

A COMPREHENSIVE STUDY OF SECOND ORDER MOCK THETA FUNCTIONS

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ABSTRACT. We consider the second order mock theta functions defined by McIntosh and define generalized functions. We give integral representation and multibasic expansion of these functions. We also show that they are F_q -functions.

1. Introduction

In his last letter to Hardy[9, pp. 354–355] four months before he died, Ramanujan wrote about functions whom he called ‘mock’ theta functions. He explained what he meant by mock theta function. It is a function $f(q)$ defined by a q -series which converges for $|q| < 1$ and which satisfies the following two conditions:

- (0) For every root of unity ζ , there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \rightarrow \zeta$ radially.
- (1) There is no single theta function which works for all ζ , i.e., for every theta function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta(q)$ is unbounded as $q \rightarrow \zeta$ radially.

He listed seventeen functions and classified them as of order three, five and seven. Later, in Ramanujan’s “Lost” Notebook there were eleven identities involving seven functions and Andrews and Hickerson[3] called these seven functions mock theta functions of order six. Recently Choi[4] considered two identities found in the “Lost” Notebook of Ramanujan involving four functions and called them mock theta functions of order ten. Ramanujan did not say what he meant by the ‘order’ of the function.

Very recently Gordon and McIntosh[6] in their brilliant paper gave a method of constructing mock theta function from ordinary theta series

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using half-shift transformation. They constructed eight functions and called them of order eight. They also gave a rigorous definition of the order of the function [6].

Recently McIntosh[7] has defined two functions and called them of order two. He has established relations between these mock theta functions and his eight order mock theta functions.

In this paper we make a comprehensive study of these second order mock theta functions. In section 4 we give Hecke type expansions for these second order mock theta functions. In a subsequent paper we will show that using these expansions these mock theta functions come out as coefficients of z^0 in certain power series expansion in z . In my paper [10], I have shown that Ramanujan's sixth order mock theta functions and Gordon and McIntosh's eighth order mock theta functions can be written as a coefficient of z^0 in the Laurent series expansion of rational functions of theta functions.

In Section 5, we define generalized functions which on specialization give these mock theta functions. We show that these generalized functions and consequently the mock theta functions are F_q -functions

In Section 6, we represent these generalized functions as an Integral.

In Section 7, we expand these generalized functions as a multibasic hypergeometric series and the same follows for these mock theta functions.

In Section 8, we represent these mock theta functions as continued fractions.

2. Notation

We shall use the following usual basic hypergeometric notations:

For $|q^k| < 1$,

$$(a; q^k)_n = (1 - a)(1 - aq^k) \dots (1 - aq^{k(n-1)}), \quad n \geq 1$$

$$(a; q^k)_0 = 1,$$

$$(a; q^k)_\infty = \prod_{m=0}^{\infty} (1 - aq^{mk}).$$

For convenience we shall write

$$(a_1, a_2, \dots, a_k; q^k)_n = (a_1; q^k)_n (a_2; q^k)_n \dots (a_k; q^k)_n.$$

When $k = 1$, we usually write $(a)_n$ and $(a)_\infty$ instead of $(a; q)_n$ and $(a; q)_\infty$, respectively.

For non-negative integers n , we have

$$(a; q^k)_n = \frac{(a; q^k)_\infty}{(aq^{nk}; q^k)_\infty}$$

and for other real n , we will take this as the definition of $(a; q^k)_n$.

$$\begin{aligned} & \phi \left[\begin{matrix} a_1, \dots, a_r : c_{1,1}, \dots, c_{1,r_1} : \dots : c_{m,1}, \dots, c_{m,r_m} \\ b_1, \dots, b_s : e_{1,1}, \dots, e_{1,s_1} : \dots : e_{m,1}, \dots, e_{m,s_m} \end{matrix} ; q, q_1, \dots, q_m ; z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} z^n \left[(-1)^n q^{\frac{n^2-n}{2}} \right]^{1+s-r} \\ & \quad \times \prod_{j=1}^m \frac{(c_{j,1}, \dots, c_{j,r_j}; q_j)_n}{(e_{j,1}, \dots, e_{j,s_j}; q_j)_n} \left[(-1)^n q_j^{\frac{n^2-n}{2}} \right]^{s_j-r_j} \end{aligned}$$

