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A NOTE ON INDECOMPOSABLE 4-MANIFOLDS

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Abstract. In this note we show that there is an anti-symplectic involution $\sigma : X \to X$ on a simply-connected, closed, non-Kähler and symplectic 4-manifold X with a disjoint union of Riemann surfaces $\amalg_{i=1}^{n} \Sigma_i, n \geq 2$ as a fixed point set. Also we show that its quotient X/σ is homeomorphic to $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$ but not diffeomorphic to $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$, $r = b_2^-(X/\sigma)$.

1. Introduction

Let (X, ω) be a closed, symplectic, 4-manifold with a symplectic structure ω. A smooth map $\sigma : X \to X$ is an anti-symplectic involution if and only if $\sigma^* \omega = -\omega$ and $\sigma^2 = \mathrm{Id}$. If X is a Kähler surface, then σ is anti-symplectic if and only if σ is anti-holomorphic, that is, $\sigma_* \circ J = -J \circ \sigma_*$ for the complex structure J on X. A typical example of an anti-holomorphic involution is a complex conjugation over a complex algebraic surface.

S. Akbulut in [1] conjectured that if X is a simply-connected, closed, symplectic 4-manifold and $\sigma : X \to X$ is an anti-symplectic involution with a smooth embedded surface as a fixed point set, then the quotient X/σ is completely decomposable, i.e.,

$$X/\sigma \cong r\mathbb{CP}^2 \sharp s \bar{\mathbb{CP}}^2 \text{ or } \sharp n(S^2 \times S^2) \text{ for some } r, s, n \in \mathbb{N}.$$

In [1], S. Akbulut showed that if X is a complex algebraic surface and the fixed point set is a real algebraic surface, then the quotient X/σ is completely decomposable for many cases.

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Note that if σ is free and X is a Kähler surface with the canonical class K_X satisfying $K_X^2 > 0$ and $b_2^+(X) > 3$, then S. Wang in [21] showed that the Seiberg-Witten invariants of X/σ are zero and so X/σ is not a symplectic 4-manifold. For the Seiberg-Witten invariants, see Section 2.

Suppose that X is a symplectic 4-manifold and σ has 2-dimensional compact submanifolds as fixed point sets. If the fixed point sets contain a Riemann surface Σ with genus $g(\Sigma) > 1$ and $[\Sigma] \neq 0 \in H_2(X : \mathbb{Z})$ then the quotient manifold X/σ with $b_2^+(X/\sigma) > 1$ has a vanishing Seiberg-Witten invariants. For details, see [5].

In this note, by using a simply-connected, closed, non-Kähler, symplectic 4-manifold X and an anti-symplectic involution $\sigma : X \to X$ with a disjoint union of Riemann surfaces $\amalg_{i=1}^n \Sigma_i, n \geq 2$ as a fixed point set, we show that the quotient X/σ is not symplectic and has vanishing Seiberg-Witten invariants for all Spin^c-structures over X/σ. Furthermore, we show that X/σ is homeomorphic to $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$ but not diffeomorphic to $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$, $r = b_2^-(X/\sigma)$.

2. Brief review of the Seiberg-Witten invariant

Let X be a closed, oriented, Riemannian 4-manifold with $b_2^+(X) \geq 1$ and let $L \to X$ be a complex line bundle satisfying $c_1(L) \equiv w_2(TX)$ mod 2. Then there is a principal Spin$^c(4)$-bundle $\xi \to X$ associated with L. By considering Spin$^c(4)$ representations on $\mathbb{C}^2$, there are $(\pm\frac{1}{2})$-twisted spinor bundles $W^\pm(\xi)$ associated with ξ. Then $W^\pm(\xi) \cong S^\pm \otimes L^{1/2}$ and $\det W^\pm(\xi) \cong L$, where $S^\pm$ is locally defined spinor bundle.

Let $\mathcal{A}(L)$ be the set of all Riemannian connections on L and $\Gamma(W^+(\xi))$ be the space of all sections of $W^+(\xi) \to X$. The gauge group $\mathcal{G}(L)$ of all bundle automorphism on L acts on the space $\mathcal{A}(L) \times \Gamma(W^+(\xi))$.

For a positive spinor field $\psi \in \Gamma(W^+(\xi))$ and a unitary connection A on L, the Seiberg-Witten equations are defined by

$$\begin{cases} F_A^+ + i\delta = q(\psi), \\ D_A\psi = 0, \end{cases}$$

where F_A^+ is the self-dual part of the curvature F_A and δ is a smooth, real valued, self-dual two-form on X. $q : C^\infty(W^+(\xi)) \to \Omega_X^+(i\mathbb{R})$ is a quadratic map defined by $q(\psi) = \psi \otimes \psi^* - \frac{||\psi||^2}{2}\mathrm{Id}$. $D_A : \Gamma(W^+(\xi)) \to$

$\Gamma(W^-(\xi))$ is the Dirac operator as defined using the connection A and the Levi-Civita connection on TX.

