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REGULARITY CRITERIA FOR TERNARY INTERPOLATORY SUBDIVISION

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Abstract. By its simplicity and efficiency, subdivision is a widely used technique in computer graphics, computer aided design and data compression. In this paper we prove a regularity theorem for ternary interpolatory subdivision scheme that can be applied to non-stationary subdivision. This theorem converts the convergence of the limit curve of a ternary interpolatory subdivision to the analysis of the rate of the contraction of differences of the polygons.

1. Introduction

Subdivision is an iterated transformation of points which generates (or represents) a (smooth) curve or surface. Due to the simplicity and efficiency, subdivision is a widely used tool in the areas including computer graphics, computer aided design and data compression. Many authors have studied on the binary subdivisions that generate double points for each iteration. Ternary scheme generates triple points so that more fast convergence is guaranteed(cf. [2], [3], [4]). Recently M.F.Hassan et al. suggest interpolating ternary stationary subdivisions and its regularity criteria for the limit curve([3], [2]).

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In this paper, we suggest a regularity criteria for interpolatory ternary subdivision scheme that can be applied even for non-stationary subdivision. We follow the Kobbelt's work for interpolating binary subdivisions presented in [6]. Instead of the binomial coefficients we use the trinomial coefficients to analyze the convergence of the limit curve.

2. Preliminary

Subdivision is one of the powerful methods for representing shape based on iterated transformations. Beginning with an initial polygon $\mathbf{P}_0 = [P_i^0]$, subdivision scheme defines a limit curve by successive iterations. Given $m-th$ level of refinement $\mathbf{P}_m = [P_i^m | i = 0, \ldots, n]$ of $n+1$ vertices, binary subdivision generates $2n+1$ vertices of polygon $\mathbf{P}_{m+1} = [P_0^{m+1}, P_1^{m+1}, \ldots, P_{2n-1}^{m+1}, P_{2n}^{m+1}]$. The vertices P_{2i}^{m+1} are called even and others odd. Interpolatory subdivision is defined by $P_i^m = P_{2i}^{m+1}$ for each i, i.e. the even vertices are fixed at each refinement step.

Ternary subdivision generates triple number of vertices

$$\mathbf{P}_{m+1} = [P_0^{m+1}, P_1^{m+1}, \ldots, P_{3n-1}^{m+1}, P_{3n}^{m+1}].$$

Following the terms of binary subdivision, every third vertices P_{3i}^{m+1} are called even. So the ternary interpolatory subdivision is characterized by $P_i^m = P_{3i}^{m+1}$. In this case, odd vertices $P_{3i+1}^{m+1}, P_{3i+2}^{m+1}$ are the newly defined vertices.(cf. [8], [9], [10])

For an approximation of derivatives, forward differences will be used. Given an arbitrary polygon $\mathbf{P}_m = [P_i^m]$, the k-th (forward) difference $\Delta^k \mathbf{P}_m$ is defined by the polygon whose vertices are

$$\Delta^k P_i^m = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k+j} P_{i+j}^m \tag{1}$$

The trinomial coefficient $\binom{n}{k}_2$, where $n \geq 0$ and $-n \leq k \leq n$, is defined by the coefficient of x^{n+k} in the expansion of $(1+x+x^2)^n$(cf.

[1], [7], [11]), i.e.,

$$(1+x+x^2)^n = \sum_{k=-n}^{n} \binom{n}{k}_2 x^{n+k}$$

Explicit formula for trinomial coefficient is given by

$$\binom{n}{k}_2 = \sum_{j=0}^{n} \frac{n!}{j!(j+k)!(n-2j-k)!}$$

and satisfy

$$\binom{n}{k}_2 = \binom{n-1}{k-1}_2 + \binom{n-1}{k}_2 + \binom{n-1}{k+1}_2 \tag{2}$$

The last identity will be used in the proof of next section.

3. Convergence theorem for ternary interpolatory subdivision

The regularity of the limit curve is the main issue in the analysis of subdivision scheme. L. Kobbelt[6] proved the convergence criteria for binary interpolatory subdivision which can be applied to non-stationary subdivisions. In this section we prove the analogous convergence theorem for ternary subdivision.

Let $[\mathbf{P}_m]$ be a sequence of polygons generated by a ternary subdivision starting from an initial polygon $\mathbf{P}_0 = [P_0^0, \ldots, P_n^0]$.

