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THE $M/G/1$ FEEDBACK RETRIAL QUEUE WITH TWO TYPES OF CUSTOMERS

Yong Wan Lee

Abstract. In M/G/1 retrial queueing system with two types of customers and feedback, we derived the joint generating function of the number of customers in two groups by using the supplementary variable method. It is shown that our results are consistent with those already known in the literature when $\delta_k = 0 (k = 1, 2)$, $\lambda_1 = 0$ or $\lambda_2 = 0$.

1. Introduction

In recent years there have been significant contributions to the retrial queueing system. Choi and Park[3] investigated an $M/G/1$ retrial queue with two types of customers in which the service distributions for both types of customers are the same. Thereafter, Falin et al.[7] investigate much the same model of Choi and Park[3], in which they assumed different service distributions for both types of customers. Recently Choi et al.[1] studied an $M/G/1$ retrial queueing system with two types of calls and finite capacity.

In this paper we deal with feedback retrial queue with two types of customers where after being served each customer either joins the retrial group or departs the system permanently. The phenomena of feedback in the retrial queueing systems are occurred in many practical situation; for instance telecommunication system where message turned out error at destination sends again.

In section 2, we describe the model. In section 3, we use the supplementary variable method to derive the joint generating function of the number of customers in the two groups and the mean queue size.

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In section 4 we show that our results are consistent with those already known in the literature when $\delta_k = 0$ $(k = 1, 2)$, $\lambda_1 = 0$ or $\lambda_2 = 0$.

2. Mathematical model

We consider a single server queueing system in which two different types of customers arrive according to independent Poisson streams with rates λ_1 and λ_2, respectively. See the block diagram of feedback retrial queue with two types of customers in Figure 1.

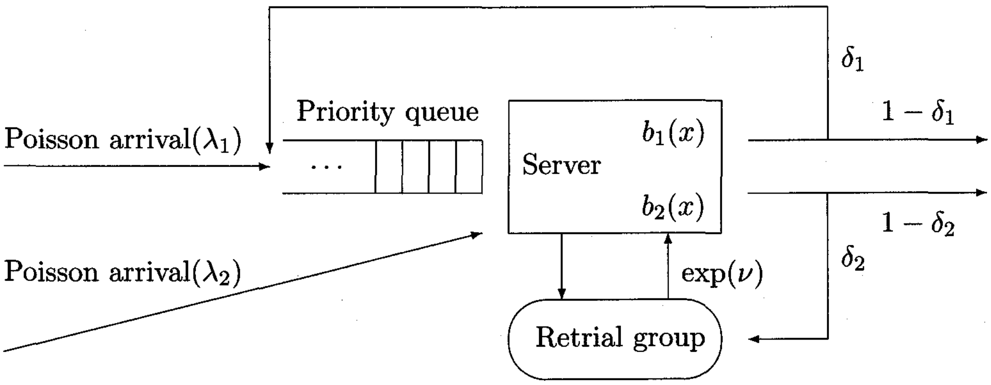

FIGURE 1. M/G/1 feedback retrial queue with two types

Customers from the Poisson flow with rate λ_1(the Poisson flow with rate λ_2, respectively) can be identified as priority customers (non-priority customers, respectively) in the system. If an arriving priority customer finds the server idle, he immediately starts to receive service. If he finds the server busy, he is queued in the priority group and then served in accordance with some discipline such as FCFS or random order.

On the other hand, when an arriving non-priority customer finds the server idle, he obtains service immediately. If he finds the server busy, he joins the retrial group in order to seek service again after a random amount of time. He persists this way until he is eventually served. All the customers in the retrial group behave independently of each other. The retrial time (the time interval between two consecutive attempts made by a customer in the retrial group) is exponentially distributed with mean $\frac{1}{\nu}$ and is independent of all previous retrial times and all other stochastic process in the system.

The service times of both types of customers are independent of each other. The service time B_k has a general distribution with p.d.f. $b_k(x)$ and mean b_k, $k = 1, 2$, where $k = 1$ is related to the priority customers and $k = 2$ is related to the non-priority customers.
Let $b_k^*(\theta) = \int_0^\infty e^{-\theta x} b_k(x)dx$ be the Laplace transform of service time B_k, $k = 1, 2$.

A priority customer who has received service departs the system with probability $1 - \delta_1$ or return to the priority group for more service with probability δ_1. A non-priority customer who has received service leaves the system with probability $1 - \delta_2$ or rejoins the retrial group with probability δ_2.