Let $SW(L)$ be the set of all solutions of the Seiberg-Witten equations. Then the gauge group $\mathcal{G}(L)$ acts on $SW(L)$ and define the moduli space $\mathcal{M}(\xi)$ by $SW(L)/\mathcal{G}(L)$ of the gauge equivalence classes of all solutions of the Seiberg-Witten equations.

For a generic self-dual two-form δ, the moduli space $\mathcal{M}(\xi)$ is a smooth manifold with its dimension $\frac{1}{4}(c_1(L)^2-(2\chi(X)+3\mathrm{sign}(X)))$ where $\chi(X)$ is the Euler characteristic of X and $\mathrm{sign}(X)$ is the signature of X. If the metric on X is chosen so that the Seiberg-Witten equations admit no reducible solutions then $\mathcal{M}(\xi)$ will be compact. Note that the irreducibility of solutions can be achieved in a path-connected subset of metrics if $b_2^+(X) > 1$.

In this situation, fix a base point x_0 in X. Let $\mathcal{G}_0(L) = \{g \in \mathcal{G}(L) | g(x_0) = 1\}$ and $\mathcal{M}_0(\xi) = SW(L)/\mathcal{G}_0(L)$. If $\dim \mathcal{M}_\delta(\xi) = 2d \geq 0$, $d \in \mathbb{Z}$ then we define the Seiberg-Witten invariant $\mathcal{S}W(\xi)$ such as

$$\mathcal{S}W(\xi) = \int_{\mathcal{M}(\xi)} c_1(\mathcal{M}(\xi)_0)^d.$$

If $\dim \mathcal{M}_\delta(\xi) < 0$ then the Seiberg-Witten invariant is defined to be zero. For details see [12].

C. H. Taubes in [18] proved the non-trivialness of the Seiberg-Witten invariants on the symplectic 4-manifold and in [19] announced an equivalence between the Seiberg-Witten invariants and the Gromov invariants.

3. Anti-symplectic involution over non-Kähler, symplectic 4-manifold

Let X_1 and X_2 be simply-connected Dolgachev surfaces given by relatively prime multiplicities $p_i, q_i \geq 1$, $i = 1, 2$. A Dolgachev surface is the result of performing two logarithmic transformations on the fibers of the basic elliptic surface $E(1)$ which is $\mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}}^2$ as being equipped with an elliptic fibration.

Let D_ϵ be a disk in $\mathbb{R}^2$ with radius $\epsilon = \frac{1}{\sqrt{\pi}}$. We identify a small tubular neighborhood $N(F_i)$ of a generic fiber F_i (Kähler torus) of X_i with $T^2 \times D_\epsilon$ so that the fibration corresponds to projection onto D_ϵ and the canonical orientations of the torus T^2 and D_ϵ map to the complex orientation.

LEMMA 3.1. *There is an anti-holomorphic involution $\sigma_i : X_i \to X_i$ such that $\sigma_i(F_i) = F_i'$ and $N(F_i) \cap N(F_i') = \emptyset$, where F_i and F_i' are generic fibers in X_i, $i = 1, 2$.*

Proof. We take two generic cubics Γ_0 and Γ_1 in $\mathbb{CP}^2$ which are the zero sets of homogeneous cubic polynomials p_0 and p_1 respectively. Let $\Gamma_0 \cap \Gamma_1 = \{P_1 \dots, P_9\}$.

For each point $w = [t_0 : t_1] \in \mathbb{CP}^1$, there is a pencil of cubic curve $\Gamma_w = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 | (t_0 p_0 + t_1 p_1)([z_0 : z_1 : z_2]) = 0\}$. Then $\{\Gamma_w | w \in \mathbb{CP}^1\}$ is a one-sheet cover of $\mathbb{C}P^2 - \{P_1, \dots, P_9\}$.

For all $Q \in \mathbb{CP}^2 - \{P_1, \dots, P_9\}$, there is unique cubic Γ_w which passes through Q and we define a map $f : \mathbb{CP}^2 - \{P_1, \dots, P_9\} \to \mathbb{CP}^1$ by $f(Q) = w \in \mathbb{CP}^1$. Then $f^{-1}(w) = \Gamma_w$.

By blowing up $\mathbb{CP}^2$ at $P_1, \dots, P_9$, we extend f to a fibration $\pi : \mathbb{CP}^2 \sharp 9 \bar{\mathbb{CP}}^2 \to \mathbb{CP}^1$, whose fibers are cubic curve and the generic fiber is a smooth elliptic curve. For details, see [10].

