Honam Mathematical J. **27** (2005), No. 4, pp. 551–559

THE CHROMATIC NUMBER OF SOME PERMUTATION GRAPHS OVER SOME GRAPHS

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Abstract. A permutation graph over a graph G is a generalization of both a graph bundle and a graph covering over G. In this paper, we characterize the F-permutation graphs over a graph whose chromatic numbers are 2. We determine the chromatic numbers of C_n-permutation graphs over a tree and the K_m-permutation graphs over a cycle.

1. Introduction

We consider only finite simple graphs. Every graph G has its vertex set $V(G)$ and its edge set $E(G)$. A *coloring* of a graph G is a mapping c from $V(G)$ to the set $\{1, 2, \ldots, k\}$ for some positive integer k. A coloring is said to be *proper* if $c(u) \neq c(v)$ for every edge uv of G. The *chromatic number* $\chi(G)$ is the smallest k such that G has a proper coloring into the set $\{1, 2, \ldots, k\}$.

Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$. A *permutation voltage assignment* ϕ of G is a function $\phi : D(G) \to S_n$ with the property

Received August 27, 2005. Revised October 25, 2005.

2000 Mathematics Subject Classification : Primary 05C15; Secondary 05C10, 55R99.

Key words and phrases : graph coloring, chromatic number, permutation graph.

The first author was supported in part by Com^{2}Mac-KOSEF, the second author was supported in part by KRF(R08-2004-000-10506-0).

that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$, where S_n is the symmetric group on n elements $\{1, \ldots, n\}$. It is known that every covering of G can be constructed from a permutation voltage assignment of G [2].

Lee and Sohn [7] defined a *permutation graph over a graph G* which is a generalization of both a graph bundle and a graph covering over G. Let F be a graph with $V(F) = \{v_1, v_2, \ldots, v_{|V(F)|}\}$. For a voltage assignment $\phi \in C^1(G; S_{|V(F)|})$ of G, we construct the F-permutation graph $G \bowtie^\phi F$ over G as follows: $V(G \bowtie^\phi F) = V(G) \times V(F)$. Two vertices (u_i, v_h) and (u_j, v_k) are adjacent in $G \bowtie^\phi F$ if either $u_i u_j \in D(G)$ and $k = \phi(u_i u_j)h$ or $u_i = u_j$ and $v_h v_k \in E(F)$.

Note that the group of all graph automorphisms $\mathrm{Aut}(F)$ of F is a subgroup of $S_{|V(F)|}$. If ϕ takes its values in $\mathrm{Aut}(F)$, then an F-permutation graph over G is just an F-bundle over G, $G \times^\phi F$ defined in [8], where the first coordinate projection $p^\phi : G \bowtie^\phi F \to G$ is the bundle projection.

In this paper, we characterize the F-permutation graphs over a graph whose chromatic numbers are 2. Let K_m be a complete graph of order m and C_n be a cycle with n vertices for $m, n \in \mathbb{N}$, we determine the chromatic numbers of C_n-permutation graphs over a tree and the K_m-permutation graphs over a cycle.

2. Results

First, we characterize F-permutation graphs over a graph whose chromatic numbers are 2. Notice that if F is non-bipartite, then every F-permutation graphs over G is non-bipartite. Let F be a connected bipartite graph with n vertices. Let ϕ be a permutation voltage assignment of G and let $W = e_1 e_2 \cdots e_n$ be a walk of length n in G. The product of voltages $\phi(e_n)\phi(e_{n-1}) \cdots \phi(e_1)$ is called the *net ϕ-voltage* of W and denoted by $\phi(W)$.

S. Hong et al. [3] characterized bipartite graph bundles over G. By modifying the method of the proof of Theorem 2.1 in [3], we have the following theorem.

Theorem 2.1. *Let F be a connected bipartite graph and let $\phi \in C^1(G;S_n)$. Then $\chi(G \bowtie^\phi F) = 2$ if and only if for each $uv \in D(G)$, $\phi(uv) \in \mathcal{P}(F) \cup \mathcal{R}(F)$ such that*

$$\phi(C) \in \begin{cases} \mathcal{P}(F) \text{ if } C \text{ is even cycle,} \\ \mathcal{R}(F) \text{ if } C \text{ is odd cycle,} \end{cases}$$

where $\mathcal{P}(F) = \{\alpha \in S_n | \alpha(V_1) = V_1, \alpha(V_2) = V_2\}$ and $\mathcal{R}(F) = \{\alpha \in S_n | \alpha(V_1) = V_2, \alpha(V_2) = V_1\}$ for the bipartition $\{V_1, V_2\}$ of $V(F)$.

