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FRAGMENTATION PROCESSES AND STOCHASTIC SHATTERING TRANSITION

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ABSTRACT. Shattering or disintegration of mass is a well known phenomenon in fragmentation processes first introduced by Kolmogorov and Filippop and extensively studied by many physicists. Though the mass is conserved in each break-up, the total mass decreases in finite time. We investigate this phenomenon in the n particle system. In this system, shattering can be interpreted such that, in uniformly bounded time on n, order n of mass is located in order $o(n)$ of clusters. It turns out that the tagged particle processes associated with the systems are useful tools to analyze the phenomenon. For the newly defined stochastic shattering based on the above ideas, we derive far sharper conditions of fragmentation kernels which guarantee the occurrence of such a phenomenon than our previous work [9].

1. Introduction

Consider the dynamics of n particles in a closed system. If i particles are joined together to form a cluster, we call it an i-cluster. Each cluster breaks up into small clusters after waiting an exponentially distributed amount of time with parameter depending on the size of the cluster. These fragmentation processes have many applications in physics, such as polymer degradation and break-up of many objects including rocks, liquid droplets, glass, etc [5, 6, 12]. The governing deterministic equation studied by many physicists is

$$\frac{d}{dt}c(x,t) = -a(x)c(x,t) + \int_x^\infty b(x|y)a(y)c(y,t)dy, \tag{1}$$

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where $c(x,t)$ is the distribution of the particles of mass x, $a(x)$ is the break up rate of the x-cluster into smaller clusters, and $b(z|x)$ is the rate of creation of z-clusters conditional on the fragmentation of the x-cluster.

Numerous exact solutions and interesting phenomena have been studied by many physicists independently of mathematicians for some time [5, 6, 14, 15]. However, the beginning of the study of this fragmentation process dates back to Kolmogorov[11]. Kolmogorov's result was soon generalized by Filippov[8] and recently by Bertoin among others [3, 4]. They consider a stochastic model of the following type. Each particle, say e, of mass $\mu(e)$ ($\mu(e)$: positive real number) waits an exponential amount of time with parameter $P(\mu(e))$ and splits into at most countable number of smaller particles e_k, $k = 1, 2, \dots$.

One of the most interesting phenomena in the fragmentation process is the shattering transition (or disintegration of mass) which is considered a counterpart of the well known gelation phenomena in the coagulation process. Though no mass is subtracted from the system during the break-up processes, for some kernels, the total mass $\int xc(x,t)dx$ decreases in finite time. This can be explained as due to the decomposition of the mass into an infinite number of particles of zero mass. Indeed, this was first introduced by Filippov, and he gave a necessary and sufficient condition for shattering under mild assumptions of the variables of the model. On the other hand, many physicists have found explicit solutions of (1) and for the case $a(x) = x^{\alpha}$ with special form of $b(z|x)$, they have shown that shattering occurs.

Based on these results, it is an interesting task to study the finite particle system of fragmentation introduced at the beginning of this section. It can be thought of as a discrete stochastic approximation of (1) or Kolmogorov and Fillipov's stochastic model, after letting the mass size of each particle be $1/n$. Moreover, it has its own advantages. Indeed, we can consider the case that higher cluster has a higher rate of break-up. Such a case can be found in the break up of glass or polymer, etc. It is also interesting that the largest cluster in the zero-range process of n particles on n sites shows such a dynamic if initially there is an n cluster in one site and the waiting time parameter $g(\cdot)$ is an increasing function of the size of the cluster. See Jeon et al.[10] for details. Note that, to the best of our knowledge, previous authors have never dealt with shattering transiton in this situation.

We assume that initially there is a single n-cluster. Shattering, then, can be characterized by saying that the time in which order n of mass (each particle has mass 1) is located in order $o(n)$ of clusters is uniformly bounded. See Jeon[9]. Let $X_t^n(i)$ represent the number of i-clusters at time t (see section 2 for the precise definition). Then, more precisely, we say stochastic shattering occurs if there exists a function $\phi(n)$ such that $\phi(n) = o(n)$ and exist $t_0 < \infty, \delta > 0$ satisfying

$$\liminf_{n\to\infty} P\left\{ \sum_{i=1}^{[\phi(n)]} iX_{t_0}^n(i) \geq \delta \right\} > 0.$$

Notice that the occurrence of shattering is a matter of the speed of the fragmentation process, and our idea is to estimate the speed by considering the tagged particle processes associated with the system, which gives a sharper results than our previous work [9]. The independence of the jump rates of the tagged particle in the n particle process makes the analysis possible. The speed of the tagged particle process can be carried out using a comparison with other simple processes. These types of stochastic dominance are justified by coupling arguments [13].