X_1 and X_2 are the results of performing two logarithmic transformations on the fibers of $\mathbb{CP}^2 \sharp 9 \bar{\mathbb{CP}}^2$ with an elliptic fibration $\pi : \mathbb{CP}^2 \sharp 9 \bar{\mathbb{CP}}^2 \to \mathbb{CP}^1$. Let $\pi_i : X_i \to \mathbb{CP}^1$ be an elliptic fibration, $i = 1, 2$.

Now we consider an anti-holomorphic involution σ_i on X_i as a complex conjugation $c : X_i \to X_i$, $i = 1, 2$. The complex conjugation c on X_i sends a point $Q \in X_i$ to $c(Q) = \bar{Q}$ and the cubic $\Gamma_{\bar{w}} = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 | (\bar{t}_0 p_0 + \bar{t}_1 p_1)[z_0 : z_1 : z_2] = 0, [\bar{t}_0 : \bar{t}_1] = \bar{w} \in \mathbb{CP}^1\}$ passes through $\bar{Q}$ where $c([t_0 : t_1]) = [\bar{t}_0 : \bar{t}_1]$. Thus $f(\bar{Q}) = \bar{w} \in \mathbb{CP}^1$ and $f^{-1}(\bar{w}) = \Gamma_{\bar{w}}$.

If $[t_0 : t_1]$ and $[\bar{t}_0 : \bar{t}_1]$ are generic points in $\mathbb{CP}^1$ with $[\bar{t}_0 : \bar{t}_1] \neq [t_0 : t_1]$ then $\pi^{-1}([t_0 : t_1])$ and $\pi^{-1}([\bar{t}_0 : \bar{t}_1])$ are two different generic torus. Let $\pi^{-1}([t_0 : t_1]) = F_i$ and $\pi^{-1}([\bar{t}_0 : \bar{t}_1]) = F_i'$, $i = 1, 2$.

For the anti-holomorphic involution σ_i with $\sigma_i(F_i) = F_i'$, there is a tubular neighborhood $N(F_i)$ of a generic fiber F_i such that σ_i sends $N(F_i)$ to a tubular neighborhood $N(F_i')$ of F_i', $i = 1, 2$. For the proof of this, see [6].

Then we can say that the small tubular neighborhoods $N(F_i)$ and $N(F_i')$ satisfy $N(F_i) \cap N(F_i') = \emptyset$, $i = 1, 2$.

Suppose in addition that $\sigma_i : X_i \to X_i$ has fixed point sets $X_i^{\sigma_i} = \amalg_{j=1}^{n_i} \Sigma_j^i$, where Σ_j^i is a Riemann surface in X_i and $\Sigma_j^i \subset X_i - (N(F_i) \amalg N(F_i'))$, $j = 1, \dots, n_i, i = 1, 2$. For the anti-holomorphic involution σ_i there is a Kähler form ω_i on X_i such that $\sigma_i^* \omega_i = -\omega_i$, $i = 1, 2$.

REMARK 3.2. Since $[t_0 : t_1] \neq [\bar{t}_0 : \bar{t}_1]$, the two generic points $[t_0 : t_1]$ and $[\bar{t}_0 : \bar{t}_1]$ are not contained in the fixed point set S^1 of the complex conjugation on $\mathbb{CP}^1$, we can say that for small tubular neighborhoods $N(F_i)$ and $N(F_i')$, the fixed point sets $X_i^{\sigma_i} = \amalg_{j=1}^{n_i} \Sigma_j^i \subset X_i - (N(F_i) \amalg N(F_i'))$, $i = 1, 2$.

The fibration on X_i determines a canonical normal framing of F_i, so there is a fiber-orientation reversing bundle isomorphism $\psi_1 : N(F_1) \to N(F_2)$, respecting the given framings and an orientation preserving diffeomorphism $\phi_1 : N(F_1) - F_1 \to N(F_2) - F_2$ by composing ψ_1 with the diffeomorphism

$$f : r \mapsto \sqrt{\epsilon^2 - r^2},\ 0 < r < \epsilon,\ \epsilon = \frac{1}{\sqrt{\pi}}$$

that turns each punctured normal fiber inside out.

Let $X_1 \sharp_{\phi_1} X_2$ be the smooth, closed, oriented 4-manifold obtained from $(X_1 - F_1) \amalg (X_2 - F_2)$ by using ϕ_1 to identify $N(F_1) - F_1$ with $N(F_2) - F_2$. Then $X_1 \sharp_{\phi_1} X_2$ is known to be a simply-connected, elliptic surface with Euler characteristic 24. For details, see Chapter 3 in [10].