3. The joint distribution of queue sizes

We define the following random variables in order to characterize our system at an arbitrary time;

$N_1(t)$ = the number of customers in the priority group (excluding the customer in the service) at time t,

$N_2(t)$ = the number of customers in the retrial group at time t,

$X(t)$ = the residual service time of the customer in the service at time t,

$$\xi(t) = \begin{cases} 0, & \text{when serve is idle at time } t, \\ 1, & \text{when server services the priority customer at time } t, \\ 2, & \text{when server services the non-priority customer at time } t. \end{cases}$$

Then the stochastic process $X(t) = (\xi(t), N_1(t), N_2(t), X(t); t \geq 0)$ is the Markovian process with state space $\{0, 1, 2\} \times \mathbb{Z}_+^2 \times \mathbb{R}_+$ and denote by (ξ, N_1, N_2, X) the limiting random variable of $(\xi(t), N_1(t), N_2(t), X(t))$. We define the related probabilities;

$$q_j = P\{\xi = 0, N_2 = j\}, \quad j = 0, 1, 2, \ldots;$$

$$p_{kij}(x)dx = P\{\xi = k, N_1 = i, N_2 = j, X \in (x, x + dx]\},$$
$$k = 1, 2, \ i, j = 0, 1, 2, \ldots, \text{ and } x \geq 0;$$

and their Laplace transforms

$$p_{kij}^*(\theta) = \int_0^\infty e^{-\theta x} p_{kij}(x)dx, \quad k = 1, 2, \ i, j = 0, 1, 2, \ldots.$$

Note that $p_{kij}^*(0) = \int_0^\infty p_{kij}(x)dx = P\{\xi = k, N_1 = i, N_2 = j\}$ is the steady state probability that there are i customers in the priority group, j customers in the retrial group and the server services the k-type customer. The usual arguments lead to the following system of difference equations;

$$(1a)\qquad \begin{aligned}(\lambda_1+\lambda_2+j\nu)q_j = &(1-\delta_1)p_{10j}(0)\\ &+(1-\delta_2)p_{20j}(0)+\delta_2 p_{20j-1}(0),\end{aligned}$$

$$(1b)\qquad \begin{aligned}-p'_{10j}(x) = &-(\lambda_1+\lambda_2)p_{10j}(x)+\lambda_1 b_1(x)q_j+\lambda_2 p_{10j-1}(x)\\ &+\delta_1 b_1(x)p_{10j}(0)+(1-\delta_1)b_1(x)p_{11j}(0)\\ &+\delta_2 b_1(x)p_{21j-1}(0)+(1-\delta_2)b_1(x)p_{21j}(0),\end{aligned}$$

$$(1c)\qquad \begin{aligned}-p'_{1ij}(x) = &-(\lambda_1+\lambda_2)p_{1ij}(x)+\lambda_1 p_{1i-1j}(x)+\lambda_2 p_{1ij-1}(x)\\ &+\delta_1 b_1(x)p_{1ij}(0)+(1-\delta_1)b_1(x)p_{1i+1j}(0)\\ &+\delta_2 b_1(x)p_{2i+1j-1}(0)+(1-\delta_2)b_1(x)p_{2i+1j}(0),\end{aligned}$$

$$(1d)\qquad \begin{aligned}-p'_{20j}(x) = &-(\lambda_1+\lambda_2)p_{20j}(x)+\lambda_2 b_2(x)q_j\\ &+(j+1)\nu b_2(x)q_{j+1}+\lambda_2 p_{20j-1}(x),\end{aligned}$$

$$(1e)\qquad \begin{aligned}-p'_{2ij}(x) = &-(\lambda_1+\lambda_2)p_{2ij}(x)+\lambda_1 p_{2i-1j}(x)+\lambda_2 p_{2ij-1}(x)\\ &+\lambda_2 p_{2ij-1}(x),\end{aligned}$$

where $i=1,2,\dots,\ j=0,1,2,\dots,\ p_{kij}=0$ for $i,j<0,\ k=1,2$ and any $x\geq 0$.

By taking Laplace transform (1b)–(1e), we obtain

$$(2b)\qquad \begin{aligned}&\{\theta-(\lambda_1+\lambda_2)\}p^*_{10j}(\theta)+\lambda_2 p^*_{10j-1}(\theta)\\ &=p_{10j}(0)-\lambda_1 b^*_1(\theta)q_j-\delta_1 b^*_1(\theta)p_{10j}(0)-(1-\delta_1)b^*_1(\theta)p_{11j}(0)\\ &\quad -\delta_2 b^*_1(\theta)p_{21j-1}(0)-(1-\delta_2)b^*_1(\theta)p_{21j}(0),\end{aligned}$$

