neq \emptyset$ for every $y \in Y$, then $G \in PK(Y, \Omega)$. If F and G are compact maps then either
(A1) *there exists $w_0 \in Y$ with $f(x, w_0) \leq \alpha$ for all $x \in \Omega$, or*
(A2) *there exists $(x_0, y_0) \in \Omega \times Y$ with $g(x_0, y_0) \leq \alpha < f(x_0, y_0)$ hold.*

THEOREM 5.10. *Let Ω be a paracompact admissible subset of a Hausdorff topological vector space and Y a convex subset of a Hausdorff topological vector space. In addition, assume Y is a normal space. Let $f, g : \Omega \times Y \to \mathbf{R}$ be such that (5.1) holds. Fix $\alpha \in \mathbf{R}$ and let*

$$G(x) = \{y \in Y : f(x, y) < \alpha\}$$

and

$$F(y) = \{x \in \Omega : g(x, y) > \alpha\}.$$

Suppose $s : Y \to Y$ is surjective. If $G(x) \neq \emptyset$ for every $x \in \Omega$, suppose $G \in PK(\Omega, Y)$. Also assume if $F(y) \neq \emptyset$ for every $y \in Y$, then $F \in s\text{-}KKM(Y, Y, \Omega)$ is a closed map. If F and G are compact maps then either
(A1) *there exists $z_0 \in \Omega$ with $f(z_0, y) \geq \alpha$ for all $y \in Y$, or*
(A2) *there exists $w_0 \in Y$ with $g(x, w_0) \leq \alpha$ for all $x \in \Omega$ hold.*

Proof. We consider three cases.

CASE (I). $G(x) \neq \emptyset$ for every $x \in \Omega$ and $F(y) \neq \emptyset$ for every $y \in Y$. Then Theorem 5.1 guarantees that there exists $(x_0, y_0) \in \Omega \times Y$ with $f(x_0, y_0) < \alpha$ and $g(x_0, y_0) > \alpha$, which contradicts (5.1).

CASE (II). $G(x) \neq \emptyset$ for every $x \in \Omega$ does not hold. Then there exists $z_0 \in \Omega$ with $G(z_0) = \emptyset$. This implies that $f(z_0, y) \geq \alpha$ for all $y \in Y$ and so (A1) holds.

CASE (III). $F(x) \neq \emptyset$ for every $x \in \Omega$ does not hold. Then there exists $w_0 \in Y$ with $F(w_0) = \emptyset$. This implies that $g(x, w_0) \leq \alpha$ for all $x \in \Omega$ and so (A2) holds.

REMARK 5.3. One can construct analytic alternatives based on Theorems 5.4–5.7. We leave the details to the reader.

Next we show how our coincidence theorems and analytic alternatives generate minimax inequalities.

THEOREM 5.11. *Let Ω be a convex admissible subset of a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological vector space. Let $f : \Omega \times Y \to \mathbf{R}$ and $\alpha = \sup_{x \in \Omega} \inf_{y \in Y} f(x, y)$ Let*

$$G(y) = \{x \in \Omega : f(x, y) > \alpha\}$$

and

$$F(x) = \{y \in Y : f(x,y) \leq \alpha\}.$$

Suppose $s : \Omega \to \Omega$ is surjective and $F \in s$-KKM(Ω, Ω, Y) is upper semicontinuous map with closed values. Also assume if $G(y) \neq \emptyset$ for every $y \in Y$, then $G \in PK(Y, \Omega)$. If F and G are compact maps, then

$$\inf_{y \in Y} \sup_{x \in \Omega} f(x,y) = \sup_{x \in \Omega} \inf_{y \in Y} f(x,y).$$

Proof. Let $\beta = \inf_{y \in Y} \sup_{x \in \Omega} f(x,y)$. Then $\alpha \leq \beta$. Notice (A2) does not hold since $f = g$. Now Theorem 5.9 guarantees that there exists $w_0 \in Y$ with $f(x, w_0) \leq \alpha$ for all $x \in \Omega$. Consequently, $\sup_{x \in \Omega} f(x, w_0) \leq \alpha$ and so $\beta \leq \alpha$. Hence

$$\inf_{y \in Y} \sup_{x \in \Omega} f(x,y) = \sup_{x \in \Omega} \inf_{y \in Y} f(x,y).$$

It is possible to generalize Theorem 5.11 using a pair of mappings f and g. Also it is not difficult to establish an analogue of Theorem 5.11 using Theorem 5.8. Instead of presenting these results we show the technique involved by obtaining a minimax inequality modelled off our analytic alternative Theorem 5.10.