By composing ϕ_1 with a cyclic permutation p of the three factors $F_1 \times S^1 = S^1 \times S^1 \times S^1$ before gluing, R. E. Gomph in [8] constructed non-Kähler, simply-connected, closed, symplectic 4-manifolds $X_1 \sharp_{\phi_1'} X_2$ by $\phi_1' = p \circ \phi_1$. More explicitly, for all $\mathbf{x} \in F_1 \times (D_\epsilon - \{0\})$, $\mathbf{x}$ can be written by

$$\mathbf{x} = ((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3}) \in F_1 \times (D_\epsilon - \{0\}) \cong N(F_1) - F_1,$$

where $(e^{i\theta_1}, e^{i\theta_2}) \in F_1 \cong T^2$ and

$$re^{i\theta_3} \in (D_\epsilon - \{0\}),\ 0 < r \leqslant \epsilon,\ 0 \leqslant \theta_i \leqslant 2\pi,\ i = 1, 2, 3.$$

Let $p : F_1 \times S^1 \to F_1 \times S^1$ be the cyclic permutation defined by $p(x_1, x_2, x_3) = (x_3, x_1, x_2)$. Then we have

$$\phi_1'(\mathbf{x}) = \phi_1'((e^{i\theta_1}, e^{i\theta_2}), re^{i\theta_3}) = ((e^{-i\theta_3}, e^{i\theta_1}), \sqrt{\epsilon^2 - r^2} e^{i\theta_2}).$$

Also the fibration on X_i determines a canonical normal framing of $\sigma_i(F_i) = F_i'$, so there is a fiber-orientation reversing bundle isomorphism $\psi_2 : N(F_1') \to N(F_2')$, respecting the given framings and an orientation preserving diffeomorphism $\phi_2 : N(F_1') - F_1' \to N(F_2') - F_2'$ defined by $f \circ \psi_2$. As above ϕ_1', we consider a twisted gluing map $\phi_2' = p \circ \phi_2 : N(F_1') - F_1' \to N(F_2') - F_2'$.

Let $X_1 \sharp_{\phi_1', \phi_2'} X_2$ be a smooth, closed, oriented 4-manifold obtained from $(X_1 - (F_1 \amalg F_1')) \amalg (X_2 - (F_2 \amalg F_2'))$ by using ϕ_1' and ϕ_2' to identify $N(F_1) - F_1$ and $N(F_2) - F_2$ and $N(F_1') - F_1'$ and $N(F_2') - F_2'$ respectively.

PROPOSITION 3.3. [6] *$X_1\sharp_{\phi_1',\phi_2'}X_2$ is a simply-connected, closed, non-Kähler, symplectic 4-manifold and there is an anti-symplectic involution σ on $X_1\sharp_{\phi_1',\phi_2'}X_2$.*

We will sketch the proof of Proposition 3.3 for the next section. Let ω_i be the Kähler form on X_i and let $\omega_0 = (\omega_1, \omega_2)$, $i = 1, 2$. By [6] and [8], there is a symplectic structure ω over $X_1\sharp_{\phi_1',\phi_2'}X_2$ for any choice of the gluing maps ϕ_1' and ϕ_2'. In fact,

$$\omega = \begin{cases} \omega_i & X_i - (N(F_i) \amalg N(F_i')),\ i = 1,2, \\ \omega_0 + t\eta & (N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2), \\ \omega_0 + t\eta' & (N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2'), \end{cases}$$

where $t \in (0, t_0]$ for sufficiently small t_0. The η and η' are closed 2-forms compactly supported in $N(F_2)$ and $N(F_2')$ respectively and they are Poincaré dual to $[F_2] \in H_2(X_2; \mathbb{R})$ and $[F_2'] \in H_2(X_2'; \mathbb{R})$ respectively.

The Dolgachev surfaces X_1 and X_2 are simply-connected and the normal circles to the fibers F_i and F_i' can be constructed along the remaining hemisphere of any sections. Thus $N(F_i) - F_i$ and $N(F_i') - F_i'$ are simply-connected respectively, $i = 1, 2$. By using the Seifert-Van Kampen Theorem, $X_1\sharp_{\phi_1',\phi_2'}X_2$ is simply-connected.

Since $X_1\sharp_{\phi_1',\phi_2'}X_2$ is obtained from $(X_1 - (F_1 \amalg F_1')) \amalg (X_2 - (F_2 \amalg F_2'))$ by using ϕ_1' and ϕ_2' to identify $N(F_1) - F_1$ and $N(F_2) - F_2$ and $N(F_1') - F_1'$ and $N(F_2') - F_2'$ respectively, $X_1\sharp_{\phi_1',\phi_2'}X_2$ is not a Kähler surface. For the twisted gluing map ϕ_i', $i = 1, 2$, and non-Kähler property, see [8].