$$(2c)\qquad \begin{aligned}&\{\theta-(\lambda_1+\lambda_2)\}p^*_{1ij}(\theta)+\lambda_1 p^*_{1i-1j}(\theta)+\lambda_2 p^*_{1ij-1}(\theta)\\ &\qquad =p_{1ij}(0)-\delta_1 b^*_1(\theta)p_{1ij}(0)-(1-\delta_1)b^*_1(\theta)p_{1i+1j}(0)\\ &\qquad\quad -\delta_2 b^*_1(\theta)p^*_{2i+1j-1}(0)-(1-\delta_2)b^*_1(\theta)p_{2i+1j}(0),\end{aligned}$$

$$(2d)\qquad \begin{aligned}&\{\theta-(\lambda_1+\lambda_2)\}p^*_{20j}(\theta)+\lambda_2 p^*_{20j-1}(\theta)\\ &\qquad =p_{20j}(0)-\lambda_2 b^*_2(\theta)q_j-(j+1)\nu b^*_2(\theta)q_{j+1},\end{aligned}$$

$$(2e)\qquad \{\theta-(\lambda_1+\lambda_2)\}p^*_{2ij}(\theta)+\lambda_1 p^*_{2i-1j}(\theta)+\lambda_2 p^*_{2ij-1}(\theta)=p_{2ij}(0).$$

We introduce the following generating function for complex z with $|z| \leq 1$,

$$Q(z_2) = \sum_{j=0}^{\infty} q_j z_2^j,$$

$$P_{ki}^*(\theta, z_2) = \sum_{j=0}^{\infty} p_{kij}^*(\theta) z_2^j, \quad k = 1, 2,$$

$$P_{ki}(0, z_2) = \sum_{j=0}^{\infty} p_{kij}(0) z_2^j, \quad k = 1, 2.$$

Multiplying equations (1a) and (2b)–(2e) by z_2^j and summing over all j, we obtain the following basic system equations;

$$(3a) \quad \begin{aligned} &(\lambda_1 + \lambda_2)Q(z_2) + \nu z_2 Q'(z_2) \\ &\qquad = (1 - \delta_1)P_{10}(0, z_2) + (1 - \delta_2 + \delta_2 z_2)P_{20}(0, z_2), \end{aligned}$$

$$(3b) \quad \begin{aligned} &\{\theta - (\lambda_1 + \lambda_2) + \lambda_2 z_2\}P_{10}^*(\theta, z_2) \\ &= (1 - \delta_1 b_1^*(\theta))P_{10}(0, z_2) - \lambda_1 b_1^*(\theta)Q(z_2) \\ &\quad - (1 - \delta_1)b_1^*(\theta)P_{11}(0, z_2) - (1 - \delta_2 + \delta_2 z_2)b_1^*(\theta)P_{21}(0, z_2), \end{aligned}$$

$$(3c) \quad \begin{aligned} &\{\theta - (\lambda_1 + \lambda_2) + \lambda_2 z_2\}P_{1i}^*(\theta, z_2) + \lambda_1 P_{1i-1}^*(\theta, z_2) \\ &\qquad = (1 - \delta_1 b_1^*(\theta))P_{1i}(0, z_2) - (1 - \delta_1)b_1^*(\theta)P_{1i+1}(0, z_2) \\ &\qquad\quad - (1 - \delta_2 + \delta_2 z_2)b_1^*(\theta)P_{2i+1}(0, z_2), \end{aligned}$$

$$(3d) \quad \begin{aligned} &\{\theta - (\lambda_1 + \lambda_2) + \lambda_2 z_2\}P_{20}^*(\theta, z_2) \\ &\qquad = P_{20}(0, z_2) - \lambda_2 b_2^*(\theta)Q(z_2) - \nu b_2^*(\theta)Q'(z_2), \end{aligned}$$

$$(3e) \quad \{\theta - (\lambda_1 + \lambda_2) + \lambda_2 z_2\}P_{2i}^*(\theta, z_2) + \lambda_1 P_{2i-1}^*(\theta, z_2) = P_{2i}(0, z_2).$$

Define the generating functions of $P_k^*(\theta, z_1, z_2)$ and $P_k(0, z_1, z_2)$ for $k = 1, 2$ as follows;

$$P_k^*(\theta, z_1, z_2) = \sum_{i=0}^{\infty} P_{ki}^*(\theta, z_2) z_1^i,$$

$$P_k(0, z_1, z_2) = \sum_{i=0}^{\infty} P_{ki}(0, z_2) z_1^i.$$

Note that $P_k^*(0, z_1, z_2) = E(z_1^{N_1} z_2^{N_2}; \xi = k)$ which is the joint generating function of (N_1, N_2) when the server services the k-type customer.