In this paper we mainly assume that clusters break up into only two small clusters, i.e., binary fragmentation. Multiple fragmentation is just a simple generalization of this model, and our method can be applied for it without big changes. (See the Remark after the proof of Theorem 2.)

2. Stochastic fragmentation processes and main theorems

In this section, we construct a system of finite state Markov chains associated with the rate constants $F^n(i,j)$, $i+j \leq n$, $n \geq 1$. In the nth Markov chain, there are n particles which form clusters. These clusters fragment at rates determined by $F^n(i,j)$ to make smaller clusters (any $(i+j)$-cluster breaks up into i-cluster and j-cluster with rate $F^n(i,j)$). After a suitable scaling, the Markov chains can be thought of as discrete, stochastic approximations to solutions of the fragmentation equation (1) [1].

NOTATION.

(a) *Let* $\mathbb{N} = \{0,1,2,\dots\}$, $\mathbb{N}^+ = \{1,2,3,\dots\}$.

(b) *Let* $E_n = \{\eta : \eta \in \mathbb{N}^{\mathbb{N}^+}, \sum_{k=1}^{\infty} k\eta(k) = n\}$.

(c) $[\cdot]$ *represents the largest integer function.*

(d) *Let* $\{e_i\}_{i=1}^{\infty}$ *be the basis of* $\mathbb{R}^{\mathbb{N}^+}$, *i.e.,* $e_i = (0,0,\dots,0,1,0,0,\dots)$, *where* 1 *is located in the* i*th coordinate.*

REMARK. Note that any $\eta \in E_n$ can be expressed by $(\eta(1), \eta(2), \dots, \eta(n))$, since all $\eta(i) = 0$ if $i > n$, or using the basis defined on the Notation (d), $\eta = \sum_{i=1}^{n} \eta(i) e_i$. Here, e_i means that there is an i-cluster. The space E_n is indeed the partition of n, and the process X_t^n which will be defined on this space can be considered as a random partition process.

Let $\{F^n(i,j)\}_{i+j \le n, n \ge 1}$ be a nonnegative sequence such that $F^n(i,j) = 0$ if $j < i$. For $i \le j$, let $\Delta_{ij}^n = (e_i + e_j - e_{i+j})$. Let X_t^n be the Markov process on E_n with generator

$$L_1^n f(\eta) = \sum_{i+j \le n} (f(\eta + \Delta_{i,j}^n) - f(\eta)) F^n(i,j) \eta(i+j) \tag{2}$$

for any bounded function defined on E_n.

We may describe the dynamics as follows:

The process waits at state η for an exponentially distributed amount of time with parameter

$$\lambda^n(\eta) \doteq \sum_{i+j \le n} F^n(i,j) \eta(i+j),$$

then jumps to state $\eta + \Delta_{ij}^n$ (or an $(i+j)$-cluster fragments to form an i and a j cluster) with probability

$$\frac{F^n(i,j)\eta(i+j)}{\lambda^n(\eta)}.$$

Since, for each n, the state space consists of finitely many points, i.e., $|E_n| < \infty$, there is a unique well defined pure jump process, say X_t^n on E_n for each n. We will call this sequence of processes $\{X_t^n\}_{n=1}^{\infty}$ the system of the stochastic fragmentation processes, and we will denote it simply by X_t^n. In general, we assume that the initial configuration $X_0^n = e_n \in E_n$, i.e., initially, there is a single n cluster.