For all $\mathbf{x} \in N(F_1) - F_1 \cong F_1 \times (D_\epsilon - \{0\})$ and $\mathbf{x}' \in N(F_1') - F_1' \cong F_1' \times (D_\epsilon - \{0\})$,

$$\sigma_1(\mathbf{x}) = \phi_2'(\sigma_1(\mathbf{x})) \text{ on } (N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2'),$$
$$\sigma_1(\mathbf{x}') = \phi_1'(\sigma_1(\mathbf{x}')) \text{ on } (N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2).$$

Then we have

$$\phi_2'(\sigma_1(\mathbf{x})) = \phi_2'((e^{-i\theta_1}, e^{-i\theta_2}), re^{-i\theta_3}) = ((e^{i\theta_3}, e^{-i\theta_1}), \sqrt{\epsilon^2 - r^2}e^{-i\theta_2}),$$
$$\begin{aligned} \sigma_2(\phi_1'(\mathbf{x}')) &= \sigma_2((e^{-i\theta_3}, e^{i\theta_1}), \sqrt{\epsilon^2 - r^2}e^{i\theta_2}) \\ &= ((e^{i\theta_3}, e^{-i\theta_1}), \sqrt{\epsilon^2 - r^2}e^{-i\theta_2}). \end{aligned}$$

Similarly, $\sigma_1(\mathbf{x}') = \phi_1'(\sigma_1(\mathbf{x}')) = \sigma_2(\phi_2'(\mathbf{x}'))$ for all $\mathbf{x}' \in N(F_1') - F_1' \cong F_1' \times (D_\epsilon - \{0\})$.

Since ψ_1 identifies the generic fiber F_1 with F_2, we have $\sigma_2 \circ \psi_1 = \psi_2 \circ \sigma_1$ on F_1. Similarly, ψ_2 identifies F_1' with F_2' and so $\psi_1 \circ \sigma_1 = \sigma_2 \circ \psi_2$ on F_1'.

Thus there is a well-defined involution σ on $X_1 \sharp_{\phi_1', \phi_2'} X_2$ such that

$$\sigma = \begin{cases} \sigma_i & X_i - (N(F_i) \amalg N(F_i')) \subset X_1 \sharp_{\phi_1', \phi_2'} X_2, \ i = 1, 2, \\ \sigma_1(\mathbf{x}') = \sigma_2(\phi_2'(\mathbf{x}')) & (N(F_1) - F_1) \sharp_{\phi_1'} (N(F_2) - F_2), \\ \sigma_1(\mathbf{x}) = \sigma_2(\phi_1'(\mathbf{x})) & (N(F_1') - F_1') \sharp_{\phi_2'} (N(F_2') - F_2'), \end{cases}$$

for all $\mathbf{x} \in N(F_1) - F_1$ and $\mathbf{x}' \in N(F_1') - F_1'$. From the construction of σ we know that the fixed point sets $(X_1 \sharp_{\phi_1', \phi_2'} X_2)^\sigma$ are $(\amalg_{j=1}^{n_1} \Sigma_j^1) \amalg (\amalg_{j=1}^{n_2} \Sigma_j^2)$.

For the symplectic structure ω on $X_1 \sharp_{\phi_1', \phi_2'} X_2$, $\sigma^* \omega = \sigma_i^* \omega_i = -\omega_i = -\omega$ on $X_i - (N(F_i) \amalg N(F_i')) \subset X_1 \sharp_{\phi_1', \phi_2'} X_2$, $i = 1, 2$. For all $\mathbf{x} \in N(F_1) - F_1$, we have

$$\begin{aligned} \sigma^* \omega(\mathbf{x}) &= \sigma^* \omega_0(\mathbf{x}) = \sigma_1^* \omega_1(\mathbf{x}) = -\omega_1(\mathbf{x}) = -\omega(\mathbf{x}), \\ \sigma^* \omega(\phi_1'(\mathbf{x})) &= \sigma^*(\omega_0 + t\eta)(\phi_1'(\mathbf{x})) = \sigma_2^*(\omega_2 + t\eta)(\phi_1'(\mathbf{x})). \end{aligned}$$

Since η is Poincaré dual to $[F_2] \in H_2(X_2; \mathbb{R})$, η can be written by $\eta = dy_3 dy_4$ for all $((y_1, y_2), (y_3, y_4)) \in F_2 \times (D_\epsilon - \{0\})$. Then $y_3 = r \cos\theta$ and $y_4 = r \sin\theta$ for the polar coordinate $(r, \theta) \in D_\epsilon - \{0\}$, $0 < r \leqslant \epsilon$, $0 \leqslant \theta \leqslant 2\pi$. Thus we have

$$\begin{aligned} \sigma_2^*(\eta) = \sigma_2^*(dy_3 dy_4) &= \sigma_2^* d(r\cos\theta) d(r \sin\theta) \\ &= d(r\cos(-\theta)) d(r\sin(-\theta)) = -\eta. \end{aligned}$$

Then we conclude that $\sigma_2^*(\omega_2 + t\eta)(\phi_1'(\mathbf{x})) = -\omega(\phi_1'(\mathbf{x}))$.