Multiplying equations (3b)–(3e) by z_1^i and summing over all i, we obtain

$$\begin{aligned}(4a)\quad &\{\theta-\lambda_1(1-z_1)-\lambda_2(1-z_2)\}P_1^*(\theta,z_1,z_2)\\ &=\{1-\delta_1 b_1^*(\theta)-\frac{(1-\delta_1)b_1^*(\theta)}{z_1}\}P_1(0,z_1,z_2)\\ &\quad-\frac{(1-\delta_2+\delta_2 z_2)b_1^*(\theta)}{z_1}P_2(0,z_1,z_2)+\frac{(1-\delta_1)b_1^*(\theta)}{z_1}P_{10}(0,z_2)\\ &\quad+\frac{(1-\delta_2+\delta_2 z_2)b_1^*(\theta)}{z_1}P_{20}(0,z_2)-\lambda_1 b_1^*(\theta)Q(z_2),\end{aligned}$$

$$\begin{aligned}(4b)\quad &\{\theta-\lambda_1(1-z_1)-\lambda_2(1-z_2)\}P_2^*(\theta,z_1,z_2)\\ &\qquad=P_2(0,z_1,z_2)-\lambda_2 b_2^*(\theta)Q(z_2)-\nu b_2^*(\theta)Q'(z_2).\end{aligned}$$

By choosing $\theta=\lambda_1(1-z_1)+\lambda_2(1-z_2)$into (4a) and (4b), we eliminate $P_1^*(\theta,z_1,z_2)$ and $P_2^*(\theta,z_1,z_2)$ from (4a) and (4b) respectively and obtain

$$\begin{aligned}(5)\quad &\{z_1-(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2)\}P_1(0,z_1,z_2)\\ &\qquad=\beta_1(z_1,z_2)\{(1-\delta_2+\delta_2 z_2)P_2(0,z_1,z_2)+\lambda_1 z_1 Q(z_2)\\ &\qquad\quad-(1-\delta_1)P_{10}(0,z_2)-(1-\delta_2+\delta_2 z_2)P_{20}(0,z_2)\},\end{aligned}$$

$$(6)\qquad P_2(0,z_1,z_2)=\beta_2(z_1,z_2)\{\lambda_2 Q(z_2)+\nu Q'(z_2)\},$$

where $\beta_k(z_1,z_2)=b_k^*(\lambda_1(1-z_1)+\lambda_2(1-z_2))$, $k=1,2$.
Now we consider the function

$$(7)\qquad h(z_1,z_2)=z_1-(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2).$$

By using Rouche's theorem it follows that for each z_2 with $|z_2|<1$, there is a unique solution $z_1=\phi(z_2)$ of the equation $h(z_1,z_2)=0$ in the unit circle, i.e.,

$$h(\phi(z_2),z_2)=\phi(z_2)-(1-\delta_1+\delta_1\phi(z_2))\beta_1(\phi(z_2),z_2)=0.$$

On the other hand, since

$$\left.\frac{\partial h(z_1,z_2)}{\partial z_1}\right|_{z_1=z_2=1}=1-(\delta_1+\lambda_1 b_1)>0,$$

we conclude that $z_1=\phi(z_2)$ is analytic on $|z_2|<1$ and is continuous at $z_2=1$ and $\phi(1)=1$ by the implicit function theorem. It is necessary to know the first and second derivatives of $\phi(z_2)$ at $z_2 = 1$ for late use. These are derived as follows

$$(8)\quad \begin{cases} \phi'(1) = \dfrac{\lambda_2 b_1}{1-(\delta_1+\lambda_1 b_1)}, \\[2ex] \phi''(1) = \dfrac{2\delta_1(1-\delta_1)\lambda_2^2 b_1^2 + (1-\delta_1)^2\lambda_2^2 E(b_1^2)}{\{1-(\delta_1+\lambda_1 b_1)\}^3}. \end{cases}$$

By substituting $z_1 = \phi(z_2)$ into (5), $P_1(0, z_1, z_2)$ is eliminated, and we get

$$(9)\quad \begin{aligned} &(1-\delta_2+\delta_2 z_2)P_2(0,\phi(z_2),z_2) + \lambda_1\phi(z_2)Q(z_2) \\ &\qquad = (1-\delta_1)P_{10}(0,z_2) + (1-\delta_2+\delta_2 z_2)P_{20}(0,z_2). \end{aligned}$$

By substituting $z_1 = \phi(z_2)$ into (6), we obtain

$$(10)\quad P_2(0,\phi(z_2),z_2) = \beta_2(\phi(z_2),z_2)\{\lambda_2 Q(z_2) + \nu Q'(z_2)\}.$$

