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A SIMPLE PROOF OF A DIFFERENCE SEQUENCE

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ABSTRACT. It is well known that the general term of a sequence can be represented by a linear combination of binomial coefficients. In this paper we give a simple proof for this fact.

The difference sequence $\Delta\mathbf{a} = (\Delta a_0, \Delta a_1, \Delta a_2, \ldots)$ of a sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ is defined by $\Delta a_i = a_{i+1} - a_i, (i = 0, 1, 2, \ldots)$. For $j = 0, 1, 2, \ldots$, the jth difference sequence $\Delta^j\mathbf{a} = (\Delta^j a_0, \Delta^j a_1, \Delta^j a_2, \ldots)$ of $\mathbf{a}$ is defined inductively by $\Delta^j\mathbf{a} = \Delta(\Delta^{j-1}\mathbf{a})$ where $\Delta^0\mathbf{a} = \mathbf{a}$. The infinite matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ \Delta a_0 & \Delta a_1 & \Delta a_2 & \cdots \\ \Delta^2 a_0 & \Delta^2 a_1 & \Delta^2 a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is called the difference matrix of $\mathbf{a}$, and the sequence $(a_0, \Delta a_0, \Delta^2 a_0, \ldots)$ is called the dual sequence of $\mathbf{a}$.

The following is a well known theorem about the relationship between a sequence and its dual sequence.

THEOREM. *Let* $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ *be a sequence and let* $\hat{\mathbf{a}} = (\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots)$ *be the dual sequence of* $\mathbf{a}$. *Then*

$$a_n = \binom{n}{0}\hat{a}_0 + \binom{n}{1}\hat{a}_1 + \binom{n}{2}\hat{a}_2 + \cdots + \binom{n}{n}\hat{a}_n \tag{1}$$

for each $n = 0, 1, 2, \ldots$.

A proof of the Theorem based on the fact that

(i) if A and B are difference matrices of sequences $\mathbf{a}$ and $\mathbf{b}$ respectively and if α, β are numbers, then $\alpha A + \beta B$ is the difference matrix of $\alpha\mathbf{a} + \beta\mathbf{b}$, and

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(ii) for a fixed nonnegative integer p, $a_n = \binom{n}{p}$ is

$$\hat{a}_i = \begin{cases} 1 & \text{if } i = p \\ 0 & \text{if } i \neq p \end{cases}$$

is found in [1, pp. 261–267]. It is also noted in [1, p.269] that

$$\sum_{i=0}^{n} a_i = \binom{n+1}{1}\hat{a}_0 + \binom{n+1}{2}\hat{a}_1 + \cdots + \binom{n+1}{n+1}\hat{a}_n \tag{2}$$

of which the validity is due to the fact that

$$\binom{0}{j} + \binom{1}{j} + \binom{2}{j} + \cdots + \binom{n}{j} = \binom{n+1}{j+1}.$$

In this note we give simple proofs for (1) and (2).

We first prove the theorem by induction on n.

If $n = 0$, then $\sum_{j=0}^{n} \binom{n}{j} \Delta^j a_0 = \binom{0}{0}\Delta^0 a_0 = a_0$, and the induction starts.

Suppose that the theorem holds for n. Then

$$a_n = \binom{n}{0}a_0 + \binom{n}{1}\Delta a_0 + \binom{n}{2}\Delta^2 a_0 + \cdots + \binom{n}{n}\Delta^n a_0.$$

Let $\mathbf{b} = \Delta \mathbf{a} = (b_0, b_1, b_2, \ldots)$. Then the dual sequence of $\mathbf{b}$ is $(\Delta a_0, \Delta^2 a_0, \Delta^3 a_0, \ldots)$ because the difference matrix of $\mathbf{b}$ can be obtained from that of $\mathbf{a}$ by removing the topmost row, and by the inductive hypothesis we have

$$b_n = \binom{n}{0}\Delta a_0 + \binom{n}{1}\Delta^2 a_0 + \binom{n}{2}\Delta^3 a_0 + \cdots + \binom{n}{n}\Delta^{n+1} a_0.$$

Now

$$\begin{aligned} a_{n+1} &= b_n + a_n \\ &= a_0 + \left(\binom{n}{0} + \binom{n}{1}\right)\Delta a_0 + \left(\binom{n}{1} + \binom{n}{2}\right)\Delta^2 a_0 + \cdots \\ &\quad + \left(\binom{n}{n-1} + \binom{n}{n}\right)\Delta^n a_0 + \Delta^{n+1} a_0 \\ &= \binom{n+1}{0}a_0 + \binom{n+1}{1}\Delta a_0 + \binom{n+1}{2}\Delta^2 a_0 + \cdots \\ &\quad + \binom{n+1}{n+1}\Delta^{n+1} a_0 \end{aligned}$$

and the theorem is proved.

To prove (2), let $s_n = a_0 + a_1 + \cdots + a_n$ $(n = 0, 1, 2, \ldots)$ and let $\mathbf{c} = (0, s_0, s_1, s_2, \ldots)$. Then, since $\mathbf{a}$ is the difference sequence of $\mathbf{c}$, the difference matrix of $\mathbf{c}$ looks like

$$\begin{bmatrix} 0 & s_0 & s_1 & s_2 & \ldots \\ a_0 & a_1 & a_2 & a_3 & \ldots \\ \Delta a_0 & * & * & * & \ldots \\ \Delta^2 a_0 & * & * & * & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and we see that the dual sequence of $\mathbf{c}$ is $(0, a_0, \Delta a_0, \Delta^2 a_0, \ldots)$. Thus by the Theorem applied to $\mathbf{c}$ we get

$$s_n = \binom{n+1}{0} 0 + \binom{n+1}{1} a_0 + \binom{n+1}{2} \Delta a_0 + \cdots + \binom{n+1}{n+1} \Delta^n a_0$$

and (2) is proved.

References

[1] R. A. Brualdi, *Introductory combinatorics*, Prentice Hall, Upper Saddle River, NJ, 1999.

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