Similarly, for all $\mathbf{x}' \in N(F_1') - F_1'$, we have

$$\begin{aligned} \sigma^* \omega(\mathbf{x}') &= -\omega_1(\mathbf{x}') = -\omega(\mathbf{x}'), \\ \sigma^* \omega(\phi_2'(\mathbf{x}')) &= \sigma_2^*(\omega_2 + t\eta')(\phi_2'(\mathbf{x}')) = -\omega(\phi_2'(\mathbf{x}')). \end{aligned}$$

Then $\sigma^* \omega = -\omega$ on $(N(F_1) - F_1) \sharp_{\phi_1'} (N(F_2) - F_2)$ and $(N(F_1') - F_1') \sharp_{\phi_2'} (N(F_2') - F_2')$.

Thus we conclude that σ is an anti-symplectic involution on $X_1 \sharp_{\phi_1', \phi_2'} X_2$ for the symplectic structure ω. For the detail proof, see [6].

4. Quotient of a non-Kähler, symplectic 4-manifold by an anti-symplectic involution

Let $X_1\sharp_{\phi_1',\phi_2'}X_2$ and σ be as in Section 3.

PROPOSITION 4.1. *Under the same constructions as above, the quotient $X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ is not symplectic and has vanishing Seiberg-Witten invariants.*

Proof. Let X_i, σ_i, F_i, F_i' be as in the proof of Proposition 3.3 and let $p_i : X_i \to X_i/\sigma_i = X_i'$ be the projection map, $i = 1, 2$. Then $p_i(F_i) = p_i(F_i') \subset X_i'$, $i = 1, 2$. Denote $p_i(F_i) = p_i(F_i')$ by $\tilde{F}_i$ and then $p_i(N(F_i)) = p_i(N(F_i'))$ is a tubular neighborhood of $\tilde{F}_i$ in X_i', $i = 1, 2$.

Since X_i is a smooth, simply-connected, Dolgachev surface and is a double cover of X_i' branched along $\amalg_{j=1}^{n_i}\Sigma_j^i$, by [3] and [20] the quotient X_i' is smooth and simply-connected, $i = 1, 2$.

The Euler characteristic and signature of X_i' are

$$\chi(X_i') = \frac{1}{2}(\chi(X_i) + \Sigma_{j=1}^{n_i}\chi(\Sigma_j^i)) = 6 + \frac{1}{2}\Sigma_{j=1}^{n_i}\chi(\Sigma_j^i),$$
$$\text{sign}(X_i') = \frac{1}{2}(\text{sign}(X_i) + \Sigma_{j=1}^{n_i}\Sigma_j^i \cdot \Sigma_j^i) = -4 + \frac{1}{2}\Sigma_{j=1}^{n_i}\Sigma_j^i \cdot \Sigma_j^i,$$
$$j = 1, \dots, n_i, \ i = 1, 2.$$

Since the Σ_j^i are the fixed point set of the anti-holomorphic involution σ_i, Σ_j^i are Lagrangian surfaces and they satisfy $\chi(\Sigma_j^i) + \Sigma_j^i \cdot \Sigma_j^i = 0, j = 1, \dots, n_i, i = 1, 2$.

Then we have $\chi(X_i') + \text{sign}(X_i') = 2 + 2b_2^+(X_i') = 2$ and so $b_2^+(X_i') = 0$, $i = 1, 2$. Thus we conclude that the quotient $X_i/\sigma_i = X_i'$ is not a symplectic 4-manifold, $i = 1, 2$.

By [2] and [3], $\tilde{F}_i \cdot \tilde{F}_i = 2F_i \cdot F_i = 2F_i' \cdot F_i' = 0, i = 1, 2$. Thus $\tilde{F}_i$ is a torus with trivial self-intersection number and we can identify tubular neighborhoods $N(\tilde{F}_i)$ with trivial normal bundles, $i = 1, 2$. Then there is a fiber-orientation reversing bundle isomorphism $\tilde{\psi} : N(\tilde{F}_1) \to N(\tilde{F}_2)$. As the construction of ϕ_i', there is a map $\tilde{\phi} : N(\tilde{F}_1) - \tilde{F}_1 \to N(\tilde{F}_2) - \tilde{F}_2$ defined by $\tilde{\phi} = p \circ f \circ \tilde{\psi}$, where f and $p : \tilde{F}_1 \times S^1 \to \tilde{F}_1 \times S^1$ are the same ones as in ϕ_i', $i = 1, 2$.

