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POSITIVE ANSWER TO THE CONJECTURE BY FONG AND ISTRATESCU

SALAH MECHERI AND AHMED BACHIR

ABSTRACT. In this note we give a positive answer to the conjecture by Fong and Istratescu.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle,\rangle$ and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. For T in $\mathcal{B}(H)$ the adjoint of T is denoted by T^*. An operator T in $\mathcal{B}(H)$ is said to be hermitian if $T = T^*$. It is well known that hermitian operators can be characterized in the following way: an operator T in $\mathcal{B}(H)$ is hermitian if and only if $\langle Tx, x\rangle$ is real. In [4], the authors gave an other characterization involving inequalities. We denote by (WN) the class of operators in $\mathcal{B}(H)$ satisfying the following inequality

$$(\mathrm{Re}T)^2 \leq |T|^2,$$

where $\mathrm{Re}T = \frac{(T+T^*)}{2}$ is the real part of T and we will write $\mathrm{Im}T = \frac{(T-T^*)}{2i}$ for the imaginary part of T. It is easy to check that

$$T = \mathrm{Re}T + i(\mathrm{Im}T)$$

and T is Hemitian if and only if $\mathrm{Im}T = 0$. We write $|T|$ for the positive square root of T^*T. Note that T is said to be hyponormal if $TT^* \leq T^*T$. This class has been introduced by Fong and Istratescu[4] whom conjectured if $T \in (WN)$ and $\sigma(T)$ (the spectrum of T) is real, then T is a hermitian operator. Note that T is hermitian if and only if

$$(\mathrm{Re}T)^2 \geq |T|^2. \tag{1.1}$$

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By reversing the inequality (1.1) we obtain the class (WN). This class contains the class of hyponormal operators. Endeed, T hyponormal implies that

$$(\mathrm{Re}T)^2 + (\mathrm{Im}T)^2 \leq |T|^2 .$$

Since $(\mathrm{Im}T)^2$ is a positive operator,

$$(\mathrm{Re}T)^2 \leq |T|^2 ,$$

that is, $T \in (WN)$. In this note we will give a positive answers to this conjecture by generalizing the case of hyponormal operators.

2. Main results

Let us listing some spectral properties of operators in (WN).

PROPOSITION 2.1. [4] *Suppose that $T \in B(H)$ is in (WN). Then we have*

(i) *If λ is a real number, then $T - \lambda$ is also in (WN).*
(ii) $\|(T-\lambda)^*x\| \leq 3\,\|T-(\lambda)x\|$, $\forall x \in H$ *and all real number* λ.
(iii) *If M is an invariant subspace of T, then $T|_M$ is also in (WN); if furthermore, $T|_M$ is hermitian, then M reduces T.*
(iv) *If λ is real eigenvalue of T, then the eigenspace* $\ker(T-\lambda)$ *reduces* T.
(v) *If λ is real and $(T-\lambda)^n x = 0$ for some $n \geq 1$, then $(T-\lambda)x = 0$.*
(vi) *If λ_1, λ_2 are two different real eigenvalues of T, then* $\ker(T-\lambda_1 I)$ *is orthogonal to* $\ker(T-\lambda_2 I)$.

REMARK 2.1. As a consequence of the previous proposition, if $(\lambda_n)_{n\in\mathbb{N}}$ is a countable sequence of real eigenvalues of T in (WN), then $H = M \oplus M^{\perp}$ and $T|_M$ is hermitian, where $M = \bigoplus_{n\geq 0} \ker(T-\lambda I)$.

LEMMA 2.1. [4] *If T is in (WN) and T is similar to a normal operator, then T is normal.*

Now we will recall Berberian's techniques.

PROPOSITION 2.2. *Let H be a complex Hilbert space. Then there exists an Hilbert space $\mathcal{H}^\circ \supset H$, and an isometric *-isomorphism*

$$\varphi : \mathcal{B}(H) \mapsto \mathcal{B}(H^\circ)\ (A \mapsto A^\circ)$$

preserving order, i.e., for all $A, B \in \mathcal{B}(H)$ and for all $\alpha, \beta \in C$ we have:
1) $\varphi(A^*) = \varphi(A)^*$;
2) $\varphi(\alpha A + \beta B) = \alpha\varphi(\alpha A) + \beta\varphi(\alpha B)$;
3) $\varphi(I_{\mathcal{H}}) = \varphi(I_{\mathcal{H}^\circ})$;

4) $\varphi(AB) = \varphi(A)\varphi(B)$;
5) $\| \varphi(A) \| = \| A \|$;
6) $\varphi(A) \leq \varphi(B)$ *if* $A \leq B$;
7) $\sigma(\varphi(A)) = \sigma(A)$, $\sigma_a(A) = \sigma_a(\varphi(A)) = \sigma_p(\varphi(A))$
8) *if A is a positive operator, then* $\varphi(A^\alpha) = |\varphi(A)|^\alpha$ *for all* $\alpha > 0$.

LEMMA 2.2. *If $T \in (WN)$, then $\varphi(T) \in (WN)$.*

Proof. If $T \in (WN)$, then $(\mathrm{Re}T)^2 \leq |T|^2$. Therefore

$$[\mathrm{Re}\varphi(T)]^2 = \varphi[(\mathrm{Re}T)^2];\ |\varphi(T)|^2 = \varphi(|(T)|^2).$$

Since φ is a isometric*-isomorphism preserving order,

$$(\mathrm{Re}T)^2 \leq |T|^2 \Rightarrow (\mathrm{Re}\varphi(T))^2 \leq |\varphi(T)|^2,$$

that is, $\varphi(T) \in (WN)$.