From (9) and (10), we obtain

$$(11)\quad \begin{aligned} &(1-\delta_1)P_{10}(0,z_2) + (1-\delta_2+\delta_2 z_2)P_{20}(0,z_2) \\ &\qquad = \{\lambda_1\phi(z_2) + (1-\delta_2+\delta_2 z_2)\lambda_2\beta_2(\phi(z_2),z_2)\}Q(z_2) \\ &\qquad + (1-\delta_2+\delta_2 z_2)\nu\beta_2(\phi(z_2),z_2)Q'(z_2). \end{aligned}$$

By equating (3a) and (11), we obtain the differential equation
(12)

$$\begin{aligned} Q'(z_2) =& \frac{1}{\nu\{(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2) - z_2\}} \\ &\times \Big[\lambda_1(1-\phi(z_2)) + \lambda_2\{1-(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2)\}\Big]Q(z_2) \end{aligned}$$

whose solution is
(13)

$$\begin{aligned} Q(z_2) = C\cdot\exp\Bigg[&-\frac{1}{\nu}\int_{z_2}^{1}\frac{1}{(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x) - x} \\ &\times\{\lambda_1(1-\phi(x)) + \lambda_2\{1-(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x)\}\}dx\Bigg]. \end{aligned}$$

Substituting (12) into (6) yields

$$(14)\quad P_2(0,z_1,z_2) = \frac{\{\lambda_1(1-\phi(z_2)) + \lambda_2(1-z_2)\}\beta_2(z_1,z_2)}{(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2) - z_2}Q(z_2).$$

Substituting (11), (12), and (14) into (5) yields

(15)
$$P_1(0,z_1,z_2) = \beta_1(z_1,z_2)\Bigg[\frac{(1-\delta_2+\delta_2 z_2)}{(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2)-z_2} \times \frac{\{\lambda_1(1-\phi(z_2))+\lambda_2(1-z_2)\}\{\beta_2(\phi(z_2),z_2)-\beta_2(z_1,z_2)\}}{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2)-z_1} + \frac{\lambda_1(\phi(z_2)-z_1)}{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2)-z_1}\Bigg]Q(z_2).$$

Finally, we will calculate $P_k^*(0,z_1,z_2)$. Letting $\theta = 0$ in (4a) and (4b) gives

$$\begin{aligned} &-\{\lambda_1(1-z_1)+\lambda_2(1-z_2)\}P_1^*(0,z_1,z_2) \\ (16)\quad &= \left(1-\delta_1-\frac{1-\delta_1}{z_1}\right)P_1(0,z_1,z_2) - \frac{1-\delta_2+\delta_2 z_2}{z_1}P_2(0,z_1,z_2) \\ &+ \frac{1-\delta_1}{z_1}P_{10}(0,z_2) + \frac{1-\delta_2+\delta_2 z_2}{z_1}P_{20}(0,z_2) - \lambda_1 Q(z_2), \end{aligned}$$

$$\begin{aligned} (17)\quad &-\{\lambda_1(1-z_1)+\lambda_2(1-z_2)\}P_2^*(0,z_1,z_2) \\ &= P_2(0,z_1,z_2) - \lambda_2 Q(z_2) - \nu Q'(z_2). \end{aligned}$$

We obtain from (16) using (11), (14), and (15)

(18)
$$P_1^*(0,z_1,z_2) = \frac{\beta_1(z_1,z_2)-1}{\lambda_1(z_1-1)+\lambda_2(z_2-1)}\Bigg[\frac{\lambda_1(\phi(z_2)-z_1)}{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2)-z_1} + \frac{(1-\delta_2+\delta_2 z_2)\{\lambda_1(1-\phi(z_2))+\lambda_2(1-z_2)\}}{(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2)-z_2} \times \frac{\beta_2(\phi(z_2),z_2)-\beta_2(z_1,z_2)}{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2)-z_1}\Bigg]Q(z_2).$$

From (12), (14), and (17), we obtain

$$(19)\quad P_2^*(0,z_1,z_2) = \frac{\{\lambda_1(1-\phi(z_2))+\lambda_2(1-z_2)\}\{1-\beta_2(z_1,z_2)\}}{(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2)-z_2} \times \frac{Q(z_2)}{\lambda_1(1-z_1)+\lambda_2(1-z_2)}.$$

To determine C, we need to find $P_k^*(0,1,1)$, $k = 1,2$. First letting $z_2 \to 1$ and then $z_1 \to 1$ in (18) and (19). Using $\phi(1) = 1$ and (8) we obtain by the L'Hospital rule that