Corollary 2.2. *Let $n(\geq 3)$ be a positive integer and let $\phi \in C^1(G;S_n)$. Then $\chi(G \bowtie^\phi C_n) = 2$ if and only if n is even and $\phi(uv) \in \mathcal{P}(C_n) \cup \mathcal{R}(C_n)$ for each $uv \in D(G)$ such that $\phi(C) \in \mathcal{P}(C_n)$ for any even cycle C of G and $\phi(C) \in \mathcal{R}(C_n)$ for any odd cycle C of G.*

Next, we determine the chromatic numbers of C_n-permutation graphs over a tree. In order to do this, we start with the following lemmas.

Lemma 2.3. *Let n be a positive integer and let P_n be a path with n vertices $v_1, v_2, \ldots, v_n$. Then for any function $\omega : V(P_n) \to \{1,2,3\}$ and any fixed $\alpha \in \{1,2,3\} - \{\omega(v_1)\}$, there exists a proper 3-coloring $c : V(P_n) \to \{1,2,3\}$ such that $c(v_1) = \alpha$ and $c(v_i) \neq \omega(v_i)$ for each $i = 2, \ldots, n$.*

Proof. We proceed by induction on the order n. If $n = 2$, then the function $c : V(P_2) \to \{1,2,3\}$ defined by $c(v_1) = \alpha$ and $c(v_2) = \{1,2,3\} - \{\alpha, \omega(v_2)\}$ is a proper 3-coloring of P_2. Assume that the lemma holds for $n = k$, that is, there exists a proper 3-coloring $c_k : V(P_n) \to \{1,2,3\}$ such that $c_k(v_1) = \alpha$, $c_k(v_i) \neq \omega(v_i)$ for each $i = 2, \ldots, k$. For $n = k+1$, we define a function $c_{k+1} : V(P_{k+1}) \to \{1,2,3\}$ by $c_{k+1}(v_i) = c_k(v_i)$ for $i = 1, \ldots, k$ and $c_{k+1}(v_{k+1}) = \{1,2,3\} - \{c_k(v_k), \omega(v_{k+1})\}$. Then c_{k+1} is a proper 3-coloring of P_{k+1}. It completes the proof. □

For a permutation $\alpha \in S_n$, an α-permutation graph $P_\alpha(G)$ consists of two copies of G, say G_x and G_y, with vertex sets $V(G_x) =$

$\{x_1, x_2, \ldots, x_n\}$ and $V(G_y) = \{y_1, y_2, \ldots, y_n\}$, along with edges $x_i y_{\alpha(i)}$ for $1 \le i \le n$.

Lemma 2.4. *Let $\alpha \in S_n$ and let $c : V(C_n) \to \{1,2,3\}$ be a proper 3-coloring. Then there exists a proper 3-coloring $\tilde{c} : V(P_\alpha(C_n)) \to \{1,2,3\}$ on $P_\alpha(C_n)$ such that $\tilde{c}|_{V(C_{n_x})} = c$.*

Proof. For each $i = 1,2,3$, let $V_i = \{\ v \in V(C_{n_y}) \mid c(\alpha^{-1}(v)) = i\}$. We define a function $\omega : V(C_{n_y}) \to \{1,2,3\}$ by $\omega(v_i) = i$ for each $v_i \in V_i$. Let $v \in V_i$ and consider the component of v in the induced graph $< V_i >$. Then the component must be a path which starts from v_s for some $s \in \{1,2,\ldots,n\}$. We can extend this path to a path of length $n-1$ in C_{n_y} which starts from v_s and ends at v_t, where $t = s-1$ or $s+1 \pmod{n}$. Moreover, v_t must be in V_j for some $j \ne i$. Now, by taking $\alpha = j$ and applying Lemma 2.3, we have a proper 3-coloring c' of the path $v_s \cdots v_t$ of length $n-1$ in C_{n_y} such that $c'(v_s) = j$ and $c'(v_t) \ne j$, which means that c' can be extended to a proper 3-coloring c'' of C_{n_y}. Now, we define $\tilde{c} : V(P_\alpha(C_n)) \to \{1,2,3\}$ by $\tilde{c}(v) = c(v)$ for each $v \in V(C_{n_x})$ and $\tilde{c}(v) = c''(v)$ for each $v \in V(C_{n_y})$. Then $\tilde{c}$ is a desired proper 3-coloring. □

Theorem 2.5. *Let T be a tree and let $\phi \in C^1(T; S_n)$. Then*

$$\chi(T \bowtie^\phi C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ & \text{and } \phi(uv) \in \mathcal{P}(C_n) \cup \mathcal{R}(C_n) \text{ for each } uv \in D(T), \\ 3 & \text{otherwise.} \end{cases}$$

Proof. It follows from Theorem 2.1 that $\chi(T \bowtie^\phi C_n) = 2$ if and only if n is even and $\phi(uv) \in \mathcal{P}(C_n) \cup \mathcal{R}(C_n)$ for each $uv \in D(T)$. Now to complete the proof, it suffices to show that $\chi(T \bowtie^\phi C_n) \le 3$ for any $\phi \in C^1(T; S_n)$. Fix $u_0 \in V(T)$, and let $c_{u_0} : C_{n_{u_0}} \to \{1,2,3\}$ be a proper 3-coloring, where $C_{n_{u_0}} = (p^\phi)^{-1}(u_0)$. Let u be a vertex of T. Then there exists a unique path $P = u_0 u_1 u_2 \cdots u_m u$ in T which connects u_0 and u.