For $x \neq 0$ and $|q| < 1$,

$$j(x, q) = (x, q/x, q; q)_\infty.$$

If a and m are integers with $m \geq 1$, then

$$\begin{aligned} J_{a,m} &= j(q^a, q^m), \\ \bar{J}_{a,m} &= j(-q^a, q^m), \\ J_m &= J_{m,3m} = (q^m; q^m)_\infty. \end{aligned}$$

3. Second order Mock theta functions

The second order mock theta function defined by McIntosh[7] are

$$\begin{aligned} A(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} (-q; q^2)_n}{(q; q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n+1} (-q^2; q^2)_n}{(q; q^2)_{n+1}} \\ B(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2+n} (-q^2; q^2)_n}{(q; q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n (-q; q^2)_n}{(q; q^2)_{n+1}} \end{aligned}$$

and

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q^2; q^2)_n^2}.$$

The function $\mu(q)$ appears in Ramanujan’s “Lost” Notebook [9, Sec 3 with $a = 1$]

4. Hecke type expansion

We shall use the following result of Bailey to find Hecke type expansion for these mock theta functions: If $\{\alpha_n\}$ and $\{\beta_n\}$ form a Bailey pair relative to a , then

$$(1) \quad \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n \alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n} = \frac{(aq)_{\infty}(aq/\rho_1\rho_2)_{\infty}}{(aq/\rho_1)_{\infty}(aq/\rho_2)_{\infty}} \sum_{n=0}^{\infty} (\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n \beta_n.$$

Two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, $n \geq 0$ form a Bailey pair relative to a , if

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}$$

for all $n \geq 0$. Letting $q \rightarrow q^2$, $\rho_1 \rightarrow \infty$ and taking $\rho_2 = -q$, $a = q^2$, in (1) we have

$$(2) \quad \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q^2)_n \alpha_n}{(-q^3; q^2)_n} = \frac{(q^4; q^2)_{\infty}}{(-q^3; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{n^2+2n}(-q; q^2)_n \beta_n.$$

If α_n and β_n are Bailey pairs, Andrews[3, (2.13) and (2.14), p.73], then

$$(3) \quad \alpha_n = \frac{q^{n^2}(bc)^n(1-aq^{2n})(a/b)_n(a/c)_n}{(1-a)(qb)_n(qc)_n} \times \sum_{j=0}^n \frac{(-1)^j(1-aq^{2j-1})(a)_{j-1}(b)_j(c)_j}{q^{\binom{j}{2}}(bc)^j(q)_j(a/b)_j(a/c)_j}$$

and

$$(4) \quad \beta_n = \frac{1}{(qb)_n(qc)_n}.$$

Letting $q \rightarrow q^2$ and taking $a = q^2$, $b = c = q$, in (3) and (4), we have

$$\alpha_n = \frac{(1-q)(1+q^{2n+1})}{(1+q)(1-q^{2n+1})} q^{2n^2+2n} \left[1 + \sum_{j=-n}^n (-1)^j q^{-(j^2+j)} \right]$$

and

$$\beta_n = \frac{1}{(q^3; q^2)_n^2}.$$

Putting these values of α_n and β_n in (2), we have

$$(5) \quad A(q) = \frac{\bar{J}_{1,2}}{J_2} \sum_{n=0}^{\infty} \frac{q^{3n^2+4n+1}}{(1-q^{2n+1})} \left[1 + \sum_{j=-n}^n (-1)^j q^{-(j^2+j)} \right],$$

which is the Hecke Type expansion for $A(q)$.