LEMMA 2.3. *Let Q be a quasinilpotent operator in (WN). If Q is a compact operator, then $Q = 0$.*

Proof. Assume $Q \neq 0$. Then there exists a non vanishing vector x in $\mathcal{H}$ such that $Qx \neq 0$. By applying Proposition 2.1 we get

$$Q^n x \neq 0, \forall n \in \mathbb{N}.$$

Because, if there exists $p \in \mathbb{N}$ such that $Q^p x = 0$, then $Qx = 0$. which contradicts our hypothesis.

Now consider the sequence $(y_n)_n$ such that

$$y_n = \frac{Q^n x}{\|Q^n x\|}, \forall n \geq 1.$$

It is clear that the sequence (y_n) is bounded with norm equal to 1. Since Q is compact, there exists a subsequence $(y_{n_k})_{n_k}$ of $(y_n)_n$ such that $Q(y_{n_k})$ converges to z (in the norm topology) with

$$Q(y_{n_k}) = \frac{Q^{n_k+1}x}{\|Q^{n_k}x\|}, \quad y_{n_k+1} = \frac{\|Q^{n_k}x\|}{\|Q^{n_k+1}x\|}Q(y_{n_k}).$$

Let C_n be a sequence such that

$$C_n = \frac{\|Q^n\|}{\|Q^{n+1}\|}, \forall n \geq 1.$$

By using the fact that Q is quasinilpotent, we prove that $\lim_n C_n = 1$. Therefore

$$\lim_{n_k} y_{n_k} = \lim_{n_k} y_{n_k+1} = \lim_{n_k} Q(y_{n_k}) = Q(\lim_{n_k} y_{n_k}) = Qz,$$

that is, $z = Q(z)$ and since $y_n \neq 0$ for all $n \geq 1$, $z \neq 0$. It follows that 1 is an eigeinvalue, which is impossible because $\sigma(Q) = \{0\}$. Finally we deduce that $Q = 0$

Now we are ready to give a positive answer to the conjecture by Fong and Istratescu. We begin by the following lemma. Note that concerning this lemma more general results in this direction can be found in [6] and [7].

LEMMA 2.4. *If T is an algebraic operator, then $\sigma(T) = \sigma_p(T)$* (point spectrum of T).

Proof. It is well known that an operator T is algebraic if and only if its spectrum consists of poles only. But a pole of an operator is always an eigenvalue. Hence for an algebraic operator the spectrum and the point spectrum coincide.

THEOREM 2.1. *If T is a compact operator in (WN) with a real spectrum, then T is hermitian.*

Proof. Let $K = \bigoplus_{\lambda \in \sigma_p(T)-\{0\}} M_\lambda$, where M_λ is the eigenspace associated to the eigenvalue λ of T. By applying Proposition 2.1, it follows that K reduces T and $T\ |_{K^\perp}$ is a compact quasi-nilpotent operator in (WN). It results from Lemma 2.3 that $T\ |_{K^\perp} = 0$. Consequently $T = T\ |_K$ is hermitian.

REMARK 2.2. Theorem 2.1 generalizes the result given by [4, Corollary 2.2] which is the following:

COROLLARY 2.1. *If T is an operator in (WN) with a real spectrum and $\dim \mathcal{H} < \infty$, then T is hermitian.*

THEOREM 2.2. *Let T be algebraic. If $T \in (WN)$ and $\sigma(T)$ is real, then T is hermitian.*

Proof. Since $\sigma(T) \subset R$, it follows from Proposition 2.1 and Lemma 2.4, that $\sigma(T) = \sigma_p(T)$. Therefore T is hermitian by Proposition(2.1(vi))

COROLLARY 2.2. *Every idempotent operator in (WN) is an orthogonal projection.*

Proof. An idempotent T satisfies $T^2 = T$, which proves that T is algebraic in (WN) with a real spectrum and the result follows by Lemma 2.2.

THEOREM 2.3. *Let $T \in \mathcal{B}(H)$ such that $P(T)$ is normal for certain polynomial P. If $T \in (WN)$ with a real spectrum, then T is hermitian.*

According to [5], there exists a reducing sub-spaces $\mathcal{H}_n$ for T such that $H = \bigoplus_{n=0}^{\infty} H_n$, $T_0 = T \mid_{H_0}$ is algebraic and $T_n = T \mid_{H_n}$, $(n \geq 1)$ similar to a normal operator. Since $T \in (WN)$ with a real spectrum, it results from Proposition 2.1(iii) that $T_n \in (WN)$ and $\sigma(T_n)$ is real for all $n \geq 0$. Now by Theorem 2.2 and Theorem 2.3 we obtain T_n $(n \geq 0)$ is hermitian. Finally T is hermitian.

THEOREM 2.4. *Let $T \in (WN)$. If $\sigma(T) = \sigma_a(T) \subset R$, where $\sigma_a(T)$ is the approximate spectrum of T, then T is hermitian.*

Proof. Since $T \in (WN)$, by Lemma 2.2 $\varphi(T) \in (WN)$. By applying Proposition 2.1(ii) we get,

$$\sigma(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T)).$$

Hence it results by Remark 2.1 that $\varphi(T)$ is hermitian. Which proves that T is hermitian.

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SALAH MECHERI, DEPARTMENT OF MATHEMATICS, KING SAUD UNIVERSITY COLLEGE OF SCIENCE, P.O.BOX 2455, RIYADH 11451, SAUDI ARABIA
E-mail: mecherisalah@hotmail.com

AHMED BACHIR, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KING KHALED UNIVERSITY, ABHA, P.O.BOX 9004, SAUDI ARABIA
E-mail: bachir-ahmed@hotmail.com