THEOREM 5.12. *Let Ω be a paracompact admissible subset of a Hausdorff topological vector space and Y a convex subset of a Hausdorff topological vector space. In addition, we assume Y is a normal space. Let $f, g : \Omega \times Y \to \mathbf{R}$ be such that (5.1) holds. For each $\alpha \in \mathbf{R}$, let*

$$G_\alpha(x) = \{y \in Y : f(x,y) < \alpha\}$$

and

$$F_\alpha(y) = \{x \in \Omega : g(x,y) > \alpha\}.$$

Suppose $s : Y \to Y$ is surjective. For each $\alpha \in \mathbf{R}$, if $G_\alpha(x) \neq \emptyset$ for every $x \in \Omega$, assume $G_\alpha \in PK(\Omega, Y)$ and if $F_\alpha(y) \neq \emptyset$ for every $y \in Y$, assume $F_\alpha \in s$-KKM(Y, Y, Ω) is closed. For each $\alpha \in \mathbf{R}$, if F_α and G_α are compact maps, then

$$\beta_0 \equiv \inf_{y \in Y} \sup_{x \in \Omega} g(x,y) \leq \sup_{x \in \Omega} \inf_{y \in Y} f(x,y) \equiv \alpha_0.$$

Proof. Let $\alpha_0 < \infty$ and $\beta_0 > -\infty$. Assume $\beta_0 > \alpha_0$. Then there exists $\alpha \in \mathbf{R}$ with

$$\alpha_0 < \alpha < \beta_0. \tag{5.2}$$

Now we apply Theorem 5.10. If (A1) holds, then there exists $z_0 \in \Omega$ with $f(z_0, y) \geq \alpha$ for all $y \in Y$. As a result, $\inf_{y \in Y} f(z_0, y) \geq \alpha$ and so $\alpha_0 \geq \alpha$, which contradicts (5.2). If (A2) holds, then there exists

$w_0 \in Y$ with $g(x, w_0) \leq \alpha$ for all $x \in \Omega$. As a result $\sup_{x\in\Omega} g(x, w_0) \leq \alpha$ and so $\beta_0 \leq \alpha$, which contradicts (5.2). Since, in both cases, we get a contradiction, so we have $\beta_0 \geq \alpha_0$.

REMARK 5.4. If $f = g$ in Theorem 5.12, then we obtain

$$\inf_{y\in Y} \sup_{x\in\Omega} f(x,y) = \sup_{x\in\Omega} \inf_{y\in Y} f(x,y).$$

Finally we obtain a minimax theorem modelled off our coincidence Theorem 5.2.

THEOREM 5.13. *Let Ω be a convex admissible subset of a Hausdorff topological vector space and Y a paracompact subset of a Hausdorff topological vector space. Let $f, g : \Omega \times Y \to \mathbf{R}$ be such that* (5.1) *holds. For each $\alpha \in \mathbf{R}$, let*

$$F_\alpha(x) = \{y \in Y : g(x,y) \geq \alpha\}$$

and for each $\beta \in \mathbf{R}$, let

$$G_\beta(y) = \{x \in \Omega : f(x,y) \leq \beta\}.$$

Suppose $s : \Omega \to \Omega$ is surjective. For each $\alpha \in \mathbf{R}$, if $F_\alpha(x) \neq \emptyset$ for every $x \in \Omega$, assume $F_\alpha \in s\text{-}KKM(\Omega, \Omega, Y)$ is a upper semicontinuous map with closed values and for each $\beta \in \mathbf{R}$, if $G_\beta(y) \neq \emptyset$ for every $y \in Y$, assume $G_\beta \in PK(Y, \Omega)$. For each $\alpha, \beta \in \mathbf{R}$, if G_α and F_β are compact maps, then

$$\beta_0 \equiv \inf_{x\in\Omega} \sup_{y\in Y} g(x,y) \leq \sup_{y\in Y} \inf_{x\in\Omega} f(x,y) \equiv \alpha_0.$$