Let $X_1'\sharp_{\tilde{\phi}}X_2'$ be the smooth, closed, oriented 4-manifold obtained from $(X_1' - \tilde{F}_1) \amalg (X_2' - \tilde{F}_2)$ by using $\tilde{\phi}$ to identify $N(\tilde{F}_1) - \tilde{F}_1$ and $N(\tilde{F}_2) - \tilde{F}_2$.

Since $\phi_2' \circ \sigma_1 = \sigma_2 \circ \phi_1'$ and $p_i((N(F_1) - F_1)\sharp_{\phi_1'}(N(F_2) - F_2)) = p_i((N(F_1') - F_1')\sharp_{\phi_2'}(N(F_2') - F_2')) = (N(\tilde{F}_1) - \tilde{F}_1)\sharp_{\tilde{\phi}}(N(\tilde{F}_2) - \tilde{F}_2)$, the quotient $X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ is diffeomorphic to $X_1'\sharp_{\tilde{\phi}}X_2'$.

Since $N(F_i)-F_i$ and $N(F_i')-F_i'$ are simply-connected and $p_i(N(F_i)-F_i) = p_i(N(F_i')-F_i') = N(\tilde{F}_i)-\tilde{F}_i$, the cylindrical end space $N(\tilde{F}_i)-\tilde{F}_i$ is simply-connected, $i = 1, 2$. By using the Seifert-Van Kampen Theorem, $X_1'\sharp_{\tilde{\phi}}X_2'$ is simply-connected.

If $X_1'\sharp_{\tilde{\phi}}X_2'$ is a symplectic 4-manifold then there is a non-trivial solution (A, ψ) of the Seiberg-Witten equations for the canonical Spin^c-structure. Since $T^3 \subset X_1'\sharp_{\tilde{\phi}}X_2'$ is a 3-dimensional torus dividing $X_1'\sharp_{\tilde{\phi}}X_2'$ into two pieces $X_1' - N(\tilde{F}_1)$ and $X_2' - N(\tilde{F}_2)$, cutting $X_1'\sharp_{\tilde{\phi}}X_2'$ along T^3, (A, ψ) sends to $(A_1 \vee A_2, \psi_1 \vee \psi_2)$, where (A_i, ψ_i) is a solution of the Seiberg-Witten equations on the space with cylindrical end $X_i' - N(\tilde{F}_i)$, $i = 1, 2$.

This means that if (A, ψ) a non-trivial solution of the Seiberg-Witten equations on $X_1'\sharp_{\tilde{\phi}}X_2'$, then at least one of (A_i, ψ_i) is a non-trivial solution of the Seiberg-Witten equations on $X_i' - N(\tilde{F}_i)$, $i = 1, 2$.

However, it is impossible. Because, by the additivity of Euler characteristic

$$\begin{aligned}\chi(X_i') &= \chi(X_i' - N(\tilde{F}_i)) + \chi(N(\tilde{F}_i)) - \chi((X_i' - N(\tilde{F}_i)) \cap N(\tilde{F}_i)) \\ &= \chi(X_i' - N(\tilde{F}_i)) + 2 - 2g = \chi(X_i' - N(\tilde{F}_i)),\end{aligned}$$

and by the Novikov additivity,

$$\begin{aligned}\mathrm{sign}(X_i') &= \mathrm{sign}(X_i' - N(\tilde{F}_i)) + \mathrm{sign}(N(\tilde{F}_i)) \\ &= \mathrm{sign}(X_i' - N(\tilde{F}_i)) + \tilde{F}_i \cdot \tilde{F}_i \\ &= \mathrm{sign}(X_i' - N(\tilde{F}_i)), \ i = 1, 2.\end{aligned}$$

Then we have $2 - 2b_1(X_i') + 2b_2^+(X_i') = 2 - 2b_1(X_i' - N(\tilde{F}_i)) + 2b_2^+(X_i' - N(\tilde{F}_i))$, and so $b_2^+(X_i' - N(\tilde{F}_i)) = 0$, $i = 1, 2$. Thus there is no non-trivial solution of the Seiberg-Witten equations over the cylindrical end space $X_i' - N(\tilde{F}_i)$, $i = 1, 2$.

Thus we conclude that there is no non-trivial solution of the Seiberg-Witten equations on $X_1'\sharp_{\tilde{\phi}}X_2'$ and the quotient $X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ is not a symplectic 4-manifold.

THEOREM 4.2. *Under the same situations with Proposition 4.1, the quotient* $X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ *is homeomorphic to* $\mathbb{CP}^2\sharp r\bar{\mathbb{CP}}^2$ *but not diffeomorphic to* $\mathbb{CP}^2\sharp r\bar{\mathbb{CP}}^2$, $r = b_2^-(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma)$.