Letting $q \rightarrow q^2$, $\rho_1 \rightarrow \infty$ and taking $\rho_2 = -q^2$, $a = q^2$, in (1), we have

$$(6) \quad \sum_{n=0}^{\infty} q^{n^2+n} \alpha_n = \frac{(q^4; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{n^2+n} (-q^2; q^2)_n \beta_n.$$

Letting $q \rightarrow q^2$ and taking $a = q^2$, $b = c = q$, in (3) and (4), we have

$$\alpha_n = \frac{(1-q)(1+q^{2n+1})}{(1+q)(1-q^{2n+1})} q^{2n^2+2n} \left[1 + \sum_{j=-n}^n (-1)^j q^{-(j^2+j)} \right]$$

and

$$\beta_n = \frac{1}{(q^3; q^2)_n^2}.$$

Putting these values of α_n and β_n in (6), we have

$$B(q) = \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{3n^2+3n} \frac{(1+q^{2n+1})}{(1-q^{2n+1})} \left[1 + \sum_{j=-n}^n (-1)^j q^{-(j^2+j)} \right],$$

which is the Hecke Type expansion for $B(q)$.

Letting $q \rightarrow q^2$, $\rho_1 \rightarrow \infty$ and taking $\rho_2 = q$, $a = 1$, in (1), we have

$$(7) \quad \sum_{n=0}^{\infty} (-1)^n q^{n^2} \alpha_n = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{n^2} (q; q^2)_n \beta_n.$$

For the Bailey pairs $\{\alpha_n\}$ and $\{\beta_n\}$, we use the result of Andrews[3, (2.14) and (2.16)], pp. 73–74] for $a = 1$

$$(8) \quad \begin{aligned} & \alpha_n(1, b, c, q) \\ &= \frac{(-1)^n q^{\binom{n+1}{2}} (1+q^n)(1-b)(1-c)}{(1-q^n b)(1-q^n c)} \\ &+ \frac{q^{n^2} (bc)^n (1-q^{2n})(1/b)_n (1/c)_n (bc-1)}{(qb)_n (qc)_n} \\ &\times \left[\frac{1}{(1-b)(1-c)} + \sum_{j=1}^{n-1} \frac{(-1)^j (1+q^j)(b)_j (c)_j}{q^{\binom{j}{2}} (bc)^{j+1} (1/b)_{j+1} (1/c)_{j+1}} \right] \end{aligned}$$

and

$$(9) \quad \beta_n = \frac{1}{(qb)_n(qc)_n}.$$

Letting $q \rightarrow q^2$ and taking $b = c = -1$, in (8) and (9), we have

$$\alpha_n = \frac{4(-1)^n q^{n^2+n}}{(1+q^{2n})}$$

and

$$\beta_n = \frac{1}{(-q^2; q^2)_n^2}.$$

Putting these values of α_n and β_n in (7), we have

$$\mu(q) = \frac{4(q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{1+q^{2n}},$$

which is another representation for Ramanujan's function $\mu(q)$.

5. F_q -functions

Truesdell[11] calls the functions which satisfy the functional equation

$$(10) \quad \frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha + 1)$$

as F -functions. The q -analogue of (10) is the q -differential equation

$$(11) \quad D_{q,z} F(z, \alpha) = F(z, \alpha + 1),$$

where

$$zD_{q,z} F(z, \alpha) = F(z, \alpha) - F(zq, \alpha).$$

We shall call the functions which satisfy the difference equation (11) as F_q -functions.