Now, by applying Lemma 2.4 $m+1$ times, we can get a proper 3-coloring $c_u : C_{n_u} \to \{1,2,3\}$ such that the coloring $\tilde{c} : V(P \bowtie^{\phi} C_n) \to \{1,2,3\}$ defined by $\tilde{c}(u, v_i) = c_u(v_i)$ for each $(u, v_i) \in V(T) \times V(C_n)$ is a proper 3-coloring such that $\tilde{c}|_{C_{n_{u_0}}} = c_{u_0}$. It completes the proof. □

Finally, we determine the chromatic numbers of K_m-permutation graphs over a cycle. Notice that a K_m-permutation graph $G \bowtie^{\phi} K_m$ over G is the same as a K_m-bundle over G. In order to compute $\chi(C_n \bowtie^{\phi} K_m)$, we introduce the following notion and a useful theorem in terms of graph bundle isomorphisms; two F-bundles $G \times^{\phi} F$ and $G \times^{\psi} F$ are *isomorphic* if there exists an isomorphism $\Psi : G \times^{\phi} F \to G \times^{\psi} F$ such that $p_{\phi} = p_{\psi} \circ \Psi$.

An isomorphism class of F-bundles over G can be characterized through the corresponding equivalence classes of functions $\phi : D(G) \to \mathrm{Aut}\,(F)$ such that $\phi(e^{-1}) = \phi(e)^{-1}$. Let $C^0(G; \mathrm{Aut}\,(F))$ denote the set of functions $f : V(G) \to \mathrm{Aut}\,(F)$. Let $C^1(G; \mathrm{Aut}(F))$ denote the set of all voltage assignments of G.

Theorem 2.6. (Kwak and Lee [4]) *Let G and F be any two graphs.*

(a) *Two F-bundles $G \times^{\phi} F$ and $G \times^{\psi} F$ are isomorphic if and only if there exists $f \in C^0(G; \mathrm{Aut}\,(F))$ such that $\psi(uv) = f(v)\phi(uv)f(u)^{-1}$ for all $uv \in D(G)$.*

(b) *Any F-bundle $G \times^{\phi} F$ over G, there exists a voltage assignment $\phi' \in C^1(G; \mathrm{Aut}(F))$ which is trivial on the directed edges of a spanning tree T of G such that $G \times^{\phi'} F$ is isomorphic to an F-bundle $G \times^{\phi} F$.*

Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$ be the vertex sets of C_n and K_m, respectively. From Theorem 2.6 (b), we can see that for any K_m-permutation graphs over C_n, $C_n \bowtie^{\phi} K_m$, there exists a permutation $\sigma \in \mathrm{Aut}(K_m) = S_n$ such that $C_n \bowtie^{\phi} K_m$ and $C_n \bowtie^{\phi_{\sigma}} K_m$ are isomorphic, where ϕ_σ is a voltage assignment in $C^1(G;\mathrm{Aut}(K_m))$ defined by

$$\phi_\sigma(e) = \begin{cases} \sigma & \text{if } e = c_{n-1}c_n, \\ \text{identity} & \text{otherwise.} \end{cases}$$

This means that any voltage assignment ϕ in $C^1(G;\mathrm{Aut}(F))$ can be identified with a voltage assignment ϕ_σ for some σ in $\mathrm{Aut}(F)$.

Any permutation $\sigma \in S_m$ determine the cycle type $\jmath(\sigma) = (\jmath_1(\sigma), \jmath_2(\sigma), \cdots, \jmath_m(\sigma))$ of σ, where $\jmath_k(\sigma)$ is the number of cycles of length k in the factorization of σ in S_m into disjoint cycles, so that $\jmath_1(\sigma)+2\jmath_2(\sigma)+\cdots+m\jmath_m(\sigma) = m$.