(20)

$$P_1^*(0,1,1) = \lim_{z_1\to 1} C\cdot\frac{\beta(z_1,1)-1}{\lambda_1(z_1-1)}\left[\frac{\lambda_1(1-z_1)}{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,1)-z_1}\right.$$
$$\left.+\frac{1-\beta_2(z_1,1)}{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,1)-z_1}\cdot\frac{-\lambda_1\phi'(1)-\lambda_2}{\delta_2+(\lambda_1\phi'(1)+\lambda_2)b_2-1}\right]$$
$$= C\cdot\frac{\lambda_1 b_1(1-\delta_2)\{1-(\delta_1+\lambda_1 b_1)\}}{\{1-(\delta_1+\lambda_1 b_1)\}[(1-\delta_2)\{1-(\delta_1+\lambda_1 b_1)\}-\lambda_2 b_2(1-\delta_1)]},$$

(21)
$$\begin{aligned} P_2^*(0,1,1) &= \lim_{z_1\to 1} C\cdot\frac{1-\beta_2(z_1,1)}{\lambda_1(1-z_1)}\frac{-\lambda_1\phi'(1)-\lambda_2}{\delta_2+(\lambda_1\phi'(1)+\lambda_2)b_2-1}\\ &= C\cdot\frac{\lambda_2 b_2(1-\delta_1)}{(1-\delta_2)\{1-(\delta_1+\lambda_1 b_1)\}-\lambda_2 b_2(1-\delta_1)}.\end{aligned}$$

From the total probability $Q(1)+P_1^*(0,1,1)+P_2^*(0,1,1)=1$, we obtain

$$C=\frac{(1-\delta_2)\{1-(\delta_1+\lambda_1 b_1)\}-\lambda_2 b_2(1-\delta_1)}{(1-\delta_1)\{1-(\delta_1+\lambda_1 b_1)\}+\lambda_1 b_1(1-\delta_2)},$$

that is, the probability that the server is idle, and

$$P_1^*(0,1,1)+P_2^*(0,1,1)$$
$$=\frac{(1-\delta_1)\{1-(\delta_1+\lambda_1 b_1)+\lambda_2 b_2\}-(1-\delta_2)(1-\delta_1-2\lambda_1 b_1)}{(1-\delta_1)\{1-(\delta_1+\lambda_1 b_1)\}+\lambda_1 b_1(1-\delta_2)}$$

as the probability that the server is busy.

Thus we have the theorem.

THEOREM 3.1. *The stationary distribution of* (ξ,N_1,N_2) *has the following generating functions*

(22)

$$Q(z_2) = E(z_2^{N_2};\xi=0)$$
$$=\frac{(1-\delta_2)\{1-(\delta_1+\lambda_1 b_1)\}-\lambda_2 b_2(1-\delta_1)}{(1-\delta_1)\{1-(\delta_1+\lambda_1 b_1)\}+\lambda_1 b_1(1-\delta_2)}$$
$$\times\exp\left[-\frac{1}{\nu}\int_{z_2}^{1}\frac{1}{(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x)-x}\right.$$
$$\left.\times\Big\{\lambda_1(1-\phi(x))+\lambda_2\{1-(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x)\}\Big\}dx\right],$$

(23)

$$P_1^*(0,z_1,z_2) = E(z_1^{N_1} z_2^{N_2}; \xi = 1)$$
$$= \frac{(1-\delta_2)\{1-(\delta_1+\lambda_1 b_1)\} - \lambda_2 b_2(1-\delta_1)}{(1-\delta_1)\{1-(\delta_1+\lambda_1 b_1)\} + \lambda_1 b_1(1-\delta_2)}$$
$$\times \frac{\beta_1(z_1,z_2)-1}{\lambda_1(z_1-1)+\lambda_2(z_2-1)} \left[\frac{\lambda_1(\phi(z_2)-z_1)}{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2)-z_1} \right]$$
$$+ \frac{(1-\delta_2+\delta_2 z_2)\{\lambda_1(1-\phi(z_2))+\lambda_2(1-z_2)\}}{\{(1-\delta_1+\delta_1 z_1)\beta_1(z_1,z_2)-z_1)\}}$$
$$\times \frac{\{\beta_2(\phi(z_2),z_2)-\beta_2(z_1,z_2)\}}{\{(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2)-z_2\}}$$
$$\times \exp\left[-\frac{1}{\nu} \int_{z_2}^{1} \frac{\lambda_1(1-\phi(x))+\lambda_2\{1-(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x)\}}{(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x)-x} dx \right]$$