We shall define the following generalized functions

$$(12) \quad A(z, \alpha) := \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2+n(\alpha+1)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2},$$

$$(13) \quad B(z, \alpha) := \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2+n\alpha} (-q^2; q^2)_n}{(q; q^2)_{n+1}^2},$$

$$(14) \quad \mu(z, \alpha) := \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (z)_n q^{n^2-n+n\alpha} (q; q^2)_n}{(-q^2; q^2)_n^2}.$$

For $z = 0, \alpha = 1$, they reduce to $A(q), B(q)$ and $\mu(q)$ functions. We shall show that these generalized functions are F_q -functions. By definition

$$\begin{aligned} zD_{q,z}A(z, \alpha) &= A(z, \alpha) - A(zq, \alpha) \\ &= \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2+n(\alpha+1)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2} \\ &\quad - \frac{1}{(zq)_\infty} \sum_{n=0}^{\infty} \frac{(zq)_n q^{n^2+n(\alpha+1)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2} \\ &= \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2+n(\alpha+1)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2} \\ &\quad - \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2+n(\alpha+1)+1} (-q; q^2)_n (1 - zq^n)}{(q; q^2)_{n+1}^2} \\ &= \frac{z}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2+n(\alpha+2)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2} \\ &= zA(z, \alpha + 1). \end{aligned}$$

Hence

$$D_{q,z}A(z, \alpha) = A(z, \alpha + 1).$$

So $A(z, \alpha)$ is a F_q -Function.

A similar proof shows that $B(z, \alpha)$ and $\mu(z, \alpha)$ are F_q -Functions.

6. Integral representation

We now give integral representation for these generalized functions and consequently for the mock theta functions Thomas[5, (1.11.1), p.19] and Jackson[5, (1.11.1), p.19] defined q -integral as

$$(15) \quad \int_0^1 f(t) d_q t = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n.$$

The limiting case of the q -beta integral [5, (1.11.7), p.19] is

$$(16) \quad \frac{1}{(q^x; q)_\infty} = \frac{(1 - q)^{-1}}{(q; q)_\infty} \int_0^1 t^{x-1} (tq; q)_\infty d_q t.$$

By definition (12)

$$A(z, \alpha) := \frac{1}{(z)_\infty} \sum_{n=0}^{\infty} \frac{(z)_n q^{n^2+n(\alpha+1)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2}.$$

Writing q^z for z and $q^\alpha = b$, we have

$$\begin{aligned} A(q^z, \alpha) &= \frac{1}{(q^z; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^z; q)_n q^{n^2+n(\alpha+1)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+n(\alpha+1)+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2 (q^{n+z}; q)_\infty} \\ &= \sum_{n=0}^{\infty} \frac{q^{(n^2+n+1)+n\alpha} (-q; q^2)_n (1-q)^{-1}}{(q; q^2)_{n+1}^2 (q; q)_\infty} \int_0^1 t^{n+z-1} (tq; q)_\infty d_q t \end{aligned}$$

by (16), with x replaced by $n + z$. Hence

$$\begin{aligned} (17) \quad A(q^z, \alpha) &= \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{z-1} (tq; q)_\infty \\ &\quad \times \sum_{n=0}^{\infty} \frac{q^{(n^2+n+1)} (-q; q^2)_n (bt)^n d_q t}{(q; q^2)_{n+1}^2}. \end{aligned}$$

But

$$A(0, \alpha) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n+1)+n\alpha} (-q; q^2)_n}{(q; q^2)_{n+1}^2}.$$

Since we have put $q^\alpha = b$,

$$A(0, b) = \sum_{n=0}^{\infty} \frac{b^n q^{n^2+n+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2}.$$

Hence

$$(18) \quad A(0, bt) = \sum_{n=0}^{\infty} \frac{(bt)^n q^{n^2+n+1} (-q; q^2)_n}{(q; q^2)_{n+1}^2}.$$

Replacing the summation on the right-side of (17) by (18), we have

$$(19) \quad A(q^z, \alpha) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{z-1} (tq; q)_\infty A(0, bt) d_q t,$$

which is the integral representation for $A(q^z, \alpha)$. Similarly the integral representation for the other two functions

$$(20) \quad B(q^z, \alpha) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{z-1} (tq; q)_\infty B(0, bt) d_q t$$

and

$$(21) \quad \mu(q^z, \alpha) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{z-1} (tq; q)_\infty \mu(0, bt) d_q t.$$