The following lemma characterizes the ternary interpolatory subdivision

Lemma 3.1. *Let $[\mathbf{P}_m]$ be a sequence of polygons. The ternary subdivision scheme by which they are generated is an interpolatory refinement scheme if and only if for all m, $k \in \mathbb{N}$, the following condition holds.*

$$\Delta^k P_i^m = \sum_{l=-k}^{k} \binom{k}{l}_2 \Delta^k P_{3i+k+l}^{m+1}, \quad i = 0, \ldots, n3^m - k, \tag{3}$$

Proof. If the condition (3) holds, it is trivial to prove that the ternary scheme is interpolatory.

Now we prove the sufficient part of the condition by induction on k. For $k = 1$,

$$\begin{aligned}
\Delta P_i^m &= P_{i+1}^m - P_i^m \\
&= P_{3i+3}^{m+1} - P_{3i}^m \\
&= P_{3i+3}^{m+1} - P_{3i+2}^m + P_{3i+2}^{m+1} - P_{3i+1}^m + P_{3i+1}^{m+1} - P_{3i}^m \\
&= \Delta P_{3i+2}^{m+1} + \Delta P_{3i+1}^{m+1} + \Delta P_{3i}^{m+1}
\end{aligned}$$

If the statement holds for some value k, by (2) we have

$$\begin{aligned}
&\Delta^{k+1} P_i^m \\
= &\ \Delta^k P_{i+1}^m - \Delta^k P_i^m \\
= &\ \sum_{l=-k}^{k} \binom{k}{l}_2 \Big\{ \left(\Delta^k P_{3i+3+k+l}^{m+1} - \Delta^k P_{3i+2+k+l}^{m+1}\right) \\
+ &\ \left(\Delta^k P_{3i+2+k+l}^{m+1} - \Delta^k P_{3i+1+k+l}^{m+1}\right) + \left(\Delta^k P_{3i+1+k+l}^{m+1} - \Delta^k P_{3i+k+l}^{m+1}\right) \Big\} \\
= &\ \sum_{l=-k}^{k} \binom{k}{l}_2 \left(\Delta^{k+1} P_{3i+2+k+l}^{m+1} + \Delta^{k+1} P_{3i+1+k+l}^{m+1} + \Delta^{k+1} P_{3i+k+l}^{m+1}\right) \\
= &\ \sum_{l=-k}^{k} \left\{ \binom{k}{l-2}_2 + \binom{k}{l-1}_2 + \binom{k}{l}_2 \right\} \Delta^{k+1} P_{3i+k+l}^{m+1} \\
= &\ \sum_{l=-k}^{k} \binom{k+1}{l-1}_2 \Delta^{k+1} P_{3i+(k+1)+(l-1)}^{m+1} \\
= &\ \sum_{l=-(k+1)}^{k+1} \binom{k+1}{l}_2 \Delta^{k+1} P_{3i+(k+1)+l}^{m+1},
\end{aligned}$$

□

Each polygon $[\mathbf{P}_m]$ and the differences $\Delta^k P_i^m$ can be considered as a piecewise linear function defined on the interval $[0, n]$ with $P_i^m =$

$\mathbf{P}_m(i3^{-m}), \Delta^k P_i^m = \Delta^k \mathbf{P}_m(h_{m,k})$ where

$$h_{m,k} = \frac{n}{n3^m - k}, \quad i = 0 \ldots, n3^m.$$

The next lemma estimates the difference between k-th differences of $[\mathbf{P}_m]$ and $[\mathbf{P}_{m+1}]$ by the $(k+1)$-th difference of $[\mathbf{P}_{m+1}]$.

Lemma 3.2. *Let $[\mathbf{P}_m]$ be a sequence of polygons by ternary interpolatory refinement scheme. Then there exists a constant σ which only depends on k such that*

$$\|\Delta^k \mathbf{P}_m - 3^k \Delta^k \mathbf{P}_{m+1}\|_\infty \le \sigma \|\Delta^{k+1} \mathbf{P}_{m+1}\|_\infty. \tag{4}$$

Proof. For the parameter values of $\Delta^k P_{3i}^{m+1}$, $\Delta^k P_i^m$ and $\Delta^k P_{3i+k}^{m+1}$, we have

$$\frac{3in}{n3^{m+1} - k} \le \frac{in}{n3^m - k} \le \frac{(3i+k)n}{n3^{m+1} - k}, \quad i = 0, \ldots, n3^m - k.$$

