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REMARK ON TWO RESULTS BY PADMANABHAM FOR EXTON'S TRIPLE HYPERGEOMETRIC SERIES X_8

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Abstract. In 1999 and 2003, Padmanabham established two results (one each) for Exton's triple hypergeometric series X_8. We aim at showing that Exton's later result can be derived from his former one.

1. Introduction and Preliminaries

In 1982, Exton [2] introduced a set of 20 triple hypergeometric series X_1 to X_{20} of which we recall here the definition of X_8:

$$
\begin{aligned}
&X_8(a,\ b,\ c\,;\ d,\ e,\ f\,;\ x,\ y,\ z) \\
&\qquad = \sum_{m,\,n,\,p=0}^{\infty} \frac{(a)_{2m+n+p}\,(b)_n\,(c)_p\,x^m\,y^n\,z^p}{(d)_m\,(e)_n\,(f)_p\,m!\,n!\,p!},
\end{aligned} \tag{1.1}
$$

where $(\alpha)_n := \Gamma(\alpha+n)/\Gamma(\alpha)$ $(\alpha \neq 0,\,-1,\,-2,\,\ldots;\ n=0,\,1,\,2,\,\ldots)$.

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The precise three-dimensional region of convergence of (1.1) is given by Srivastava and Karlsson [8, p. 101, Entry 41a]:

$$4r = (s+t-1)^2, \quad |x| < r, \quad |y| < s, \quad \text{and} \quad |z| < t$$

where the positive quantities r, s and t are associated radii of convergence. For details about this function and many other three-variables hypergeometric functions, one refers to Srivastava and Karlsson [8].

Exton [2] gave the following Laplace integral representation of (1.1):

$$\begin{aligned} &X_8(a,\, b,\, c\,;\, d,\, e,\, f\,;\, x,\, y,\, z) \\ &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-u}\, u^{a-1}\, {}_0F_1(-;\, d\,;\, u^2\, x)\, {}_1F_1(b\,;\, e\,;\, u\, y)\, {}_1F_1(c\,;\, f\,;\, u\, z)\, du, \end{aligned} \tag{1.2}$$

provided $\Re(a) > 0$.

It may be remarked in passing that X_8 reduces to Horn's function H_4 when $z \to 0$ and the Appell's function F_2 when $x \to 0$.

Srivastava and Panda [9, p. 423, Eq.(26)] presented a definition of a general double hypergeometric function:

$$\begin{aligned} &F^{p\,:\,q\,;\,k}_{l\,:\,m\,;\,n}\left[\begin{array}{lll} (a_p) &: (b_q) &; (c_k) \;; \\ (\alpha_l) &: (\beta_m) &; (\gamma_n) \;; \end{array} x,\, y\right] \\ &= \sum_{r,\,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s \;\; x^r\, y^s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s \;\; r!\, s!}, \end{aligned} \tag{1.3}$$

where the several cases of convergence conditions are given in [7, p. 64]. Note that Srivastava and Panda's function (1.3) is more general than the one defined by Kampé de Fériet [3] (*cf.* Appell et Kampé de Fériet [1, p. 150, Eq.(29)].

In 1999, Padmanabham [4] obtained the following result for Exton's triple hypergeometric series X_8:

$$
\begin{aligned}
&X_8(a,\ b,\ c\,;\ d,\ e,\ f\,;\ x,\ y,\ z) \\
(1.4)\qquad &= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(e)_n\,n!}\, y^n\, {}_3F_2 \left[\begin{matrix} -n,\ 1-n-e,\ b \\ 1-n-b,\ f \end{matrix} ;\ -\frac{z}{y} \right] \\
&\times {}_2F_1 \left[\begin{matrix} \tfrac{1}{2}a+\tfrac{1}{2}n,\ \tfrac{1}{2}a+\tfrac{1}{2}n+\tfrac{1}{2} \\ d \end{matrix} ;\ 4x \right].
\end{aligned}
$$

In 2003, Padmanabham [4] established the following result for X_8:

(1.5)

$$
\begin{aligned}
&X_8(a,\ b,\ b\,;\ d,\ c,\ c\,;\ x,\ -x,\ x) \\
&= F^{2:2;0}_{0:3;1} \left[\begin{matrix} \tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2} & :\ b,\ c-b & ;\ --- & ; \\ ---- & :\ c,\ \tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2} & ;\ d & ; \end{matrix}\ x^2,\ 4x \right]
\end{aligned}
$$

with the help of the following classical Dixon's theorem [6, p. 92] for the well poised ${}_3F_2(1)$:

$$
\begin{aligned}
&{}_3F_2 \left[\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix} ;\ 1 \right] \\
(1.6)\qquad &= \frac{\Gamma\left(1+\tfrac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\tfrac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\tfrac{1}{2}a-b\right)\Gamma\left(1+\tfrac{1}{2}a-c\right)\Gamma(1+a-b-c)} \\
&\qquad (\Re\,(a-2b-2c) > -2)\,.
\end{aligned}
$$

The object of this note is to show how the identity (1.5) can be derived by starting with (1.4).