(24)

$$P_2^*(0,z_1,z_2) = E(z_1^{N_1} z_2^{N_2} : \xi = 2)$$
$$= \frac{(1-\delta_2)\{1-(\delta_1+\lambda_1 b_1)\} - \lambda_2 b_2(1-\delta_1)}{(1-\delta_1)\{1-(\delta_1+\lambda_1 b_1)\} + \lambda_1 b_1(1-\delta_2)}$$
$$\times \frac{\{\lambda_1(1-\phi(z_2))+\lambda_2(1-z_2)\}\{1-\beta_2(z_1,z_2)\}}{\{\lambda_1(1-z_1)+\lambda_2(1-z_2)\}\{(1-\delta_2+\delta_2 z_2)\beta_2(\phi(z_2),z_2)-z_2\}}$$
$$\times \exp\left[-\frac{1}{\nu} \int_{z_2}^{1} \frac{\lambda_1(1-\phi(x))+\lambda_2\{1-(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x)\}}{(1-\delta_2+\delta_2 x)\beta_2(\phi(x),x)-x} dx \right]$$

4. Special cases

(a) When $\delta_k = 0$ $k = 1,2$, our model becomes the $M/G/1$ retrial queueing system with priority customers studied by Falin, Artalejo and Martin[7]. In this case, equations (22), (23), and (24) reduce to

$$Q(z_2) = E(z_2^{N_2}; \xi = 0) = \{1-(\lambda_1 b_1 + \lambda_2 b_2)\}$$
$$\times \exp\left[-\frac{1}{\nu} \int_{z_2}^{1} \frac{\lambda_1(1-\phi(x))+\lambda_2(1-b_2^*(\lambda_1+\lambda_2-\lambda_1\phi(x)-\lambda_2 x))}{b_2^*(\lambda_1+\lambda_2-\lambda_1\phi(x)-\lambda_2 x)-x} dx \right]$$

$$P_1^*(0,z_1,z_2) = E(z_1^{N_1} z_2^{N_2}; \xi = 1)$$
$$= \left[\frac{\lambda_1(\phi(z_2)-z_1)}{b_1^*(\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)-z_1} + \frac{\lambda_1(1-\phi(z_2))+\lambda_2(1-z_2)}{b_1^*(\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)-z_1} \right.$$
$$\left. \times \frac{b_2^*(\lambda_1+\lambda_2-\lambda_1\phi(z_2)-\lambda_2 z_2)-b_2^*(\lambda_1+\lambda_2-\lambda_1 z_1-\lambda_2 z_2)}{b_2^*(\lambda_1+\lambda_2-\phi(z_2)z_1-\lambda_2 z_2)-z_2} \right] Q(z_2),$$

$$P_2^*(0,z_1,z_2) = E(z_1^{N_1} z_2^{N_2}; \xi = 2)$$
$$= \frac{\lambda_1(1-\phi(z_2)) + \lambda_2(1-z_2)}{\lambda_1(1-z_1) + \lambda_2(1-z_2)} \frac{1 - b_2^*(\lambda_1 + \lambda_2 - \lambda_1 z_1 - \lambda_2 z_2)}{b_2^*(\lambda_1 + \lambda_2 - \lambda_1\phi(z_2) - \lambda_2 z_2) - z_2} Q(z_2),$$

which agree with Theorem 3 in Falin, Artalejo and Martin[7].

(b) When $b_k(x) = b(x)$ and $\delta_k = 0$, $k = 1,2$, our model becomes the $M/G/1$ retrial queue with Bernoulli schedule with $\lambda_1 = q\lambda$ and $\lambda_2 = p\lambda (q = 1-p)$ studied by Choi and Park[3].

In this case $\phi(z_2) = b^*(\lambda - \lambda q\phi(z_2) - \lambda p z_2)$, equations (22), (23), and (24) reduce to

$$E(z_2^{N_2}; \xi = 0) = (1-\lambda b)\exp\left[-\frac{\lambda}{\nu}\int_{z_2}^1 \frac{1-\phi(x)}{\phi(x)-x}dx\right],$$

$$E(z_1^{N_1} z_2^{N_2}; \xi \neq 0) = (1-\lambda b)\frac{b^*(\lambda - q\lambda z_1 - p\lambda z_2) - 1}{qz_1 + pz_2 - 1}$$
$$\times \frac{\phi(z_2) - z_1}{b^*(\lambda - q\lambda z_1 - p\lambda z_2) - z_1}\frac{1-z_2}{\phi(z_2)-z_2}\exp\left[-\frac{\lambda}{\nu}\int_{z_2}^1 \frac{1-\phi(x)}{\phi(x)-x}dx\right],$$

which agree with Theorem in Choi and Park [3].