In this system of processes, we can define the stochastic shattering phenomenon using the idea that in finite time at least δn, for some $\delta > 0$, amount of mass is located in the $o(n)$ order of clusters. More precisely,

DEFINITION 1. For given fragmentation kernels $F^n(i,j)$, we say stochastic shattering occurs if there exists a function $\phi(n)$ such that $\phi(n) = o(n)$ and $t_0 < \infty$, $\delta > 0$, satisfying

$$\liminf_{n \to \infty} P\left\{ \sum_{i=1}^{[\phi(n)]} i X_{t_0}^n(i) \ge \delta n \right\} > 0.$$

As mentioned in section 1, this finite particle system can be thought of as the discrete approximation of (1). Indeed, let $\tilde{E}_n$ be the space obtained from E_n by normalizing the coordinates by n, e.g., the kth coordinate becomes k/n-coordinate. That is,

$$\tilde{E}_n = \left\{ \sum_{i=1}^{n} \eta(i) e_{i/n} : \eta \in E_n \right\},$$

where $\{e_{i/n}\}$'s are the new basis and $e_{i/n}$ means that there is an (i/n)-cluster. Now, let us define the system of the scaled fragmentation process Y_t^n on $\tilde{E}_n$ with generator

$$L_2^n f(\eta) = \sum_{i+j \le n} (f(\eta + \tilde{\Delta}_{i,j}^n) - f(\eta)) \frac{F^n(i,j)}{F^n(n)} \eta(i+j) \tag{3}$$

for any bounded function defined on $\tilde{E}_n$, where $\eta \in \tilde{E}_n$, $\tilde{\Delta}_{i,j}^n = e_{i/n} + e_{j/n} - e_{(i+j)/n}$ and $F^n(n) = \sum_{1 \le i \le n-1} F^n(i, n-i)$.

In Y_t^n, a k-cluster becomes a k/n-cluster, and the jump rate $F^n(k)$ becomes $F^n(k)/F^n(n)$ so that the n-cluster is normalized to a 1-cluster and the fragmentation rate of this 1-cluster is normalized to 1. If $F^n(i,j)$ are chosen so that

$$\sum_{i+j \le k} F^n(i,j)/F^n(n) \to a(x)$$

and

$$\sum_{i-\epsilon n \le j \le i+\epsilon n} F^n(j, k-j)/F^n(k) \to \int_{z-\epsilon}^{z+\epsilon} b(y|x) dy,$$

where $x = \lim_{n,k\to\infty} k/n$, $z = \lim_{n,i\to\infty} i/n$, and $a(x)$ and $b(z|x)$ are defined in (1). Then Y_t^n is thought to be a discrete stochastic approximation of (1). Therefore, the mass located on the $o(n)$ cluster becomes zero in the limit. However, the rigorous proof of the convergence does not seem to be established. See [1] for physical derivation.

From now on, to make the notation simple, we will omit the largest integer symbol $[\cdot]$ if there is no difference in calculating the asymptotics. That is, if there is a number which is not in an integer form but should be, then notice that $[\cdot]$ is omitted. For example, in the following Condition 1, ϵk means $[\epsilon k]$ and $k/2$ is in fact $[k/2]$.

Condition 1. *There exist $0 < \epsilon < 1/2$ and $0 < \gamma \leq 1$ such that*

$$\sum_{l=\epsilon k}^{k/2} F_k^n(l) \geq \gamma$$

for all k, where $F^n(k) = \sum_{1 \leq i \leq k-1} F^n(i, k-i)$.

Remark. This assumption implies that if a k-cluster breaks up, then with positive probability it becomes two clusters of size bigger than or equal to ϵk. Therefore, this corresponds to the condition of $\int_\epsilon^{x/2} b(z|x)dz \geq \gamma$ in (1). This condition excludes the dust evaporation phenomena. For example, if $F^n(i, k-i) = f(k)\delta_i^1$, where $\delta_a^b = 1$ if $a = b$ and 0 if $a \neq b$ (i.e., Becker-Döring type fragmentation [2]), it does not satisfy Condition 1. In this case, any cluster emits only a single particle, which is invisible in the system ($o(n)$ order). Consequently, for some $F^n(k)$ (e.g., $F^n(k) = k^\alpha, \alpha > 1$), a huge cluster, even though we can not detect any fragmentation, reduces its mass as time passes. Obviously, there is no deterministic counterpart of this phenomenon.

Let l_0 be the first number which makes $\lambda^{l_0} \leq 1$.

Theorem 1. *Suppose $F^n(k)$ is decreasing and*

$$\limsup_{n\to\infty} \sum_{k=0}^{\infty} \frac{l}{F^n(\lambda^k n)} < \infty,$$

where $\lambda = 1 - \epsilon$. Then under Condition 1 stochastic shattering occurs.