2. Derivation of (1.5) from (1.4)

Replacing c by b, e and f by c, y by $-x$, and z by x in (1.4), we have

$$\begin{aligned} X_8 &:= X_8(a,\, b,\, b\,;\, d,\, c,\, c\,;\, x,\, -x,\, x) \\ &= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n!}\,(-x)^n\, {}_3F_2\left[\begin{matrix} -n,\ b,\ 1-n-c \\ 1-n-b,\ c \end{matrix};\, 1\right] \\ &\quad \times {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}n,\ \frac{1}{2}a+\frac{1}{2}n+\frac{1}{2} \\ d \end{matrix};\, 4x\right]. \end{aligned} \tag{2.1}$$

Applying Dixon's theorem (1.6) to ${}_3F_2(1)$ in (2.1), we obtain

$$X_8 = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n!}\,(-x)^n\, {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}n,\ \frac{1}{2}a+\frac{1}{2}n+\frac{1}{2} \\ d \end{matrix};\, 4x\right] \mathcal{A}(b,\, c\,;\, n), \tag{2.2}$$

where, for convenience,

$$\mathcal{A}(b,\, c\,;\, n) := \frac{\Gamma(c)\,\Gamma(1-b-n)\,\Gamma\left(1-\frac{1}{2}n\right)\,\Gamma\left(c-b+\frac{1}{2}n\right)}{\Gamma(c-b)\,\Gamma(1-n)\,\Gamma\left(c+\frac{1}{2}n\right)\,\Gamma\left(1-b-\frac{1}{2}n\right)}.$$

By making use of Legendre's duplication formula for the Gamma function:

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2\alpha) = 2^{2\alpha-1}\,\Gamma(\alpha)\,\Gamma\left(\alpha+\frac{1}{2}\right),$$

we have

$$\mathcal{A}(b,\, c\,;\, n) = \frac{\Gamma(c)\,\Gamma(1-b-n)\,\Gamma\left(c-b+\frac{1}{2}n\right)}{\Gamma(c-b)\,\Gamma\left(c+\frac{1}{2}n\right)\,\Gamma\left(1-b-\frac{1}{2}n\right)} \cdot \frac{2^n\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}n\right)},$$

from which we see that

$$\mathcal{A}(b,\, c\,;\, n) = 0 \tag{2.3}$$

whenever n is an odd positive integer.

Considering (2.3), we can rewrite X_8 in (2.2) as follows:

$$\begin{aligned} X_8 &= \sum_{n=0}^{\infty} \frac{(a)_{2n}\,(b)_{2n}\,x^{2n}}{(c)_{2n}\,(2n)!}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}a+n,\ \frac{1}{2}a+\frac{1}{2}+n \\ d \end{matrix};\, 4x\right] \\ &\quad \times \frac{\Gamma(c)\,\Gamma(1-b-2n)\,\Gamma(c-b+n)}{\Gamma(c-b)\,\Gamma(c+n)\,\Gamma(1-b-n)} \cdot \frac{2^{2n}\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-n\right)}. \end{aligned} \tag{2.4}$$

Now, in (2.4), using the following well-known identities:

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^n}{(1-\alpha)_n}$$

and

$$(\alpha)_{2n}=2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha}{2}+\frac{1}{2}\right)_n,$$

we get

(2.5)
$$X_8=\sum_{n=0}^{\infty}\frac{(b)_n\,(c-b)_n\,\left(\frac{1}{2}a\right)_n\,\left(\frac{1}{2}a+\frac{1}{2}\right)_n}{(c)_n\,\left(\frac{1}{2}c\right)_n\,\left(\frac{1}{2}c+\frac{1}{2}\right)_n\,n!}\,x^{2n}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a+n,\ \frac{1}{2}a+\frac{1}{2}+n\\ d\end{matrix};4x\right].$$

Finally express ${}_2F_1$ in (2.5) as a series and use the identity

$$(\alpha)_n\,(\alpha+n)_m=(\alpha)_{m+n},$$

we have

$$X_8=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}a\right)_{m+n}\,\left(\frac{1}{2}a+\frac{1}{2}\right)_{m+n}\,(b)_n\,(c-b)_n\,x^{2n}\,(4x)^m}{(c)_n\,\left(\frac{1}{2}c\right)_n\,\left(\frac{1}{2}c+\frac{1}{2}\right)_n\,(d)_m\,n!\,m!},$$

which, upon using (1.3), becomes (1.5). This completes our desired proof.

We conclude by *noting* that the result (1.5) in Padmanabham's paper [5] contains several misprints.

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