(c) When $\lambda_1 = 0$ and $\delta_k = 0$, $k = 1,2$, our model becomes the $M/G/1$ retrial queue. In this case $N_1 = 0$. Equations (22), (23), and (24) reduce to

$$E(z_2^{N_2}; \xi = 0) = (1-\lambda_2 b_2)\exp\left[-\frac{\lambda_2}{\nu}\int_{z_2}^1 \frac{1-b_2^*(\lambda_2 - \lambda_2 x)}{b_2^*(\lambda_2 - \lambda_2 x) - x}dx\right],$$

$$E(z_2^{N_2}; \xi = 2) = (1-\lambda_2 b_2)\frac{1 - b_2^*(\lambda_2 - \lambda_2 z_2)}{b_2^*(\lambda_2 - \lambda_2 z_2) - z_2}$$
$$\times \exp\left[-\frac{\lambda_2}{\nu}\int_{z_2}^1 \frac{1-b_2^*(\lambda_2 - \lambda_2 x)}{b_2^*(\lambda_2 - \lambda_2 x) - x}dx\right],$$

which agree with Theorem 6 in Falin[5]. The generating function of queue length in the system is $E(z_2^{N_2}; \xi = 0) + z_2 E(z_2^{N_2}; \xi = 2)$ which equals

$$\frac{(1-\lambda_2 b_2)(1-z_2) b_2^*(\lambda_2 - \lambda_2 z_2)}{b_2^*(\lambda_2 - \lambda_2 z_2) - z_2}\exp\left[-\frac{\lambda_2}{\nu}\int_{z_2}^1 \frac{1-b_2^*(\lambda_2 - \lambda_2 x)}{b_2^*(\lambda_2 - \lambda_2 x) - x}dx\right].$$

This result agree with (3.11) in Yang and Templeton[9].

(d) When $\lambda_2 = 0$ and $\delta_k = 0$, $k = 1,2$, our model becomes the ordinary $M/G/1$ queue. In this case $\phi(z_2) = b_1^*(\lambda_1 - \lambda_1 z_1)$, $N_2 = 0$. Equations (22) and (23) reduce to

$$P(\xi = 0) = 1 - \lambda_1 b_1,$$

$$E(z_1^{N_1}; \xi = 1) = (1 - \lambda_1 b_1)\frac{b_1^*(\lambda_1 - \lambda_1 z_1) - 1}{z_1 - b_1^*(\lambda_1 - \lambda_1 z_1)}.$$

The generating function of the queue length in the system is $Q(1) + z_1 P^*(0, z_1, 1)$ which equals

$$\frac{(1 - \lambda_1 b_1)(z_1 - 1)b_1^*(\lambda_1 - \lambda_1 z_1)}{z_1 - b_1^*(\lambda_1 - \lambda_1 z_1)}.$$

This is the Pollaczek-Khinchin formula.

(e) When $\lambda_1 = 0$, our model becomes the $M/G/1$ feedback retrial queue. In this case $\phi(z_2) = b_1^*(\lambda_2 - \lambda_2 z_2)$, $N_1 = 0$. Equation (22) and (23) reduce to

$$\begin{aligned} E(z_2^{N_2}; \xi = 0) = &\{1 - (\delta_2 + \lambda_2 b_2)\} \\ &\times \exp\left[-\frac{\lambda_2}{\nu}\int_{z_2}^{1} \frac{1 - (1 - \delta_2 + \delta_2 x)b_2^*(\lambda_2 - \lambda_2 x)}{(1 - \delta_2 + \delta_2 x)b_2^*(\lambda_2 - \lambda_2 x) - x} dx \right], \end{aligned}$$

$$\begin{aligned} E(z_2^{N_2}; \xi = 2) = &\frac{\{1 - (\delta_2 + \lambda_2 b_2)\}\{1 - b_2^*(\lambda_2 - \lambda_2 z_2)\}}{(1 - \delta_2 + \delta_2 z_2)b_2^*(\lambda_2 - \lambda_2 z_2) - z_2} \\ &\times \exp\left[-\frac{\lambda_2}{\nu}\int_{z_2}^{1} \frac{1 - (1 - \delta_2 + \delta_2 x)b_2^*(\lambda_2 - \lambda_2 x)}{(1 - \delta_2 + \delta_2 x)b_2^*(\lambda_2 - \lambda_2 x) - x} dx \right]. \end{aligned}$$

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DEPARTMENT OF MATHEMATICS EDUCATION, ANDONG NATIONAL UNIVERSITY, KYUNG BUK 760-749, KOREA
E-mail: ywlee@andong.ac.kr