Proof. Since the simply-connected, non-Kähler, closed, symplectic 4-manifold $X_1\sharp_{\phi_1',\phi_2'}X_2$ is a double cover of $X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ branched along $\amalg_{j=1}^{n_1}\Sigma_j^1$ and $\amalg_{j=1}^{n_2}\Sigma_j^2$, the quotient $X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ is smooth and simply-connected. The Euler characteristic and the signature of the quotient are

$$\begin{aligned}\chi(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) &= \chi(X_1'\sharp_{\tilde{\phi}}X_2') \\ &= \chi(X_1') + \chi(X_2') - 2\chi(T^2) \\ &= 12 + \frac{1}{2}(\Sigma_{j=1}^{n_1}\chi(\Sigma_j^1) + \Sigma_{j=1}^{n_2}\chi(\Sigma_j^2)),\end{aligned}$$

$$\begin{aligned}\mathrm{sign}(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) &= \mathrm{sign}(X_1'\sharp_{\tilde{\phi}}X_2') \\ &= \mathrm{sign}(X_1') + \mathrm{sign}(X_2') \\ &= -8 + \frac{1}{2}(\Sigma_{j=1}^{n_1}\Sigma_j^1\cdot\Sigma_j^1 + \Sigma_{j=1}^{n_2}\Sigma_j^2\cdot\Sigma_j^2).\end{aligned}$$

Then we have

$$\chi(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) + \mathrm{sign}(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) = 4,$$

$$b_2^+(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) = 1$$

and

$$b_2^-(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) = 9 + (\Sigma_{j=1}^{n_1}(1 - g(\Sigma_j^1)) + \Sigma_{j=1}^{n_2}(1 - g(\Sigma_j^2))),$$

where $g(\Sigma_j^i)$ is the genus of Σ_j^i, $i = 1, 2$.

Let

$$b_2^-(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) = 9 + (\Sigma_{j=1}^{n_1}(1 - g(\Sigma_j^1)) + \Sigma_{j=1}^{n_2}(1 - g(\Sigma_j^2))) = r.$$

Since $X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ is a simply-connected, smooth 4-manifold with

$$b_2^+(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) = 1 \quad \text{and} \quad b_2^-(X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma) = r,$$

$X_1\sharp_{\phi_1',\phi_2'}X_2/\sigma$ is homeomorphic to $\mathbb{CP}^2\sharp r\bar{\mathbb{CP}}^2$ by the Freedman's classification theorem of simply-connected, oriented, closed 4-manifolds [7].

The quotient $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$ is a Kähler surface and there is a non-trivial solution of the Seiberg-Witten equations for the Spin^c-structure ξ with $c_1(\det \xi = L) = c_1(\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2)$ and $c_1(L) \cup [\omega_g] < 0$ where ω_g is a self-dual harmonic form whose cohomology class lies in the component of the open positive cone in $H^2(\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2 : \mathbb{R})$ which contains H, a basis of $H^2(\mathbb{CP}^2)$. For details, see [12].

However, over the quotient $X_1 \sharp_{\phi_1', \phi_2'} X_2/\sigma$, there is no non-trivial solution of the Seiberg-Witten equations by Proposition 4.1 and so the Seiberg-Witten invariants are zero for all Spin^c-structures over $X_1 \sharp_{\phi_1', \phi_2'} X_2/\sigma$.

Since the Seiberg-Witten invariants of $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$ and $X_1 \sharp_{\phi_1', \phi_2'} X_2/\sigma$ are different, $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$ and $X_1 \sharp_{\phi_1', \phi_2'} X_2/\sigma$ are not diffeomorphic.

REMARK 4.3. For the anti-holomorphic involution σ on $X_1 \sharp_{\phi_1', \phi_2'} X_2$, we have shown that the quotient $X_1 \sharp_{\phi_1', \phi_2'} X_2/\sigma$ is not diffeomorphic to $\mathbb{CP}^2 \sharp r \bar{\mathbb{CP}}^2$, $r = b_2^-(X_1 \sharp_{\phi_1', \phi_2'} X_2/\sigma)$.

On the other hand, to find a counter example of the Akbulut's conjecture given in the introduction, we have to find an anti-symplectic involution σ on the simply-connected, non-Kähler, symplectic 4-manifold $X_1 \sharp_{\phi_1', \phi_2'} X_2$ with a smooth non-empty embedded surface as a fixed point set. In this paper, we have assumed that the σ on $X_1 \sharp_{\phi_1', \phi_2'} X_2$ has at least two embedded surfaces as fixed point sets. However if we investigate and apply the constructions of σ and $X_1 \sharp_{\phi_1', \phi_2'} X_2/\sigma$, it will be helpful to find an anti-holomorphic involution on a simply-connected, closed, non-Kähler, symplectic 4-manifold obtained from Dolgachev surface which has a Riemann surface as a fixed point set and understand the above conjecture.

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