7. Multibasic expansions

Using the summation formula [5, (3.6.7), p.71] and [8, Lemma 10, p.57], we have the bibasic expansion

$$(22) \quad \sum_{k=0}^{\infty} \frac{(1-ap^kq^k)(1-bp^kq^{-k})(a, b; p)_k (c, a/bc; q)_k q^k}{(1-a)(1-b)(q, aq/b; q)_k (ap/c, bcp; p)_k} \sum_{m=0}^{\infty} \alpha_{m+k} \\ = \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m (cq, aq/bc; q)_m}{(ap/c, bcp; p)_m (q, aq/b; q)_m} \alpha_m.$$

COROLLARY. Letting $q \rightarrow q^3$, $c \rightarrow \infty$ in (22), we have

$$(23) \quad \sum_{k=0}^{\infty} \frac{(1-ap^kq^{3k})(1-bp^kq^{-3k})(a, b; p)_k q^{\frac{3k^2+3k}{2}}}{(1-a)(1-b)(q^3, aq^3/b; q^3)_k b^k p^{\frac{k^2+k}{2}}} \sum_{m=0}^{\infty} \alpha_{m+k} \\ = \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m q^{\frac{3m^2+3m}{2}}}{(q^3, aq^3/b; q^3)_m b^m p^{\frac{m^2+m}{2}}} \alpha_m.$$

We shall use this corollary to give a multibasic expansion of these generalized functions and since for $\alpha = 1$, $x = 0$, they reduce to the second order mock theta functions, these multibasic expansion for $\alpha = 1$ and $x = 0$ will be for the mock theta functions.

(i) Putting $a = \frac{x}{q}$, $p = q$, and taking

$$\alpha_n = \frac{q^{n\alpha+1}(-q; q^2)_n (q^3; q^3)_n (xq^2/b; q^3)_n b^n}{(q; q^2)_{n+1}^2 (bq; q)_n}$$

in (23), we get after a little simplification,

$$(24) \quad (x)_\infty A(x, \alpha) \\ = \sum_{k=0}^{\infty} \frac{(1-xq^{4k-1})(1-bq^{-2k})(x; q)_{k-1}(-q; q^2)_k q^{k^2+k(\alpha+1)+1}}{(1-bq^k)(q; q^2)_{k+1}^2} \\ \times \phi \left[\begin{matrix} q, 0 : -q^{2k+1}, 0 : xq^{3k+2}/b, q^{3k+3} : \\ bq^{k+1} : q^{2k+3}, q^{2k+3} : 0, 0 : \end{matrix} ; q, q^2, q^3, bq^\alpha \right].$$

(ii) Putting $a = \frac{x}{q}$, $p = q = b$ and taking

$$\alpha_n = \frac{q^{n\alpha}(-q^2; q^2)_n(q^3; q^3)_n(xq; q^3)_n}{(q; q^2)_{n+1}(q^2; q)_n}$$

in (23), we have

$$(25) \quad (x)_\infty B(x, \alpha) \\ = \sum_{k=0}^{\infty} \frac{(1-xq^{4k-1})(1-q^{-2k+1})(x; q)_{k-1}(-q^2; q^2)_k q^{k^2+k\alpha}}{(1-q^{k+1})(q; q^2)_{k+1}^2} \\ \times \phi \left[\begin{matrix} q, 0 : -q^{2k+2}, 0 : xq^{3k+1}, q^{3k+3} : \\ q^{k+2} : q^{2k+3}, q^{2k+3} : 0, 0 : \end{matrix} ; q, q^2, q^3; q^\alpha \right].$$