In the deterministic analogue, the n-cluster of the stochastic model is normalized to a 1-cluster. Obviously, k-cluster corresponds to k/n-cluster in the deterministic case. Therefore, $a(x) = 1/x^\alpha$ in (1) matches $F^n(k) = (n/k)^\alpha$ in the stochastic model. In this case, we have:

Corollary 1. *If $F^n(k) = (n/k)^\alpha$, $\alpha > 0$, then under Condition 1 stochastic shattering occurs.*

More detailed picture can be given by:

Corollary 2. *If $F^n(k) = (\log(n/k))^\alpha + 1, \alpha > 1$, then under Condition 1 stochastic shattering occurs.*

Now consider a condition which is stronger than Condition 1:

CONDITION 2. *There exists* $\epsilon > 0$ *such that*

$$\sum_{i=1}^{\epsilon k} F_k^n(i) = 0$$

for all k.

THEOREM 2. *Suppose* $F^n(k)$ *is decreasing and*

$$\liminf_{n\to\infty} \sum_{k=0}^{\infty} \frac{1}{F^n(\lambda^k)} = \infty,$$

then under Condition 2 stochastic shattering does not occur.

COROLLARY 3. *If* $F^n(k) = (\frac{n}{k})^\alpha$, $\alpha \le 0$, *then under Condition 2 stochastic shattering does not occur.*

Also we have:

COROLLARY 4. *If* $F^n(k) = (\log(\frac{n}{k}))^\alpha + 1$, $\alpha \le 1$, *then under Condition 2 stochastic shattering does not occur.*

Note that F^n does not need to be a decreasing function. For example:

THEOREM 3. *If* $F^n(k) = k^\alpha$, $\alpha > 0$, *then under Condition 1 stochastic shattering occurs.*

3. Tagged particle processes

Consider the tagged particle process $Z_t^n \doteq (X_t^n, Y_t^n)$ defined on $E_n \times \mathbb{N}$, where Y_t^n is the position of the tagged particle. The process waits at state (η, k) for an exponentially distributed amount of time with parameter

$$\lambda^n(\eta) = \sum_{i+j\le n} F^n(i,j)\eta(i+j),$$

then jumps to state $(\eta^{i,j}, k)$ ($\eta^{i,j} \doteq \eta + e_{i+j} - e_i - e_j$) with probability

$$\frac{F^n(i,j)\eta(i+j)}{\lambda^n(\eta)},$$

if $i + j \ne k$, to state $(\eta^{i,j}, k)$ with probability

$$\frac{F^n(i,j)(\eta(i+j) - 1)}{\lambda^n(\eta)},$$

if $i+j=k$, to state $(\eta^{i,j}, i)$ with probability

$$\frac{iF^n(i,j)\eta(i+j)}{k\lambda^n(\eta)},$$

if $i+j=k$, to state $(\eta^{i,j}, j)$ with probability

$$\frac{jF^n(i,j)\eta(i+j)}{k\lambda^n(\eta)}.$$

We can express the process formally by the corresponding generator L_3^n defined on the set of bounded functions on $E_n \times \mathbb{N}$ such that

$$\begin{aligned} L_3^n f((n,k)) = & \sum_{i+j\le n, i+j\ne k} (f((\eta^{i,j},k)) - f((\eta,k)))F^n(i,j)\eta(i+j) \\ & + \sum_{i+j=k} (f((\eta^{i,j},k)) - f((\eta,k)))F^n(i,j)(\eta(k)-1) \\ & + \sum_{i=1}^{k-1}(f((\eta^{i,k-i},i)) - f((\eta,k)))\frac{i}{k}F^n(i,k-i). \end{aligned} \tag{4}$$

Note that this L_3^n satisfies the positive maximum principle in Theorem 2.2 in page 165 of [7]. Therefore, by the Theorems 4.1 and 5.4 in the same chapter of [7] about martingale problem, there is unique Markov process Z_t^n such that, for any bounded function f on $E_n \times \mathbb{N}$,

$$f(Z_t^n) - \int_0^t G_n f(Z_s^n) ds$$

is a martingale.

Though the formal definition is complicated, observe that the Y_t^n does not depend on the state of X_t^n. Indeed, Y_t^n is a time homogeneous Markov chain with transition probability

$$P\{Y_{t+h}^n = i \mid Y_t^n = k\} = \lambda_{k,i}^n h + o(h),$$

where $\lambda_{k,i}^n = F_k^n(i)F^n(k)i/k$.