Proof. Let $\alpha_0 < \infty$ and $\beta_0 > -\infty$. Assume $\beta_0 > \alpha_0$. Then there exists $\beta \in \mathbf{R}$ with $\alpha_0 < \beta < \beta_0$. In addition there exists $\epsilon > 0$ with

$$\alpha_0 < \beta < \beta + \epsilon \equiv \alpha < \beta_0. \tag{5.3}$$

We consider three cases.

CASE (I). $F_\alpha(x) \neq \emptyset$ for every $x \in \Omega$ and $G_\beta(y) \neq \emptyset$ for every $y \in Y$. Now Theorem 5.2 guarantees that there exists $(x_0, y_0) \in \Omega \times Y$ with $f(x_0, y_0) \leq \beta$ and $g(x_0, y_0) \geq \alpha$. This together with (5.1) implies

$$\alpha \leq g(x_0, y_0) \leq f(x_0, y_0) \leq \beta,$$

which contradicts (5.3).

CASE (II). $G_\beta(y) \neq \emptyset$ for every $y \in Y$ does not hold. Then there exists $w_0 \in Y$ with $f(x, w_0) > \beta$ for all $x \in \Omega$ and so

$$\inf_{x\in\Omega} f(x, w_0) \geq \beta.$$

As a result, $\alpha_0 \geq \beta$, which contradicts (5.3).

CASE (III). $F_\alpha(x) \neq \emptyset$ for every $x \in \Omega$ does not hold. Then there exists $z_0 \in \Omega$ with $g(z_0, y) < \alpha$ for all $y \in Y$ and so

$$\sup_{y \in Y} g(z_0, y) \leq \alpha.$$

Consequently $\beta_0 \leq \alpha$, which contradicts (5.3).

References

[1] R. P. Agarwal and D. O'Regan, *Collectively fixed point theorems*, Nonlinear Anal. Forum **7** (2002), 167–179.

[2] ———, *Coincidence theory for $\mathcal{U}_c^\kappa$ maps and inequalities*, J. Nonlinear Convex Anal. **5** (2004), 265–274.

[3] ———, *Fixed point theorems for S-KKM maps*, Appl. Math. Lett. **16** (2003), 1257–1264.

[4] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, Springer Verlag, Berlin, 1994.

[5] T. H. Chang, Y. Y. Huang and J. C. Jeng, *Fixed point theorems for multifunctions in S-KKM class*, Nonlinear Anal. **44** (2001), 1007–1017.

[6] T. H. Chang, Y. Y. Huang, J. C. Jeng, and K. H. Kuo, *On S-KKM property and related topics*, J. Math. Anal. Appl. **229** (1999), 212–227.

[7] T. H. Chang and C. L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl. **203** (1996), 224–235.

[8] D. O'Regan, N. Shahzad and R. P. Agarwal, *Approximation and Furi–Pera type theorems for the S-KKM class*, Vietnam J. Math. **32** (2004), 451–465.

[9] S. Park, *Fixed points, intersection theorems, variational inequalities and equilibrium theorems*, Int. J. Math. Math. Sci. **24** (2000), 79–93.

DONAL O'REGAN, DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF IRELAND, GALWAY, IRELAND
E-mail: donal.oregan@nuigalway.ie

NASEER SHAHZAD, DEPARTMENT OF MATHEMATICS, KING ABDUL AZIZ UNIVERSITY, P.O. BOX 80203, JEDDAH 21589, SAUDI ARABIA
E-mail: nshahzad@kaau.edu.sa

RAVI P. AGARWAL, DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA INSTITUTE OF TECHNOLOGY, MELBOURNE, FLORIDA 32901, U.S.A.
E-mail: agarwal@fit.edu