(iii) Putting $a = \frac{x}{q}$, $p = q$, $b = q^2$ and taking

$$\alpha_n = \frac{(-1)^n(x; q^3)_n(q^3; q^3)_n(q; q^2)_n q^{n\alpha}}{(-q^2; q^2)_n^2(q^3; q)_n}$$

in (23), we have

$$(26) \quad (x)_\infty \mu(x, \alpha) \\ = \sum_{k=0}^{\infty} \frac{(-1)^k(1-xq^{4k-1})(1-q^{-2k+2})(x; q)_{k-1}(q; q^2)_k q^{k^2-k+k\alpha}}{(1-q^{k+2})(-q^2; q^2)_k^2} \\ \times \phi \left[\begin{matrix} q, 0 : q^{2k+1}, 0 : xq^{3k}, q^{3k+3} : \\ q^{k+3} : -q^{2k+2}, -q^{2k+2} : 0, 0 : \end{matrix} ; q, q^2, q^3; (-q)^\alpha \right].$$

8. Continued fraction

McIntosh[7] has shown

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2; q^2)_n}{(q; q^2)_{n+1}}, \\ B(q) = \sum_{n=0}^{\infty} \frac{q^n(-q; q^2)_n}{(q; q^2)_{n+1}}.$$

We can write

$$(27) \quad A(q) = \frac{q}{1-q} \sum_{n=0}^{\infty} \frac{q^n(-q^2; q^2)_n(q^2; q^2)_n}{(q^2; q^2)_n(q^3; q^2)_n} \\ = \frac{q}{1-q} {}_2\phi_1 \left[\begin{matrix} -q^2, q^2 \\ q^3 \end{matrix} ; q^2; q \right]$$

and

$$(28) \quad \begin{aligned} B(q) &= \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^n (-q; q^2)_n (q^2; q^2)_n}{(q^2; q^2)_n (q^3; q^2)_n} \\ &= \frac{1}{1-q} {}_2\varphi_1 \left[\begin{matrix} -q, q^2 \\ q^3 \end{matrix} ; q \right]. \end{aligned}$$

We have put $A(q)$ and $B(q)$ as a ${}_2\varphi_1$ series and we have continued fraction representations for ${}_2\varphi_1$ series hence $A(q)$ and $B(q)$ can be represented as continued fractions.

We shall show the representation for $A(q)$ as a continued fraction. The representation for $B(q)$ will be on similar lines. Consider the Heine’s continued fraction expansion [2, (9.4), p.20]

$$\frac{{}_2\varphi_1(a, b; c; q, z)}{{}_2\varphi_1(a, bq; cq; q, z)} = 1 - \frac{a_1 z}{1} \frac{a_2 z}{1-1} \frac{a_3 z}{1-\dots},$$

where

$$a_{2k+1} = \frac{q^k(1 - aq^k)(b - cq^k)}{(1 - cq^{2k})(1 - cq^{2k+1})}$$

and

$$a_{2k} = \frac{q^{k-1}(1 - bq^k)(a - cq^k)}{(1 - cq^{2k-1})(1 - cq^{2k})}.$$

Making $q \rightarrow q^2$ and putting $a = q^2, b = -q^2, c = q^3, z = q$ we have

$$\frac{(1-q)}{q} \frac{A(q)}{{}_2\varphi_1(q^2, -q^4; q^5; q^2, q)} = 1 - \frac{a_1 z}{1} \frac{a_2 z}{1-1} \frac{a_3 z}{1-\dots},$$

where

$$a_{2k+1} = -\frac{q^{2k+2}(1 - q^{2k+2})(1 + q^{2k+1})}{(1 - q^{4k+3})(1 - q^{4k+5})}$$

and

$$a_{2k} = \frac{q^{2k}(1 + q^{2k+2})(1 - q^{2k+1})}{(1 - q^{4k+1})(1 - q^{4k+3})}.$$

Similarly we can have a continued fraction representation for $B(q)$.

9. Conclusion

We have made a very comprehensive study of these mock theta functions. We shall give a modular transformation in a later paper.

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