Lemma 2.7. *Let $m(\geq 4)$ and let σ be a permutation in S_m. Then there exists a permutation $\tau \in S_m$ such that $\tau(i) \neq i$ and $\tau(i) \neq \sigma(i)$, for each $i = 1, 2, \ldots, m$. Moreover, the number of such τ's about given σ is*

$$m! + (-1)^k \sum_{k=1}^{m-1} \left(\sum_{\substack{r,\, s \in \{0,1,\ldots,k\} \\ r+s=k}} 2^s \binom{\jmath_1(\sigma)}{r} \binom{m-\jmath_1(\sigma)}{s} \right) (m-k)!,$$

where $\jmath_1(\sigma)$ is the number of cycles of length 1 in the cycle decomposition of $\sigma \in S_m$ into disjoint cycles.

Proof. We can easily show the existence of such a permutation τ by Theorem 7.4 [1]. Moreover, by the inclusion-exclusion principle, we can get the number of such permutations τ's from a given permutation σ. □

Lemma 2.7 has no meaning unless $m \geq 4$. Notice that Lemma 2.7 fails for $m = 3$. For example, let $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$. Then there is no τ in S_3 such that $\tau(i) \neq i$ and $\tau(i) \neq \sigma(i)$ for each $i = 1, 2, 3$.

Since we assume all graphs are simple, all cycles C_n in the following theorem have length bigger than 2.

Theorem 2.8. *Let m, n be positive integers.*

(1) $\quad m = 1, \quad \chi(C_n \bowtie^{\phi_\sigma} K_1) = \begin{cases} 2 & \text{if } n\text{: even,} \\ 3 & \text{if } n\text{: odd,} \end{cases}$

(2) $\quad m = 2, \quad \chi(C_n \bowtie^{\phi_\sigma} K_2) = \begin{cases} 2 & \text{if } n\text{: even and } \sigma\text{=identity,} \\ & \text{or } n : \text{odd and } \sigma = (12), \\ 3 & \text{otherwise,} \end{cases}$

(3) $\quad m \geq 3, \quad \chi(C_n \bowtie^{\phi_\sigma} K_3) = \begin{cases} 4 & \text{if } m = 3 \text{ and } \sigma = (12), (13) \text{ or} \\ & \quad (23), \\ m & \text{otherwise,} \end{cases}$

where $\sigma \in S_m$.

Proof. If $m = 1$, then $\chi(C_n \bowtie^{\phi_\sigma} K_1)$ is just C_n. Notice that $\chi(C_n) = 3 = m + 2$ if n is odd and $\chi(C_n) = 2 = m + 1$ if n is even. Hence, we have the value of the case. If $m = 2$, using the results of Theorem 2.1, it is not hard to find the conditions which induce $\chi(C_n \times^{\phi_\sigma} K_2) = 2$. Moreover, since $C_n \bowtie^{\phi_\sigma} K_2$ is neither a complete graph nor an odd cycle, $\chi(C_n \bowtie^{\phi_\sigma} K_2) \leq \Delta(C_n \bowtie^{\phi_\sigma} K_2) = 3$, where $\Delta(G)$ is the maximum degree of a graph G. If $m = 3$, Klavžar and Mohar proved in [6].

In the other cases, since $C_n \bowtie^{\phi} K_m$ contains K_m as a subgraph, we have $\chi(C_n \bowtie^{\phi} K_m) \geq m$. It suffices to show that $C_n \bowtie^{\phi} K_m$ has an m-coloring $c : V(C_n \bowtie^{\phi} K_m) \to \{1, 2, \ldots, m\}$. First, we color the fiber of u_h by

$$c(u_h, v_i) = \begin{cases} i & \text{if } h = \text{odd}, h \neq n, \\ d(i) & \text{if } h = \text{even}, h \neq n, \end{cases}$$

where d is a derangement in S_m. Next, we aim to find m-colorings of the fibers of remaining vertices of C_n (i.e., (u_n, v_i), $i = 1, \ldots, m$) such that the combination of these coloring and a suitable coloring of the fiber of u_n form an m-coloring of the bundle. To find such a suitable coloring, we divide the following two cases.

1) $n =$ odd. Let $\alpha = d\sigma^{-1}$. Then, by Lemma 2.7, there exits a permutation τ in S_m such that $\tau(i) \neq i$ and $\tau(i) \neq \alpha(i) = d\sigma^{-1}(i)$ for

each $i = 1,2, \ldots, m$. Now, we color the fiber of u_n by $c(u_n, v_i) = \tau(i)$. It is not hard to see that τ is a desired m-coloring of the fiber of u_n.

2) $n =$ even. By Lemma 2.7, there exists a permutation τ in S_m such that $\tau(i) \neq i$ and $\tau(i) \neq \sigma^{-1}(i)$ for each $i = 1,2, \ldots, m$. Now, we color the fiber of u_n by $c(u_n, v_i) = \tau(i)$, $i = 1, \ldots, m$. Then τ is a desired m-coloring of the fiber of u_n. □

Acknowledgements. The authors would like to thank the referee for helpful discussion.

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