Thus it is suffices to consider the distance between $\Delta^k P_i^m$ and $3^k \Delta^k P_{3i+r}^{m+1}$ for $r = 0, \ldots, k$. By Lemma 3.1 we have

$$\begin{aligned} |\Delta^k P_i^m - 3^k \Delta^k P_{3i+r}^{m+1}| &= \left| \sum_{l=-k}^{k} \binom{k}{l}_2 \Delta^k P_{3i+k+l}^{m+1} - 3^k \Delta^k P_{3i+r}^{m+1} \right| \\ &\le \sigma \|\Delta^{k+1} \mathbf{P}_{m+1}\|_\infty, \end{aligned} \tag{5}$$

because the trinomial coefficients sum to 3^k. □

Lemma 3.3. *Let $[\mathbf{P}_m]$ be a sequence of polygons by ternary interpolatory refinement scheme. If there exists a $q < 3^k$ such that*

$$\sum_{m=0}^{\infty} \|q^m \Delta^{k+1} \mathbf{P}_m\|_\infty < \infty,$$

then

$$\sum_{m=0}^{\infty} \|q^m \Delta^k \mathbf{P}_m\|_\infty < \infty.$$

Proof. By the Lemma 3.2, there exists $\sigma > 0$ such that

$$\begin{aligned}\|\Delta^k \mathbf{P}_m\|_\infty &\leq 3^{-k}\|\Delta^k \mathbf{P}_{m-1}\|_\infty + \sigma\|\Delta^{k+1}\mathbf{P}_m\|_\infty \\ &\leq 3^{-2k}\|\Delta^k \mathbf{P}_{m-2}\|_\infty + 3^{-k}\sigma\|\Delta^{k+1}\mathbf{P}_{m-1}\|_\infty + \sigma\|\Delta^{k+1}\mathbf{P}_m\|_\infty \\ &\cdots\cdots \\ &\leq 3^{-mk}\|\Delta^k \mathbf{P}_0\|_\infty + \sigma\sum_{i=1}^{m} 3^{(i-m)k}\|\Delta^{k+1}\mathbf{P}_i\|_\infty.\end{aligned}$$

Let $r = q3^{-k}$, then $r < 1$. For large $N \in \mathbb{N}$, we have

$$\sum_{m=0}^{N} \|q^m \Delta^k \mathbf{P}_m\|_\infty \leq \sum_{m=0}^{N} r^m \|\Delta^k \mathbf{P}_0\|_\infty + \sigma \sum_{m=1}^{N}\sum_{i=0}^{m} r^{m-i}\|q^i \Delta^{k+1}\mathbf{P}_i\|_\infty$$

As $N \to \infty$, we have

$$\sum_{m=0}^{\infty} \|q^m \Delta^k \mathbf{P}_m\|_\infty \leq \frac{1}{1-r}\left(\|\Delta^k \mathbf{P}_0\|_\infty + \sigma\sum_{m=1}^{\infty}\|q^m \Delta^{k+1}\mathbf{P}_m\|_\infty\right) < \infty$$

□

Following the Kobbelt's result in [6] carefully with the ternary coefficient 3^{km} in mind, we have the following lemma.

Lemma 3.4. *Let* $[\mathbf{P}_m]$ *be a sequence of polygons by ternary interpolatory refinement scheme. Then*

$$\text{(6)} \qquad \lim_{m\to\infty} \mathbf{P}_m = g \in C^k,\ \frac{d^k}{dx^k}g = f \Leftrightarrow \lim_{m\to\infty} 3^{km}\Delta^k \mathbf{P}_m = f \in C^0.$$

Now we can obtain a sufficient condition for the convergence of ternary subdivision.

Theorem 3.1. *Let* $[\mathbf{P}_m]$ *be a sequence of polygons generated by a ternary interpolatory subdivision scheme. If*

$$\text{(7)} \qquad \sum_{m=0}^{\infty} \|3^{km}\Delta^{k+l}\mathbf{P}_m\|_\infty < \infty,$$

for some $l \in \mathbb{N}$, *then the sequence* $[\mathbf{P}_m]$ *uniformly converges to a* k*-times continuously differentiable limit curve.*

Proof. By Lemma 3.3, we need only to prove for $l = 1$. By Lemma 3.2,

$$\|3^{km}\Delta^k\mathbf{P}_m - 3^{k(m+1)}\Delta^k\mathbf{P}_{m+1}\|_\infty \leq \sigma\|3^{km}\Delta^{k+1}\mathbf{P}_{m+1}\|_\infty$$

So the sequence $[\|3^{km}\Delta^k\mathbf{P}_m\|_\infty]$ is a Cauchy sequence and converges to a continuous limit. Lemma 3.4 completes the proof. □

One of the important problems in geometric modeling and computer graphics is to generate a fair curve from a polygon. A method for this is to define an energy functional that represents the fairness and find the energy minimizer of the energy functional. As in [5] for the binary subdivision, it is expected that we can find a functional representing a fairness for ternary subdivision and the convergence(or regularity) of the subdivision can be analyzed by theorem 3.1.

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