3. Proofs of main theorems

We begin by proving the following Lemma.

LEMMA 1. *For X_t^n and Y_t^n given above, we have*

$$EX_t^n(j) = \frac{n}{j} P(Y_t^n = j). \tag{5}$$

Proof. For the n particles in the initially given n-cluster, we give an order, say 1, 2, ..., n. Let $Y_t^{n,k}$ be the location of kth particle at time t. Then $Y_t^{n,k}$ are identically distributed and have the same distribution with Y_t^n. For any $1 \leq j \leq n$,

$$jX_t^n(j) = \sum_{k=1}^{n} 1_{\{j\}}(Y_t^{n,k}).$$

Therefore,

$$EjX_t^n(j) = E\sum_{k=1}^{n} 1_{\{j\}}(Y_t^{n,k}) = \sum_{k=1}^{n} P(Y_t^{n,k} = j) = nP(Y_t^n = j),$$

hence we have (5).

Proof of Theorem 1. Let $\lambda = 1 - \epsilon$ and let

$$\mu_l^n = \gamma F^n(\lambda^{l-1} n).$$

Consider the pure birth process Z_t^n satisfying

$$P\{Z_{t+h}^n = l + 1 \mid Z_t^n = l\} = \mu_l^n h + o(h).$$

Then we can prove that the process Z_t^n is stochastically dominated by Y_n^t in the sense that before Z_t^n hits the state l_0, Y_t^n hits a state $i \leq 1/\lambda$. This will be done by showing that there exists a coupling. See Lindvall[13] for details about coupling methods. Let $A^0 = \{n\}$ and

$$A^l = \{i \in N : \lambda^l n \leq i < \lambda^{l-1} n\}$$

for $l = 1, 2, \dots, l_0$, where l_0 is the smallest number which makes $\lambda^l n \leq \phi(n)$. Note that $F_k^n(m)F^n(k) \geq \mu_l^n$ if $m \in A^{l+1}$. Let $D = \{1, 2, \dots, n\} \times \{0, 1, \dots, l_0\}$ and consider the coupled process C_t defined on D, with jump rates

$$(k, l) \to \begin{cases} (m, l) & \text{with rate } F_k^n(m)F^n(k) \text{ if } m \in A^l \setminus A^{l+1}, \\ (m, l) & \text{with rate } F_k^n(m)F^n(k) - \mu_l^n \text{ if } m \in A^{l+1}, \\ (m, l+1) & \text{with rate } \mu_l^n \text{ if } m \in A^{l+1}. \end{cases}$$

By the construction, we can write $C_t = (Y_t^n, Z_t^n)$ and this coupling shows the dominance as desired. It only remains to show that under the condition the process Z_t^n hits l_0 in finite time uniformly on n. Indeed, define J_n successively by $J_0 = 0$, and for $l \geq 1$,

$$J_l^n = \inf\{t > J_{l-1}^n : Z_{t-}^n \neq Z_t^n\}$$

and

$$T_i = \inf\{t \geq T_{i-1} \mid Z_t^n \neq i\}.$$

Then $J_{l_0}^n = T_1 + T_2 + \cdots + T_{l_0}$ and

$$E(J_{l_0}^n) = \sum_{l=0}^{l_0} \frac{1}{\mu_l^n} = \frac{1}{\gamma} \sum_{l=0}^{l_0} \frac{1}{F^n(\lambda^l n)} \leq M$$

for some constant M which does not depend on n. Let $t_0 = 3M$, then

$$P\left\{Y_{t_0} = \frac{1}{\lambda}\right\} \geq P\{J_{l_0}^n \leq t_0\} = 1 - P\{J_{l_0}^n \leq t_0\} \geq \frac{2}{3}.$$

From Lemma 1, we have

$$\sum_{i \leq 1/\lambda} EiX_{t_0}^n(i) \geq \frac{2}{3}n.$$

Therefore,

$$P\left\{\sum_{i \leq 1/\lambda} iX_{t_0}^n(i) \geq \frac{1}{3}n\right\} \geq \frac{1}{3},$$

since, if not,

$$E\sum_{i \leq 1/\lambda} iX_{t_0}^n(i) < n\frac{1}{3} + \frac{1}{3}n\frac{2}{3} = \frac{5}{9},$$

and contradiction follows. Therefore by the choice of $\phi(n) = 1/\lambda$ and $\delta = 1/3$ in the Definition 1, we are done.

Proof of Corollary 1. For $F^n(k) = (n/k)^\alpha$, $\alpha > 0$, we have

$$\sum_{l=0}^{l_0} \frac{l}{F^n(\lambda^l n)} \leq \sum_{l=0}^{\infty} \frac{l}{(\frac{1}{\lambda^l})^\alpha} = \sum_{l=0}^{\infty} l\lambda^{\alpha l} \leq M$$

for some M independent of n. Therefore $F^n(k)$ meets the conditions of Theorem 1.

Proof of Corollary 2. For $F^n(k) = (\log(n/k))^\alpha + 1,\ \alpha > 10$, we have

$$\sum_{l=0}^{l_0} \frac{l}{F^n(\lambda^l n)} \leq \sum_{l=0}^{\infty} \frac{l}{(\log \frac{1}{\lambda^l})^\alpha + 1} = \sum_{l=0}^{\infty} \frac{1}{l^\alpha(-\log\lambda) + 1} \leq M$$

for some M independent of n. Therefore $F^n(k)$ meets the conditions of Theorem 1.

Proof of Theorem 2. For any given $\phi(n) = o(n)$, let l_0 be the smallest integer satisfying $\epsilon^{l_0} n \leq \phi(n)$. Note that $l_0 \to \infty$ and $n \to \infty$, since $l_0 \geq (\log\phi(n) - \log n)/\log\epsilon$. Consider the pure birth process Z_t^n satisfying

$$P\{Z_{t+h}^n = l+1 \mid Z_t^n = l\} = \mu_l^n h + o(h),$$

where $\mu_l^n = F^n(\epsilon^l n)$. Note that

$$\mu_l^n \geq \sum_{m\in A^{l+1}} F_k^n(m) F^n(k).$$

Now we can prove that the process Z_t^n stochastically dominates Y_t^n in the sense that Z_t^n hits the state l_0, before Y_t^n hits the state 1. Again, this will be done by showing that there exists a coupling. For same A^0, A^l and D, consider the coupled process C_t defined on D, with jump rates

$$(k,l) \to \begin{cases} (m,l) & \text{with rate } F_k^n(m)F^n(k) \ \text{ if } m \in A^l \setminus A^{l+1}, \\ (m,l+1) & \text{with rate } F_k^n(m)F^n(k) \ \text{ if } m \in A^{l+1}, \\ (k,l+1) & \text{with rate } \mu_l^n - \sum_{m\in A^{l+1}} F_k^n(m)F^n(k). \end{cases}$$

We can also write $C_t = (Y_t^n, Z_t^n)$ and this coupling shows the dominance as desired.

Now let us show that under the condition the process Z_t^n hits l_0 in finite time uniformly on n. Indeed, define J_n successively by $J_0 = 0$, and for $l \geq 1$,

$$J_l^n = \inf\{t > J_{l-1}^n : Z_{t-}^n \neq Z_t^n\}$$

and

$$T_i = \inf\{t \geq T_{i-1} \mid Z_t^n \neq i\}.$$

Then $J_{l_0}^n = T_1 + T_2 + \cdots + T_{l_0}$ and

$$E(e^{-J_{l_0}^n}) = E\left(\prod_{k=1}^{l_0} e^{-T_k}\right) = \prod_{k=1}^{l_0} Ee^{-T_k} = \prod_{k=1}^{l_0} \frac{\mu_k^n}{1+\mu_k^n} = \prod_{k=1}^{l_0} \frac{1}{1+1/\mu_k^n}.$$

Since the last term tends to infinity as n tends to 0, for any t_0 and $\epsilon > 0$ there exists N so that for any $n \geq N$,

$$P\{J_{l_0}^n < t_0\} < \epsilon.$$

Consequently, the conditions of Definition 1 is not fulfilled.

REMARK. The above proofs do not depend on the fact that the fragmentation is binary. Therefore, we can apply the same method for the multiple fragmentation.

Proofs of Corollaries 3 and 4. It is clear by the same way of the proofs of Corollaries 1 and 2.

Proof of Theorem 3. In the proof of Theorem 1, the only difference is the estimation of jump rates on each block. By the similar calculation of Corollary 1, we are done.

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